MA711 **Problem Set 5** **November 29, 2007**

These problems are due at class time on Tuesday,December 4. No late papers accepted. Problems must be written up legibly and in the order given below so as to facilitate grading. **Also please: Just one problem per page of paper. I need more space for comments.** *In proving something, you can use without proof anything from Royden's text that precedes the given problem. Also earlier assigned problems can be invoked—even if you didn't get them—but unassigned ones cannot be invoked unless you solve them.* **From this point on, you must work on and write up your proofs entirely on your own.** *Each regular problem is worth 10 points for a total of 40 points.*

1. Let $(X, \mathcal{E}, \mu)$ be an abstract measure space. Fatou's Lemma says that if $\langle f_n \rangle$ is a sequence of nonnegative μ-measurable functions such that (a) $f_n \to f$ a.e., then (b) $\int f\, d\mu \leq \underline{\lim} \int f_n\, d\mu$. Show that if (a) is replaced by (a*) $f_n \xrightarrow{\mu} f$, then (b) still follows. [Since Fatou's Lemma is the basis for all the convergence theorems, it follows that in all of them pointwise convergence can be replaced by convergence in measure. This is Royden, p. 96 #21 except for abstract measure spaces; it comes with a hint, which, if you use it, requires proving #20 for abstract measure spaces.]

2. Let $(X, \mathcal{E}, \mu)$ be any measure space. Prove that for any $f \in L^p$, $1 \leq p < \infty$, and for any $\varepsilon > 0$ there is a simple function $\phi \in L^p$ (which therefore necessarily vanishes off a set of finite measure) such that $||f - \phi||_p < \varepsilon$. (Hence the simple functions belonging to L^p form a dense subset.) [This is Royden #41, p. 287.]

3. Decide whether Royden #16, p. 126, is valid for any measure space $(X\mathcal{E}, \mu)$. If it is, prove it within that context. If it is not, give a counter example and also prove the result as stated for $L^p[0, 1]$.

4. Consider the L^p spaces corresponding to Lebesgue measure on $X = [0, 1]$. Show that $L^\infty[0, 1] \subset L^p[0, 1]$ for every $p \geq 1$ and that

$$||f||_\infty = \lim_{p \to \infty} ||f||_p \quad \forall \quad f \in L^\infty.$$

Please note: your proof should establish that $\lim_{p\to\infty} ||f||_p$ exists, *i.e.*, you cannot simply assume the limit exists and show it equals $||f||_\infty$. [This is Royden #2, p. 119.]

Optional Extra Credit Problems (5 points each, all or nothing)

A. Royden, p. 287 # 42.

B. Show that $R^p[0, 1]$ is not complete, where $R^p[0, 1]$ consists of all functions f such that $(R) \int_0^1 |f(x)|^p dx < \infty$. (Thus $R^p[0, 1]$ consists of all bounded Riemann-integrable functions togther with unbounded functions for which the improper Riemann integral of $|f|^p$ exists over $[0, 1]$.) You may find it helpful to use the theorem that a bounded function f is Riemann-integrable iff its points of discontinuity form a set of measure zero, i.e. iff f is continuous a.e.