

## The Ascending and Descending Chain Propositions

These propositions are frequently useful in applications of the theory to various problems.

**Ascending Chain Proposition:** Let  $\langle E_n \rangle$  denote an ascending chain of measurable sets, i.e.,

$$E_1 \subset E_2 \subset E_3 \subset \cdots \subset E_n \subset E_{n+1} \subset \cdots .$$

Then

$$m \left( \bigcup_{n=1}^{\infty} E_n \right) = \lim_{n \rightarrow \infty} m(E_n).$$

Proof: This follows from the countable additivity of  $m$  on  $\mathcal{M}$ , as you will prove.

**Descending Chain Proposition:** Let  $\langle E_n \rangle$  denote a descending chain of measurable sets, i.e.,

$$E_1 \supset E_2 \supset E_3 \supset \cdots \supset E_n \supset E_{n+1} \supset \cdots .$$

Then if  $m(E_n) < \infty$  for some  $n = N$  (and hence for all  $n \geq N$ ),

$$m \left( \bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \rightarrow \infty} m(E_n).$$

Proof: In Royden. One of your exercises is to show that the proposition fails to hold when there is no value  $n = N$  for which  $m(E_N) < \infty$ —and hence  $m(E_n) = \infty$  for all values of  $n$ .