to y. These parts (equivalence classes) are known as **orbits** of G on X. The orbit of x contains all the members of X which are of the form $g(x)$ for some $g \in G$, and it is usually written as Gx. Explicitly,

$$Gx = \{y \in X \mid y = g(x) \text{ for some } g \in G\}.$$

Intuitively, the orbit Gx contains all the objects which are indistinguishable from x under the action of G. For example, when G is the group of automorphisms of the graph shown in Fig. 21.3 the vertex set is partitioned into two orbits, $\{1, 3, 5\}$ and $\{2, 4, 6\}$, and we have

$$G1 = G3 = G5 = \{1, 3, 5\}, \qquad G2 = G4 = G6 = \{2, 4, 6\}.$$

Exercises 21.2

1 Write down all the automorphisms of the graph shown in Fig. 21.1. (There are only two of them!) Show that the group of automorphisms induces a partition of the vertex set into three orbits.

2 Let G be the group of automorphisms of the tree shown in Fig. 21.4a, acting on the set X of vertices. Determine the orbits of G on X.

3 Repeat Ex. 2 for the tree shown in Fig. 21.4b.

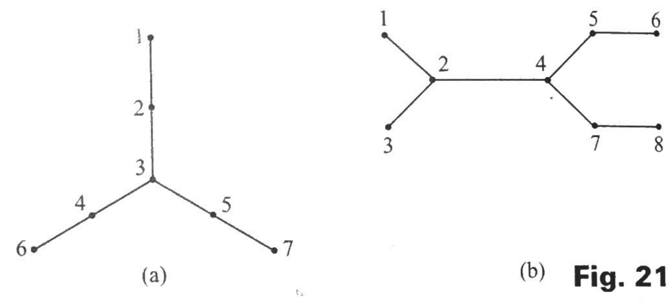

Fig. 21.4
Find the orbits.

There are two obvious numerical problems concerning orbits: we need to know how to find the size of each orbit, and how to find the number of orbits. The solution of these problems involves the combination of group-theoretical ideas on the one hand and elementary counting techniques on the other.

When G is a group of permutations of a set X we shall denote by $G(x \to y)$ the set of elements of G which take x to y; that is

$$G(x \to y) = \{g \in G \mid g(x) = y\}.$$

In particular, when $x = y$ the set $G(x \to x)$ contains all the permutations γ in G which **fix** x: that is, the permutations such that $\gamma(x) = x$. The set $G(x \to x)$ is called the **stabilizer** of x, and is written G_x. We note that if γ_1 and γ_2 are in G_x then

$$\gamma_1\gamma_2(x) = \gamma_1(x) = x,$$

and so $\gamma_1\gamma_2$ also belongs to G_x. Thus G_x is actually a *subgroup* of G.

Theorem 21.2 Let G be a group of permutations of X and suppose that h belongs to $G(x \to y)$. Then

$$G(x \to y) = hG_x,$$

the left coset of G_x with respect to h.

Proof If α is in the left coset hG_x we have $\alpha = h\beta$ for some β in G_x. Thus

$$\alpha(x) = h\beta(x) = h(x) = y,$$

and so α belongs to the set $G(x \to y)$. Conversely, if γ is in $G(x \to y)$ then

$$h^{-1}\gamma(x) = h^{-1}(y) = x$$

so that $h^{-1}\gamma = \delta$ where δ is in the stabilizer G_x. Thus $\gamma = h\delta$ is in hG_x. We have proved that every member of hG_x is also in $G(x \to y)$ and conversely, so it follows that the two sets are the same. □

Theorem 21.2 is best regarded as a rule for finding the size of the set $G(x \to y)$. We recall that any coset of a given subgroup has the same size as the subgroup itself, and so $|hG_x| = |G_x|$. Thus whenever there is an element h of G which takes x to y (in other words, whenever y is in the orbit Gx) we have the equation

$$|G(x \to y)| = |G_x| \quad (y \in Gx).$$

This holds for any y in Gx. On the other hand, if y is not in the orbit Gx then, by definition, there are no permutations taking x to y, and so

$$|G(x \to y)| = 0 \quad (y \notin Gx).$$

Exercises 21.2 (continued)

4 In the *Example* given in Section 21.1 verify explicitly that
(i) $|G(2 \to 6)| = |G_2|$, (ii) $|G(3 \to 1)| = |G_3|$.

5 Let G be a group of permutations of a set X and let k be an element of $G(x \to y)$. Prove that $G(x \to y)$ is equal to the *right* coset $G_y k$, and deduce that if u and v are any two elements in the same orbit of G then $|G_u| = |G_v|$.

