MA711 **Problem Set 4** **November 15, 2007**

These problems are due at class time on Tuesday, November 20. No late papers accepted. Problems must be written up legibly and in the order given below so as to facilitate grading. **Also please: Just one problem per page of paper. I need more space for comments.** *In proving something, you can use without proof anything from Royden's text that precedes the given problem. Also earlier assigned problems can be invoked—even if you didn't get them—but unassigned ones cannot be invoked unless you solve them.* **From this point on, you must work on and write up your proofs entirely on your own.** *Each regular problem is worth 10 points for a total of 60 points.*

1. Let g be an absolutely continuous monotone function on $[0,1]$. Show that if $E \subset [0,1]$ is such that $m(E) = 0$, then $m(g[E]) = 0$. This is Royden, p. 111 # 18.]

2. (a) Show that if $f \in AC[\varepsilon, 0]$ for every $\varepsilon > 0$ and if f is continuous at $x = 0$, then it need not be the case that $f \in AC[0,1]$. (b) Show that if f is as in part (a) but has the additional property that $f \in BV[0,1]$, then $f \in AC[0,1]$. Hint: There is an short, easy way to do this and a long, hard way, which involves a lot of logical pitfalls. Many who take the long route end up in a pit—so look for the short way! [This is essentially Royden, p. 110#12.]

3. Let $(X, \mathcal{E}, \mu)$ be an abstract measure space. (a) Show that if f is integrable then $S = \{x \in X : f(x) \neq 0\}$ is of σ-finite measure, i.e., is a countable union of sets of finite measure. (b) Show that if $f \geq 0$ is integrable, then there exists a sequence of simple functions $\langle \psi_n \rangle$ such that $\psi_n(x)$ monotonically increases to $f(x)$ for each $x \in X$ and, in addition, each ψ_n vanishes off a set of finite measure. *NOTE: In doing this problem, you can assume Prop. 7, p. 260.* [This is Royden, p. 268 #21a–b.]

4. Let $X = [0,1]$ and let $\mathcal{E}$ consist of all sets $E \subset X$ such that either E or $\widetilde{E} = [0,1] \sim E$ is countable (meaning empty, finite, or countably infinite). (a) Show that $\mathcal{E}$ is a σ-algebra. (b) Define ν on $\mathcal{E}$ by $\nu(E) = 0$ if E is countable and $\nu(E) = 1$ if $\widetilde{E}$ is countable. Prove that ν is a measure on $\mathcal{E}$, i.e., since it is obvious that $\nu(E) \geq 0$ and that $\nu(\emptyset) = 0$, prove that ν is countably additive. [Non-Royden problem from Assignment 14.]

5. Let $(X, \mathcal{E}, \mu)$ be a measure space, and suppose ν is a measure on $\mathcal{E}$ that is an indefinite integral, i.e., $\nu(E) = \int_E g\,d\mu$ for all $E \in \mathcal{E}$, where $g \geq 0$ is measurable. Prove that if f is any nonnegative measurable function, then $\int f\,d\nu = \int fg\,d\mu$. [This is based on Royden p. 268 #22b, which comes with a hint.]

6. Let μ, ν and λ be σ-finite measures on $(X, \mathcal{E})$. Show that the Radon Nikodym derivatives have the following properties (all functions below being assumed measurable):

(a) $$\nu << \mu \quad \text{and} \quad f \geq 0 \quad \Rightarrow \int f\,d\nu = \int f \left[\frac{d\nu}{d\mu}\right] d\mu;$$

(b) $$\left[\frac{d(\nu_1 + \nu_2)}{d\mu}\right] = \left[\frac{d\nu_1}{d\mu}\right] + \left[\frac{d\nu_2}{d\mu}\right];$$

(c) $$\nu << \mu << \lambda \quad \Rightarrow \quad \left[\frac{d\nu}{d\lambda}\right] = \left[\frac{d\nu}{d\mu}\right]\left[\frac{d\mu}{d\lambda}\right].$$ [Royden, p. 281 #34.]

See Reverse Side for Optional Extra Credit Problems

Optional Extra Credit Problems

You may, if you wish, hand in solutions to one or more of the following problems. Each is worth 5 points, graded on an all-or-nothing basis, i.e., <u>no partial credit</u>.

E1. Problem B from Assignment 11, Part II.

E2. Problem C from Assignment 11, Part II.

E3. Royden, p. 111 #21.

E4. Royden, p. 111 #22 (See remark in Assignment 12 about a possible typo in statement of problem.)

E5. Give a simple example of a function f defined on $[0,1]$ such that f is absolutely continuous in the usual sense, but not in the stronger sense that permits overlapping intervals. (See Assignment 12 for more details.)

E6. Royden, p. 71 #28 modified as explained in Assignment 12.