MA711 **Problem Set 1** **September 13, 2007**

These problems are due at classtime on Tuesday, September 18. No late papers accepted. Problems must be written up legibly and in the order given below so as to facilitate grading. In proving something, you can use without proof anything from Royden's text that precedes the given problem. Also earlier assigned problems can be invoked—even if you didn't get them. **From this point on, you must work on & write up your proofs entirely on your own.** *Each regular problem is worth 10 points for a total of 30 points. Each extra credit problem is worth 3 points. Thus a grade of 42/30 or 140% is theoretically possible if all the regular and the extra credit problems are correctly done. Since the extra credit problems are not worth much point-wise, make sure you have done a good job on the required problems before dealing with them.*

1. Show that $f^{-1}[\bigcup_\lambda A_\lambda] = \bigcup_\lambda f^{-1}[A_\lambda]$. [This is Royden, p. 16 #17a.]

2. Let $\langle x_n \rangle$ and $\langle y_n \rangle$ denote two sequences of nonnegative real numbers (to simplify matters). Use Royden's definition of $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ as augmented by me with the intermediary sequences $s_n(a) = \sup_{k \geq n} a_k$ and $i_n(a) = \inf_{k \geq n} a_k$ to prove that

$$\overline{\lim}\, x_n + \underline{\lim}\, y_n \;\leq\; \overline{\lim}\, (x_n + y_n).$$

[This is a special case of part of Royden, p. 39 #16. Be careful with expressions that could be infinite.]

3. Prove Prop. 5: (a) Given any set A and any $\varepsilon > 0$, there is an open set O such that $A \subset O$ and $m^*(O) \leq m^*(A) + \varepsilon$. (b) There is a set G which is a G_δ set—i.e., G is a countable intersection of open sets—such that $A \subset G$ and $m^*(A) = m^*(G)$. [This is Royden, p. 58 #6 and it is easy.]

Optional Extra Credit Problems (graded on an all-or-nothing basis)

A. Let $\mathbb{Q}^*$ denote the set of all rational numbers of the form $\dfrac{p}{2q}$, where p is odd and p and q are relatively prime. Prove that $\mathbb{Q}^*$ is a dense subset of $\mathbb{R}$.

B. Let f be defined and bounded on $[a,b]$. Use Darboux's theorem on Riemann integrabity (Historical Introduction [Web], p. 20) to show that if f is increasing on $[a,b]$, i.e., $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$, then f is Riemann integrable on $[a,b]$, since the upper and lower Riemann integrals of f are equal. You can use without proof the fact that since f is increasing, the left- and right-hand limits $f(x-)$ and $f(x+)$ exist at every x (excluding of course $f(a-)$ and $f(b+)$ since $f(x)$ is given as defined on $[a,b]$).

C. Prove Dini's Theorem: Let f be defined on $[a,b]$ and differentiable there with $|f'(x)| \leq B$ for all $x \in [a,b]$. (a, b, B all finite real numbers). Then if $f'(x) = 0$ on a dense subset of $[a,b]$, either f is a constant function, or f' is not Riemann-integrable on $[a,b]$.

D. Let $\mathbb{Q}_1$ denote the set of rational numbers in $[0,1]$. Show that if $I_1, \ldots, I_n$ are any open intervals such that $\mathbb{Q}_1 \subset \bigcup_{j=1}^n I_j$, then $\sum_{j=1}^n \ell(I_j) \geq 1$. [This is Royden, p. 58 #5, which the computer did not select as a required problem.]