6 Let $X = \mathbb{Z}_5$ and suppose that G is the cyclic group of permutations of X generated by the permutation π defined by the rule $\pi(x) = 2x$. Write down the elements of G in cycle notation and determine the orbits of G on X.

21.3 The size of an orbit

In this section we shall establish a fundamental relationship between the size of an orbit Gx and the size of the stabilizer G_x. We shall need the results obtained in the previous section, together with the techniques for counting sets of pairs developed in Section 10.2.

Let G be a group of permutations of a set X, and let x be a chosen element of X. The set S of pairs (g, y) such that $g(x) = y$ can be described by means of a table as in Section 10.2.

	$\cdots$ y $\cdots$	
$\vdots$ g $\vdots$	$\checkmark$ means that (g, y) is in S	$r_g(S)$
	$c_y(S)$	

The result can also be used to compute the order of a group of permutations, provided that we can calculate the size of an orbit and the corresponding stabilizer.

Example Let T be a regular tetrahedron in three-dimensional space (Fig. 21.5). Find the order of the group of rotational symmetries of T.

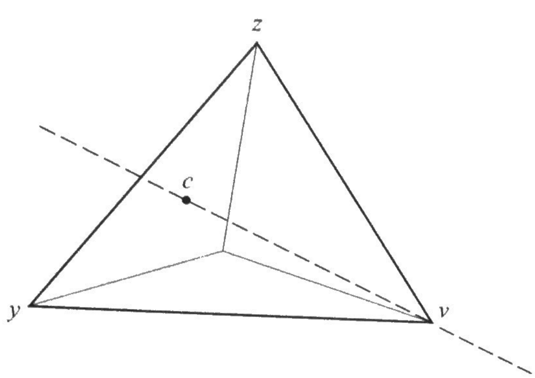

Fig. 21.5
A regular tetrahedron.

Solution Let G be the group of permutations of the corners which correspond to rotational symmetries, and let z be any corner of T. Given any other corner y there is an edge yz of T and there are two faces of T which are bounded by yz. Let c be the centroid of one of these faces and let v be the opposite corner of T (Fig. 21.5). Then a rotation of 120° about the axis cv (in the appropriate sense) takes z to y. Hence the orbit Gz contains all four vertices, and we have $|Gz| = 4$.

The only rotational symmetries which fix z are the rotations through 0°, 120°, and 240° about the altitude passing through z, and so the stabilizer G_z has order 3. Hence

$$|G| = |Gz| \times |G_z| = 4 \times 3 = 12. \qquad \square$$

Exercises 21.3

1 Label the corners of a regular tetrahedron T as 1, 2, 3, 4. Write down the permutations corresponding to the twelve rotational symmetries of T and verify that the group obtained is the alternating group A_4.

2 Let X denote the set of corners of a cube and let G denote the group of permutations of X which correspond to rotations of the cube. Show that:

(i) G has just one orbit on X;

(ii) if z is any corner, then $|G_z| = 3$;

(iii) $|G| = 24$.

3 Let V be the vertex-set of the graph Γ shown in Fig. 21.6, and let G be the automorphism group of Γ. Determine the orbits of G on V, and compute the orders of G_a, G_b, and G.

