MA711 Problem Set 6, Part I (20 Points) December 4, 2006

Problem Set 6 is in two Parts, each worth 20 points for a total of 40 points. In addition there are 3 optional extra credit problems worth a total of 20 points. All the problems (both parts) are due no later than **11:00 AM Saturday, December 15**. *Place them in my mailbox or under my office door. No late papers accepted. Problems must be written up* **legibly, in the order given below , with each problem on a separate page** *so as to facilitate grading. In proving something, you can use without proof anything from Royden's text that precedes the given problem. Also earlier assigned problems can be invoked—even if you didn't get it.* **You must work on and write up your proofs entirely on your own.** *Each required problem on Part I is worth 10 points.*

1. Let $(X, \mathcal{E}, \mu)$ and $(Y, \mathcal{F}, \nu)$ be any two complete measure spaces. Suppose $g(x)$ is μ-integrable and $h(y)$ is ν-integrable. Show that $f(x, y) = g(x)h(y)$ is integrable with respect to $\lambda = \mu \times \nu$ and that

$$\int f(x, y) d(\mu \times \nu) = \left(\int g(x)\, d\mu \right) \left(\int h(y)\, d\nu \right).$$

Please note: (i) Before you show f is integrable, you must show it is measurable with respect to $\lambda = \mu \times \nu$; (ii) $(X, \mathcal{E}, \mu)$ and $(Y, \mathcal{F}, \nu)$ are not assumed to be sigma finite. [This is Royden #22, p. 311.]

2. Let $(X, \mathcal{E}, \mu)$ and $(Y, \mathcal{F}, \nu)$ both denote Lebesgue measure m on the Lebesgue measurable subsets of $\mathbb{R}$. Then the product measure $\lambda = \mu \times \nu$ defines in this case two-dimensional Lebesgue measure and will be denoted by m_2. Let f denote a non-negative Lebesgue integrable function on $\mathbb{R}$. The sigma algebra $\mathcal{L}$ of m_2-measurable subsets of $X \times Y = \mathbb{R}^2$ are called the Lebesgue measurable subsets of $\mathbb{R}^2$.

(a) Show that the following sets are m_2-measurable:

$$E(f) = \{(x, y) : 0 < y < f(x)\} \quad \text{and} \quad \bar{E}(f) = \{(x, y) : 0 \le y \le f(x)\}.$$

(b) Prove that $m_2[E(f)] = m_2[\bar{E}(f)] = \int f(x)\, dx$.

Note: (b) says that the integral of a nonegative function equals the area of the region between its graph and the x-axis (regardless of whether the boundary lines are included). [This is part of Royden p. 312 #31.]

Optional Extra Credit Problems (5 points each, no partial credit)

A. Let m_2 denote Lebesgue measure in $\mathbb{R}^2$ as in problem 2 above. The Borel sets in $\mathbb{R}^2$ constitute, by definition, the smallest σ-algebra of subsets of $\mathbb{R}^2$ that contains all open sets. Prove that every Borel set is m_2-measurable, i.e., is in the σ-algebra $\mathcal{L}$ obtained by applying the Carathéodory extension procedure with $\mu = \nu = m$. [Cf. Royden, p.310 #21.]

B. Infinite Products of Probability Spaces: Royden, p. 313, # 33

MA711　　Problem Set 6, Part II (20 Points)　　December 6, 2006

Problem Set 6 is in two Parts, each worth 20 points for a total of 40 points. In addition there are 3 optional extra credit problems worth a total of 20 points. All the problems (both parts) are due no later than **11:00 AM Saturday, December 15**. *Place them in my mailbox or under my office door. No late papers accepted. Problems must be written up* **legibly, in the order given below , with each problem on a separate page** *so as to facilitate grading. In proving something, you can use without proof anything from Royden's text that precedes the given problem. Also an earlier assigned problems can be invoked—even if you didn't get it.* **You must work on and write up your proofs entirely on your own.** *Problem 3 below is worth 20 points, 5 points for each part.*

3. In what follows let F be increasing and right-hand continuous, and let μ_F be the associated Lebesgue-Stieltjes measure restricted to $\mathcal{B}$, the σ-algebra of Borel subsets of $\mathbb{R}$. Also m denotes Lebesgue measure but restricted to $\mathcal{B}$. [This is Royden #17, p. 303, albeit clarified and expanded.]

(a) Show that $\mu_F << m$ if and only if $F \in \mathrm{AC}[a, b]$ for all finite $[a, b]$.

(b) If $\mu_F << m$, show that the Radon-Nikodym derivative of μ_F with respect to Lebesgue measure m is just the ordinary derivative of F, i.e., show that $\frac{d\mu_F}{dm} = F'$ a.e (in the sense of Lebesgue measure).

(c) Show that if $F \in \mathrm{AC}[a, b]$ for all finite intervals $[a, b]$, then $\int f d\mu_F = \int f F' dm$ for every nonnegative Borel-measurable function f and also for every $f \in L^1(\mu_F)$. (This shows that when F has the absolute continuity property, Lebesgue-Stieltjes integrals reduce to Lebesgue integrals, which could be expressed symbolically by $d\mu_F = F'(x)dx$.)

(d) Suppose that F is also *bounded* (as is the case, e.g., for a cumulative distribution function associated to a random variable). Show that for such F the following are equivalent:

(i) $\mu_F << m$;

(ii) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $m(B) < \delta$ ($B \in \mathcal{B}$) implies $\mu_F(B) < \varepsilon$;

(iii) $F \in \mathrm{AC}(-\infty, \infty)$, where this means that F satisfies the ε-δ definition of absolute continuity with respect to the interval $I = (-\infty, \infty)$, i.e., the nonoverlapping intervals involved in the ε-δ definition are not constrained to lie in a finite interval $[a, b]$.

Optional Extra Credit Problem (10 points, no partial credit)

This is the most difficult of the extra-credit problems and so is worth 10 points—but do not attempt unless all other work is completed.

C. With the same set-up as in #3 above: Prove that $\mu_F \perp m$ if and only if F is singular, i.e., if and only if $F' = 0$ a.e. (in sense of m).