MA 711 Assigned Problem:

Recall that the connection between point functions $f(x)$ and set functions $\nu(E)$ is given by $\nu[p,q] = f(q) - f(p)$. Expressed in terms of ν the total, positive and negative variations of f on $[a,b]$ are:

$$T_a^b(f) = \sup_{\mathcal{P}} \sum_{i=1}^{n} |\nu(I_i)|, \quad P_a^b(f) = \sup_{\mathcal{P}} \sum_{i=1}^{n} [\nu(I_i)]^+, \quad N_a^b(f) = \sup_{\mathcal{P}} \sum_{i=1}^{n} [\nu(I_i)]^-,$$

where $\mathcal{P}$ is the partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$ into (almost disjoint) subintervals $I_i = [x_{i-1}, x_i]$.

With this in mind let $\mathcal{E}$ denote a sigma algebra of subsets of a non-empty set X, and let ν be a signed measure on $\mathcal{E}$. Fix $E \in \mathcal{E}$, and let $\mathcal{P}$ denote a partition of E into a finite number of disjoint subsets $E_i \in \mathcal{E}$, so $E = \cup_{i=1}^n E_i$, and $E_i \cap E_j = \emptyset$ for $i \neq j$. Then the total, positive and negative variations of ν on E are defined, respectively, by:

$$|\nu|(E) = \sup_{\mathcal{P}} \sum_{i=1}^{n} |\nu(E_i)|, \quad \nu^+(E) = \sup_{\mathcal{P}} \sum_{i=1}^{n} [\nu(E)]^+, \quad \nu^-(E) = \sup_{\mathcal{P}} \sum_{i=1}^{n} [\nu(E)]^-.$$

Show that $|\nu| = \nu_1 + \nu_2$, $\nu^+ = v_1$ and $\nu^- = \nu_2$, where ν_1 and ν_2 are defined with respect to a Hahn Decomposition $X = P \cup N$ by $\nu_1(E) = \nu(E \cap P)$ and $\nu_2 = -\nu(E \cap N)$. Hence conclude that the total, positive and negative variations of a signed measure are measures and that $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$, which is analogous to $f(b) - f(a) = P_a^b(f) - N_a^b(f)$ and $T_a^b(f) = P_a^b(f) + N_a^b(f)$.