1 Periodogram Method

The periodogram is defined as

$$I(\nu) = \frac{1}{2\pi N} \left| \sum_{j=1}^{N} X(j) e^{ij\nu} \right|^2 , \tag{1.1}$$

where ν is the frequency, N is the length of the series, and X is the time series. In the finite variance case, $I(\nu)$ is an estimator of the spectral density of X, and a series with long-range dependence will have a spectral density proportional to $|\nu|^{1-2H}$ for frequencies close to the origin. A log-log regression thus provides an estimate of d.

We thus expect a log-log plot of the periodogram versus the frequency to display a straight line with a slope of $1 - 2H$. In practice we use only the lowest 10% of the frequencies for the calculation, since this behavior holds only for frequencies close to zero. See [1], [4], [2].

2 Modified Periodogram

A modification of the periodogram method compensates for the fact that on a log-log plot, most of the frequencies fall on the far right, thus exerting a very strong influence on the least-squared line fitted to the periodogram. The frequency axis is divided into logarithmically equally spaced boxes, and the periodogram values corresponding to the frequencies inside the box are averaged. Several of the values at very low frequencies are left untouched, since there are so few of them to begin with. Because the periodogram is very scattered we use a robustified least-squares (the *ltsreg* in Splus), formally a "least-trimmed squares regression" which minimizes the sum of the $q \simeq n/2$ smallest squared residuals. In general, results do not seem an improvement on the original periodogram method. See [4], [2].

3 Periodogram with a continuum of cut-offs

In this version, the periodogram is calculated for several values of the cut-off frequency. (It is also called the "continuous periodogram".) In the plot, extra short-range dependence is reflected by a transition stage before the plot levels out. The value at which the plot flattens out is where the estimate of H should be taken. See [3].

4 Averaged (Cumulative) Periodogram

Instead of using the periodogram itself, we can calculate the cumulative periodogram, and then plot that on a log-log plot. The results should be proportional to $|\nu|^{2-2H}$.

The advantage of this method is that it is somewhat smoother than the periodogram itself, although more approximations have to be used. There is an additional source of bias due to the fact that the periodogram is discrete, and thus instead of integrating, which would result in $|\nu|^{2-2H}$, we have to sum. To account for this, we propose a method where we define the following statistic:

$$F_L(\nu_k) := \frac{2\pi}{N} \sum_{j=L+1}^{k} I(\nu_j), \quad L < k, \tag{4.1}$$

ν_L representing a lower cut-off frequency. For small enough k,

$$F_L(\nu_k) \sim C_1(\nu_k^{2-2H} - \nu_L^{2-2H}). \tag{4.2}$$

Thus, one picks values of L and M, calculates $F_L(\nu_k)$ for $L < k \leq M$, and performs a least-squares fit of $F_L(\nu_k)$ versus $C_1(\nu_k^{2-2H} - \nu_L^{2-2H})$. The results are thus not graphical, since the function is no longer linear on a log-log plot.

The periodogram and modified periodogram are discussed in Taqqu, Teverovsky and Willinger [4] and Taqqu and Teverovsky [2], as well as, originally, by Geweke and Porter-Hudak, [1]. The periodogram with a continuum of cut-offs is discussed in Taqqu and Teverovsky [3].

References

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