μ-Integrals and μ-Integrability

Let $\mathcal{M}$ denote an algebra of subsets of $\mathbb{R}$ that contains all intervals I (open, closed, half open, bounded, unbounded).

Suppose further that a set function $\mu : \mathcal{M} \to \mathbb{R}^*$ has been defined such that

(i) $\mu(I) = \ell(I) \equiv$ length of I

(ii) μ is finitely additive on $\mathcal{M}$ in the sense that if $E_1, \ldots, E_n$ belong to $\mathcal{M}$ and are pairwise disjoint ($E_i \cap E_j = \emptyset$ for $i \neq j$), then $\mu(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$.

Def. A μ-partition $\mathcal{P}$ of $[a,b]$ is a a finite sequence $E_1, \ldots, E_n$ of pairwise disjoint sets from $\mathcal{M}$ whose union is $[a,b]$: $[a,b] = \bigcup_{i=1}^n E_i$.

Remark. Since $\mathcal{M}$ contains all intervals it follows that among the μ-partitions are all the *Riemann partitions*, i.e., $a = x_0 < x_1 < \cdots < x_n = b$ and $E_1 = [a, x_1]$, $E_2 = (x_1, x_2]$, ..., $E_n = (x_{n-1}, b]$.

Lower & Upper μ-integrals. Let f be defined and bounded on $[a,b]$. Given any μ-partition $\mathcal{P} : [a,b] = \bigcup_{i=1}^n E_i$, let

$$L(\mathcal{P}) = \sum_{i=1}^n b_i \mu(E_i), \quad b_i = \inf_{x \in E_i} f(x), \quad \underline{\int_a^b} f\, d\mu = \sup_{\mathcal{P}} L(\mathcal{P})$$

$$U(\mathcal{P}) = \sum_{i=1}^n B_i \mu(E_i), \quad B_i = \sup_{x \in E_i} f(x), \quad \overline{\int_a^b} f\, d\mu = \inf_{\mathcal{P}} U(\mathcal{P}).$$

Def. Say that f is μ-integrable if $\underline{\int_a^b} f\, d\mu = \overline{\int_a^b} f\, d\mu$ and this common value is $\int_a^b f\, d\mu$.

Prop. A: For any bounded function f defined on $[a,b]$ the Riemann and μ-integrals are related by

$$(1) \qquad \underline{\int_a^b} f(x)\, dx \leq \underline{\int_a^b} f\, d\mu \leq \overline{\int_a^b} f\, d\mu \leq \overline{\int_a^b} f(x)\, dx.$$

Thus (1) implies every Riemann integrable function is μ-integrable but that there might exist μ-integrable functions that are not Riemann integrable.

Proof: The two outer inequalities are immediate consequences of the definitions and the fact that all Riemann partitions are μ-partitions. The middle inequality follows provided

$$(2) \qquad \forall\ \mathcal{P}_1,\ \mathcal{P}_2,\ L(\mathcal{P}_1) \le U(\mathcal{P}_2),$$

since then $\sup_{\mathcal{P}_1} L(\mathcal{P}_1) = \underline{\int}_a^b f\,d\mu \le U(\mathcal{P}_2)$ for all $\mathcal{P}_2$, which implies $\underline{\int}_a^b f\,d\mu \le \inf_{\mathcal{P}_2} U(\mathcal{P}) = \overline{\int}_a^b f\,d\mu$.

Inequality (2) follows from the fact that if $\mathcal{P}_1$ is $E_1, \ldots, E_m$ and $\mathcal{P}_2$ is $F_1, \ldots, F_n$, then the refinement $\mathcal{P}_1 \wedge \mathcal{P}_2$ given by $E_i \cap F_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ satisfies $L(\mathcal{P}_1 \wedge \mathcal{P}_2) \ge L(\mathcal{P}_1)$ and $U(\mathcal{P}_1 \wedge \mathcal{P}_2) \le U(\mathcal{P}_2)$.

For example, <u>by the finite additivity (ii) of μ</u>

$$\begin{aligned} L(\mathcal{P}_1) &= \sum_{i=1}^m b_i \mu(E_i) = \sum_{i=1}^m b_i \mu\Big(\bigcup_{j=1}^n E_i \cap F_j\Big) \\ &= \sum_{i=1}^m \sum_{j=1}^n b_i \mu(E_i \cap F_j) \le \sum_{i=1}^m \sum_{j=1}^n b_{ij} \mu(E_i \cap F_j) \end{aligned},$$

since $b_{ij} \equiv \inf_{x \in E_i \cap F_j} f(x) \ge b_i$ for all $j = 1, \ldots, n$. Thus

$$L(\mathcal{P}_1) \le L(\mathcal{P}_1 \wedge \mathcal{P}_2) \le U(\mathcal{P}_1 \wedge \mathcal{P}_2) \le U(\mathcal{P}_2),$$

the third inequality following by reasoning analogous to that for the first.