1 Local Whittle Method

Unlike the Whittle estimator, the local Whittle estimator is a semi-parametric estimator, in that it only specifies the parametric form of the spectral density when ν is close to zero, namely,

$$f(\nu) \sim G(H)|\nu|^{1-2H} \qquad \text{as } \nu \to 0. \tag{1.1}$$

The local Whittle estimator is also based on the periodogram. Its computation involves the introduction of an additional parameter m, which is an integer less than $N/2$, and satisfying, as $N \to \infty$, $1/m + m/N \to 0$. For a spectral density satisfying (1.1), the discrete analogue of the function Q in the Whittle estimator is

$$Q(G, H) := \frac{1}{m} \sum_{j=1}^{m} \left(\frac{I(\nu_j)}{G\nu_j^{1-2H}} + \log G\nu_j^{1-2H} \right) \tag{1.2}$$

where $\nu_j = 2\pi j/N$. Note the two differences with the Whittle estimator: (a) the frequencies are integrated (summed) not up to $2\pi[(N-1)/2]/N$ but only up to $2\pi m/N$; (b) the spectral density f is replaced by its asymptotic form (1.1).

Replacing in (1.2) the constant G by its estimate $\hat{G} = \frac{1}{m}\sum_{j=1}^{m} \frac{I(\nu_j)}{\nu_j^{1-2H}}$, one obtains

$$R(H) := Q(\hat{G}, H) - 1 = \log\left(\frac{1}{m}\sum_{j=1}^{m} \frac{I(\nu_j)}{\nu_j^{1-2H}}\right) - (2H-1)\frac{1}{m}\sum_{j=1}^{m} \log \nu_j. \tag{1.3}$$

The value $\hat{H}$ which minimizes $R(H)$ converges in probability to the actual value H_0, as $N \to \infty$. Robinson [1] shows that

$$m^{1/2}(\hat{H} - H_0) \to_d \text{Normal } (0, 1/4), \qquad \text{as } N \to \infty. \tag{1.4}$$

This last relation yields asymptotic ("theoretical") values for the standard deviations, confidence intervals, etc.

The choice of m can be quite important. The larger the value of m, the faster $\hat{H}$ converges to H_0 (see (1.4)). On the other hand, if the series is not "ideal", e.g. if it is FARIMA$(1, d, 1)$, then we would want m to be small. In such a case we only want to use the frequencies which are close to zero, since at larger frequencies the short-range behaviour of the series will affect the form of the spectral density.

The Local Whittle estimator was developed by Robinson [1] and was further studied, along with the Whittle and Aggregated Whittle estimators, by Taqqu and Teverovsky [2], [3].

References

[1] P. M. Robinson. Gaussian semiparametric estimation of long range dependence. *The Annals of Statistics*, 23:1630–1661, 1995.

[2] M. S. Taqqu and V. Teverovsky. Robustness of Whittle-type estimates for time series with long-range dependence. Preprint, 1995.

[3] M. S. Taqqu and V. Teverovsky. Semi-parametric graphical estimation techniques for long-memory data. In P. M. Robinson and M. Rosenblatt, editors, *Athens Conference on Applied Probability and Time Series Analysis. Volume II: Time Series Analysis in Memory of E. J. Hannan*, pages 420–432, New York, 1996. Springer-Verlag. Lecture Notes in Statistics, **115**.