MA711 **Problem Set 3** **October 25, 2007**

These problems are due at class time on Tuesday, October 30. No late papers accepted. Problems must be written up legibly and in the order given below so as to facilitate grading. In proving something, you can use without proof anything from Royden's text that precedes the given problem. Also earlier assigned problems can be invoked—even if you didn't get them. **From this point on, you must work on and write up your proofs entirely on your own.** *Each regular problem is worth 10 points for a total of 40 points.*

1. Let $f(x,t)$ be defined and bounded on the square $Q = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq 1\}$, and suppose that for each fixed t the function $x \to f(x,t)$ is a measurable function of x. Assume in addition that for each $(x,t) \in Q$ the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists and that

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \leq B \quad \text{for all } (x,t) \in Q.$$

Prove that

$$(\star) \qquad \frac{d}{dt}\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial f}{\partial t}(x,t)\,dx.$$

Note: This requires proving that the derivative on the left in $(\star)$, i.e., the derivative of $F(t) = \int_0^1 f(x,t)\,dx$ actually exists and that its value is the value given by the right-hand side of $(\star)$. [This is Royden, p. 94 #19.]

2. (a) Show that if f is integrable over $E \in \mathcal{M}$, then so is $|f|$ and $\left|\int_E f\right| \leq \int_E |f|$. (b) Suppose f is (Lebesgue) integrable on $[0,\infty)$ and that the improper Riemann integral

$$(R)\int_0^\infty f \stackrel{\text{def}}{=} \lim_{n\to\infty}\int_0^n f$$

exists. Show that $(R)\int_0^\infty f = \int_0^\infty f$. [This is the gist of Royden, p. 93 #10.]

3. Let $f \in L^1(E)$. Show that for any $\varepsilon > 0$ there exists a simple function φ such that $\int_E |f - \varphi| < \varepsilon$. [This is Royden, p. 93 #15a.]

4. Let $f \in BV[a,b]$. Prove that $\int_a^b |f'| \leq T_a^b(f)$. [This is Royden, p. 104 #11.]

Optional Extra Credit Problems

You may, if you wish, hand in solutions to one or more of the following problems. Each is worth 5 points, graded on an all-or-nothing basis, i.e., no partial credit.

A. If $\langle f_n \rangle$ is a sequence of functions on $[a,b]$ that converges to $f(x)$ for all $x \in [a,b]$, then $T_a^b(f) \leq \underline{\lim}\, T_a^b(f_n)$. [This is Royden, p. 104 #9.]

B. Let f be an increasing function defined on $[a,b]$. Show that for any $r < s$ the set $E_{rs} = \{x \in [a,b] : D^+f(x) < r < s < D_-f(x)\}$ has outer measure zero. [This is one of the four cases comprising the proof that $f'(x)$ exists a.e. One was done in lecture and another is given by Royden, p. 100.]