

Assignment 13B: L^p Spaces

1. Let (X, \mathcal{E}, μ) be any measure space. Prove that for any $f \in L^p(\mu)$, $1 \leq p < \infty$, and for any $\varepsilon > 0$ there is a simple function $\phi \in L^p(\mu)$ (which therefore necessarily vanishes off a set of finite measure) such that $\|f - \phi\|_p < \varepsilon$. (Hence the simple functions belonging to $L^p(\mu)$ form a dense subset.) [This is Royden #41, p. 287.]
2. Let (X, \mathcal{E}, μ) be any measure space, and let $f_n, f \in L^p(\mu)$, $1 \leq p < \infty$. (a) Show that if $f_n \rightarrow f$ in $L^p(\mu)$ (i.e. $\lim_{n \rightarrow \infty} \|f - f_n\|_p = 0$), then there is a subsequence $f_{n_k} \rightarrow f$ pointwise a.e. (b) Show by an example that the result of (a) cannot be strengthened so as to have $f_n(x) \rightarrow f(x)$ for almost all x .
3. Consider the L^p spaces corresponding to Lebesgue measure on $X = [0, 1]$. Show that $L^\infty \subset L^p$ for every $p \geq 1$ and that

$$\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p \quad \forall f \in L^\infty.$$

Please note: your proof should also establish that $\lim_{p \rightarrow \infty} \|f\|_p$ exists, *i.e.*, you cannot simply assume the limit exists and show it equals $\|f\|_\infty$. [This is Royden #2, p. 119.]

4–6. In each of the three Royden problems given below decide whether the L^p result to be proved for Lebesgue measure on $[0, 1]$ is actually valid for the the L^p space associated to an arbitrary measure space (X, \mathcal{E}, μ) . If it is, provide the general proof. If it is not, give the proof in the special case described by Royden and also give a counter-example to show it is not true for all (X, \mathcal{E}, μ) : p. 126: #10, #11, #16.