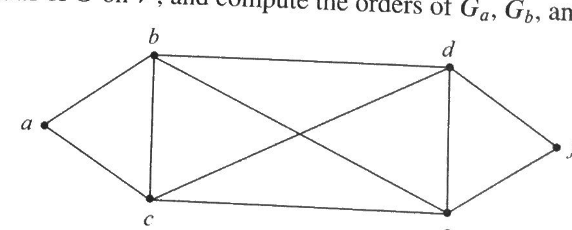

Fig. 21.6
Illustrating Ex. 21.3.3.

(c) There are eight reflections in axes joining the mid-points of opposite sides, and each of them has no fixed configurations.

(d) There are eight reflections in axes passing through opposite corners. The positions of the three black beads are unchanged (as a threesome) by such a reflection only if one of the beads occupies one of the two corners lying on the axis, and the other pair occupies one of the seven pairs of corners symmetrically placed with respect to this axis. Hence there are $2 \times 7 = 14$ fixed configurations for each reflection of this kind.

It follows that the number of different necklaces is

$$\tfrac{1}{32}(560 + (8 \times 14)) = 21. \qquad \square$$

Exercises 21.4

1 Show that there are just five different necklaces which can be constructed from five white beads and three black beads. Sketch them.

2 Suppose identity cards are manufactured from square cards ruled with a 4×4 grid, with two holes punched. How many different cards can be produced in this way?

3 Let V be the vertex set of the binary tree shown in Fig. 21.9, and let G be the group of automorphisms of the tree. Write down the elements of G (as permutations of V), and verify that Theorem 21.4 holds in this case.

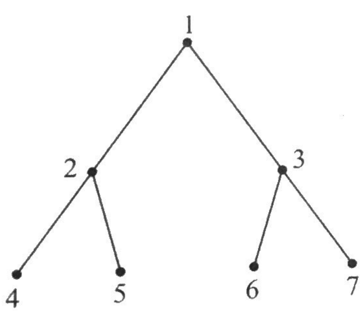

Fig. 21.9
Illustrating Ex. 21.4.3 and Ex. 21.4.4.

4 Let X denote the set of 'coloured trees' which result when each vertex of the tree in Fig. 21.9 is assigned one of the colours red or blue. How many different coloured trees of this kind are there?

5 Let G be a group of permutations of X, and let $I(x)$ be an expression which is constant on each orbit of G, so that

$$I(g(x)) = I(x) \quad \text{for all} \quad g \in G, \quad x \in X.$$

Let D be a set of representatives, one from each orbit, and let $E = \{(g, x) \mid g(x) = x\}$, as in the proof of Theorem 21.4. By evaluating the sum

$$\sum_{(g,x)\in E} I(x)$$

in two different ways, show that

$$\sum_{x\in D} I(x) = \frac{1}{|G|} \sum_{g\in G} \sum_{x\in F(g)} I(x).$$

(This is a 'weighted' version of Theorem 21.4, and reduces to it when $I(x) = 1$ for all $x \in X$. It will be used in Chapter 27.)

21.5 Representation of groups by permutations

Suppose we are given a group G, not specifically a group of permutations, and a set X. A **representation** of G by permutations of X assigns to each element g of G a permutation $\hat{g}$ of X, in such a way that the composition of group elements is compatible with the composition of the corresponding permutations. In other words,

$$\widehat{g_1 g_2} = \hat{g}_1 \hat{g}_2 \quad \text{for all } g_1 \text{ and } g_2 \text{ in } G.$$

21.7 Miscellaneous Exercises

1 Let G be any subgroup of S_n. Prove that either all the permutations in G are even, or exactly half of them are even.

2 Show that A_n is the only subgroup of index two in S_n.

3 Let π be a permutation which has cycles of lengths $n_1, n_2, \ldots, n_r$. Show that the order of π is the least common multiple of $n_1, n_2, \ldots, n_r$.

4 Construct subgroups of S_5 which have orders 1, 2, 3, 4, 5, 6, 8, 10, 12, 20, 24, 60, 120.

5 Investigate the possibility of subgroups of S_5 whose orders are not among the numbers listed in the previous exercise.

6 Let G be the group of automorphisms of the graph $K_{3,3}$ (as defined in Ex. 17.1.2) and let v be any vertex of the graph. Calculate $|G_v|$ and $|G|$.

