MA711 **Problem Set 2** **October 4, 2007**

These problems are due at classtime on Thursday, October 11. No late papers accepted. Problems must be written up legibly and in the order given below so as to facilitate grading. In proving something, you can use without proof anything from Royden's text that precedes the given problem. Also earlier assigned problems can be invoked—even if you didn't get them. **From this point on, you must work on & write up your proofs entirely on your own.** *Each regular problem is worth 10 points for a total of 40 points. Each extra credit problem is worth 5 points.*

1. Let E be a subset of $\mathbb{R}$ such that $m^*(E) < \infty$. Prove that (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c), where:

   (a) E is measurable

   (b) Given $\varepsilon > 0$ there is an open set $O \supset E$ with $m^*(O \sim E) < \varepsilon$

   (c) Given $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$

   [This is Royden, p.64 # 13a.]

2. Prove the Ascending Chain Proposition: If $\langle E_n \rangle$ is a sequence of measurable sets such that $E_n \subset E_{n+1}$ for all $n \in \mathbb{Z}^+$, then

$$m(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n).$$

[This is from Assignment 4.]

3. (a) Show that any function g that is continuous on $(-\infty, \infty)$ is measurable there. (b) Show that if f is a measurable real-valued function defined on some measurable set E and if g is as in (a), then the composite function $g \circ f$ is also a measurable real-valued function on E. [This is essentially Royden, p. 71 #25.]

4. Prove the following generalization of Fatou's Lemma: If $\langle f_n \rangle$ is a sequence of nonnegative measurable functions on a measurable set E, then

$$\int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n.$$

Optional Extra Credit Problems (graded on an all-or-nothing basis)

A. Prove Proposition 22. [This is Royden, p. 71 #23.]

B. Use the Monotone convergence Theorem to prove that if $f \geq 0$ is defined and measurable on $(-\infty, \infty)$ and is also integrable in the sense that $\int_{-\infty}^{\infty} f < \infty$, then $F(x) = \int_{-\infty}^{x} f$ is continuous. [This is Royden, p. 89 #5, and was discussed in Assignment 7B.]

C. Prove that if A is any set with $m^*(A) < \infty$, then A contains a nonmeasurable subset. [This is Royden, p. 66 #16 and comes with a hint containing assertions that must be proved in order to use the hint.]

D. Let $F \subset [0, 1]$ denote the generalized Cantor set (aka the "fat Cantor set") defined in Royden, p. 64 #14b. Show that (i) F is closed and (ii) $[0, 1] \sim F$ is dense in $[0, 1]$—equivalently: F is nowhere dense in $[0, 1]$]. (iii) Show that $m(F) = 1 - \alpha$, where $\alpha \in (0, 1)$ is the parameter in the construction of F.