MA 711 The Theorems of Fubini and Tonelli

Let $(X, \mathcal{E}, \mu)$ and $(Y, \mathcal{F}, \nu)$ be two **complete** measure spaces and $(X\times Y, \mathcal{L}(\mathcal{E}\times\mathcal{F}), \lambda = \mu\times\nu)$ the associated product measure space. **Suppose throughout what is to follow that** $f(x, y)$ **is a** λ**-measurable function**. In the table below are three theorems that establish an equality of the form

$$\int_{X\times Y} f(x,y)\, d(\mu\times\nu) = \int_X \left\{ \int_Y f(x,y)\, d\nu \right\} d\mu = \int_Y \left\{ \int_X f(x,y)\, d\mu \right\} d\nu \tag{1}$$

under various additional assumptions about f, μ, and ν.

The two main theorems positing (1) are the theorems of Fubini and Tonelli. As we shall see, they both follow fairly easily from a third theorem, which I have dubbed below "Fubini Lite". In stating the conclusions of each theorem in the table below, I have used the following notations:

$$f_x(y) = f(x,y) \quad (f \text{ as a function of } y \text{ for } x \text{ fixed}); \tag{2}$$

$$h(x) = \int f_x(y)\, d\nu = \int f(x,y)\, d\nu \quad (\text{for } x \text{ such that the integral exists}) \tag{3}$$

The Three Theorems

Fubini Lite	Fubini	Tonelli
If: $f \in L^1(\lambda)$ & $f \geq 0$	If: $f \in L^1(\lambda)$	If: $f \geq 0$ & μ, ν σ-finite
Then:	Then:	Then:
$f_x(y) \geq 0$ is ν-m'ble a.a.x	$f_x(y)$ is ν-int'ble a.a.x	$f_x(y) \geq 0$ is ν-m'ble a.a.x
$h(x) \geq 0$ is μ-m'ble	$h(x)$ is μ-int'ble	$h(x) \geq 0$ is μ-m'ble
$\int h(x)\, d\mu = \int f\, d\lambda$	$\int h(x)\, d\mu = \int f\, d\lambda$	$\int h(x)\, d\mu = \int f\, d\lambda$
Similar conclusions hold	with the roles	of μ and ν reversed