Assignment 7: *Make sure you understand this.*

Recall that if φ is any nonnegative simple function, I defined $\int_E \varphi$ as follows: Let $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ denote the canonical representation of φ. (Hence $A_i = \{x : \varphi(x) = a_i > 0\}$ and $a_i \neq a_j$ for $i \neq j$, so $A_i \cap A_j = \emptyset$ for $i \neq j$.) Then by definition

$$\int_E \varphi = \sum_{i=1}^n a_i m(A_i \cap E). \tag{1}$$

This is not quite what Royden does in Chapter 4. The key to the properties of integrals of simple functions is the following

Lemma. *Let φ be a nonnegative simple function and $E \in \mathcal{M}$. Then if φ is represented in the form $\varphi = \sum_{j=1}^{\ell} c_j \chi_{E_j}$, where the E_j are pairwise disjoint measuable sets, then*

$$\int_E \varphi = \sum_{j=1}^{\ell} c_j m(E_j \cap E). \tag{2}$$

PROOF: The idea is that $A_i = \bigcup_{j'} E_{j'}$ where j' is such that $c_{j'} = a_i$. Thus $A_i \cap E = \cup_{j'} E_{j'} \cap E$ (disjoint union) and so $m(A_i \cap E) = \sum_{j'} m(E_{j'} \cap E)$, and (2) then follows readily from (1).

The following properties then follow readily from (2). (All Greek-letter functions are simple and nonnegative)

$$\int_E (a\varphi + b\psi) = a \int_E \varphi + b \int_E \psi \text{ for all } \quad a, b \geq 0. \tag{3}$$

$$\int_{A \cup B} \varphi = \int_A \varphi + \int_B \varphi \quad \text{if} \quad A \cap B = \emptyset. \tag{4}$$

$$\varphi(x) \leq \psi(x) \quad \forall \quad x \in E \Rightarrow \int_E \varphi \leq \int_E \psi. \tag{5}$$

Remarks: (3) follows by getting representations of φ and ψ using refinements of the canonical sets for φ and ψ, respectively, so that

$$\varphi = \sum_{j=1}^{\ell} c_j \chi_{E_j} \quad \text{and} \psi = \sum_{j=1}^{\ell} d_j \chi_{E_j}, \tag{6}$$

where the E_j are the same for φ and ψ. (I explained this once before in the context of m-integrals.) Then (6) and (2) yield (3), as well as (5). (4) follows directly from (2)—or even from (1).