

Two Problems on m -integrals

(1) Suppose that f and g are m -integrable on $E = [a, b]$. Show that if $f(x) \leq g(x)$ for all $x \in E$, then $\int_E f \, dm \leq \int_E g \, dm$.

(2) Let f be m -integrable on $E = [a, b]$. Show that

$$\int_E f \, dm = \sup_{\varphi \leq f} \int_E \varphi \, dm.$$

NOTE: “ $\sup_{\varphi \leq f}$ ” means $\varphi(x) \leq f(x)$ for all $x \in E$ and that φ is a simple function.

HINTS: (a) Show that for every partition \mathcal{P} of E there is a simple function $\varphi_{\mathcal{P}}$ such that $L(\mathcal{P}) = \int_E \varphi_{\mathcal{P}} \, dm$. What can you conclude? (b) Show that for any simple function φ such that $\varphi(x) \leq f(x)$ for all $x \in E$ there is a partition \mathcal{P}_{φ} of E such that $\int_E \varphi \, dm \leq L(\mathcal{P}_{\varphi})$. What can you conclude?