7 Show that the group of automorphisms of Petersen's graph has order 120.

8 Suppose that each corner of a regular tetrahedron is assigned one of the colours red, white, and blue. Find the number of distinguishable tetrahedra of this kind.

9 Show that 57 different cubes can be constructed by painting each *face* of a cube red, white, or blue.

10 Let Y be the set of partitions of an 8-set into two parts of size 4. Let A_8 act on Y in the manner corresponding to its action on the 8-set. Show that there is just one orbit in this action, and calculate the order of a stabilizer.

11 Suppose that G is a group of permutations of a set X which is abelian and has just one orbit on X. Show that $|G| = |X|$.

12 Show that the number of different ID cards with an $n \times n$ square grid and two holes (as in Section 21.4) is

$$\tfrac{1}{16}(n^4 + 6n^2 - 4n) \qquad \text{if } n \text{ is even,}$$
$$\tfrac{1}{16}(n^4 + 8n^2 - 8n - 1) \qquad \text{if } n \text{ is odd.}$$

13 Suppose that ID cards are made in the shape of an equilateral triangle, with n equidistant lines ruled parallel to each edge (on both sides of the card) and a hole punched in one of the small triangles so formed. If n is an odd number of the form $6m + 1$ or $6m + 3$ show that the number of ID cards which can be constructed is $\frac{1}{6}(n+2)(n+3)$. What is the result when $n = 6m+5$?

14 How many different necklaces can be made from seven black beads and three white beads?

15 A circular disc is divided on one side into eight equal sectors, five of which are to be painted blue and three red. How many different discs can be made in this way?

16 Let p be a prime and define T_p to be the set of permutations of $\mathbb{Z}_p$ given by functions of the form

$$t(x) = ax + b \quad (x \in \mathbb{Z}_p)$$

where a and b are in $\mathbb{Z}_p$ and $a \neq 0$. Show that T_p is a group of order $p(p-1)$. Also prove that given any two ordered pairs (x_1, y_1) and (x_2, y_2) of distinct elements of $\mathbb{Z}_p$, there is a unique permutation t in T_p such that $t(x_1) = x_2$ and $t(y_1) = y_2$.

17 Let X be any set of transpositions in S_n, and define a graph $\Gamma(X)$ as follows. The vertices are the integers $1, 2, \ldots, n$, and ij is an edge if and only if there is a transposition in X which switches i and j. Show that $\Gamma(X)$ is connected if and only if any element of S_n can be expressed as a composite of transpositions belonging to X.

18 An **inversion** in a permutation π of $\mathbb{N}_n$ is a pair (i, j) such that $i < j$ and $\pi(i) > \pi(j)$. Show that if $q(\pi)$ is the total number of inversions in σ, then $\operatorname{sgn}(\pi) = (-1)^{q(\pi)}$.

In the next three exercises we shall consider a finite group G of rotations in three-dimensional space, each rotation having an axis which passes through the origin.

19 Let S be a sphere with the centre at the origin and suppose ρ is in G. The axis of ρ meets S in two points called the *poles* of ρ. Show that if x is a pole of ρ and $y = \theta(x)$ for some θ in G, then y is a pole of $\theta\rho\theta^{-1}$. Deduce that G acts on the set P of poles.

20 A pole x is said to have order m if it is the pole of a rotation of order m. Show that if x has order m_x then the size of the orbit of x is $|G|/m_x$. Deduce that

$$2(|G| - 1) = \sum \frac{|G|}{m_x}(m_x - 1),$$

where the sum is taken over the orbits of G on P.

21 Show that the formula obtained in the previous exercise can be rewritten as

$$2\left(1 - \frac{1}{|G|}\right) = \sum \left(1 - \frac{1}{m_x}\right).$$

Deduce that there are either two or three orbits, and that the possibilities are as given in the following table.

$\lvert G\rvert$:	n	$2n$	12	24	60
Order of poles:	n, n	$2, 2, n$	$2, 3, 3$	$2, 3, 4$	$2, 3, 5$