Some Properties of Extended Real-Valued Functions

Recall that f is an extended real-valued function if $f(x) = \pm\infty$ at some points $x \in D$, where $D \in \mathcal{M}$ is the domain of definition of f.

If f and g are extended real-valued functions on $D \in \mathcal{M}$, then the product $fg(x) = f(x)g(x)$ makes sense even if the right-hand side is $(\pm\infty) \cdot (\pm\infty)$. Indeed:

Proposition 1: *If f and g are extended real-valued measurable functions on $D \in \mathcal{M}$, then so is their product fg.*

If f and g are extended real-valued measurable functions on $D \in \mathcal{M}$, then $f(x) + g(x) = \pm(\infty - \infty)$—which is meaningless—is a possibility at some $x \in D$. However, this turns out not to be a problem provided the sets

$$A = \{x \in D : f(x) = \pm\infty\} \quad \text{and} \quad B = \{x \in D : g(x) = \pm\infty\}$$

have measure zero.[1] Thus the values of f and g are finite, except on sets of measure zero, or, as one also says, f and g are finite-valued *almost everywhere* (a.e. for short).

Proposition 2: *If f and g are measurable extended real-valued functions on $D \in \mathcal{M}$, then if they are finite almost everywhere, $f + g$ is measurable on D no matter how $f(x) + g(x)$ is defined when $f(x) + g(x) = \pm(\infty - \infty)$.*

Proofs of Propositions 1–2 are outlined in Royden's Problem #22, p. 71, which has not been assigned but is worth a look.

[1] A and B are measurable sets. E.g., $A = A_{-\infty} \cup A_{\infty}$, where, e.g., $A_{\infty} = \bigcap_{n=1}^{\infty} E_n$, where $E_n = \{x \in D : f(x) \geq n\} = E(f \geq n)$ is a measurable set, which means $A_{\infty} \in \mathcal{M}$.