We present here the fractional ARIMA (FARIMA) time series. These are linear processes of the form

$$X_i = \sum_{j=0}^{\infty} c_{i-j}\epsilon_j, \tag{0.1}$$

with "innovations" ϵ_j that are i.i.d. random variables. The innovations will be either Gaussian, non-Gaussian with finite variance or they may have infinite variance, in particular, a stable distribution.

1 Gaussian FARIMA series

Gaussian FARIMA(p, d, q) processes are widely used in the modelling of time series. A Gaussian FARIMA$(0, d, 0)$ process is defined by $X_i = \Delta^{-d}\epsilon_i,\ \ i \geq 1$, where the ϵ_i are independent, identically distributed Gaussian random variables with mean 0, and where Δ is the differencing operator $\Delta\epsilon_i = \epsilon_i - \epsilon_{i-1}$. For fractional d, we interpret Δ^{-d} by using the formal power series expansion,

$$\Delta^{-d} = \sum_{i=0}^{\infty} b_i(-d)B^i, \tag{1.1}$$

where B is the backward operator, $B\epsilon_i = \epsilon_{i-1}$,

$$b_i(-d) = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}, \quad i = 1, 2, \ldots.$$

and Γ denotes the gamma function. In this case, $c_i = b_i(-d)$ in (0.1). Observe that $c_i \sim (\Gamma(d))^{-1}i^{d-1}$ as $i \to \infty$. The autocovariance function of the process $X = \{X_i, i \geq 1\}$ satisfies, for $-1/2 < d < 1/2$,

$$\gamma(h) = EX_iX_{i+h} \sim C_d h^{2d-1} \quad \text{as} \quad h \to \infty \tag{1.2}$$

where $C_d = \pi^{-1}\Gamma(1-2d)\sin \pi d$. Thus, the autocovariance decreases like a power function for large lags. The spectral density (Fourier transform of γ) of a Gaussian FARIMA$(0, d, 0)$ process is given by:

$$f(\nu) = C_d\Big(2\sin\frac{\nu}{2}\Big)^2 \sum_{k=-\infty}^{\infty} \frac{1}{|\nu + 2\pi k|^{2d+2}} \sim C_d|\nu|^{-2d} \text{ as } \nu \to 0, \tag{1.3}$$

The spectral density behaves like a power function at the origin. Long-range dependence corresponds to $d > 0$, because in this case, the series spectral density diverges to infinity at the origin.

A Gaussian FARIMA$(0, d, 0)$ is a particular case of Gaussian FARIMA(p, d, q) processes. These are defined through the equation

$$X_i - \phi_1 X_{i-1} - \cdots - \phi_p X_{i-p} = \Delta^{-d}\epsilon_i - \theta_1\Delta^{-d}\epsilon_{i-1} - \cdots - \theta_q\Delta^{-d}\epsilon_{i-q}, \tag{1.4}$$

where the $\Delta^{-d}\epsilon_i$ are as above and the ϕ_i and the θ_i are the autoregressive and the moving average coefficients respectively. The coefficients $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ and the exponent d determine the c_i's in (0.1). One still has $c_i \sim Ci^{d-1}$ as $i \to \infty$, where C is a constant. The Gaussian FARIMA(p, d, q) series follow the same asymptotic relations for their autocovariance and spectral density as do the Gaussian FARIMA$(0, d, 0)$ processes, although the actual functions are more complicated, and have additional short-term components. These short-term components can influence significantly the estimates of d. For more details on FARIMA series, see Samorodnitsky and Taqqu [3], Sections 7.12 and 7.13.

2 FARIMA series with other finite variance innovations

In addition to Gaussian innovations ϵ_i, one can also consider innovations ϵ_i which, for example, have *Exponential* and *Lognormal* distributions. An Exponential random variable ϵ has the cumulative distribution function,

$$F(x) = P(\epsilon \le x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-\lambda x} & x \ge 0. \end{cases} \tag{2.1}$$

A Lognormal random variable ϵ has the cumulative distribution function,

$$F(x) = \Phi\left(\frac{\ln(x) - \mu}{\sigma}\right), \tag{2.2}$$

where Φ is the cumulative distribution function of a standard Normal random variable.

3 FARIMA series with infinite variance innovations

For infinite variance innovations ϵ, one can use, for example, symmetric and skewed stable distributions and Pareto distributions. These are characterized by a parameter α and satisfy

$$P(\epsilon > x) \sim Cx^{-\alpha}, \quad \text{as } x \to \infty, \tag{3.1}$$

implying that the variance is infinite if $\alpha < 2$. The *symmetric α-stable* distribution has the characteristic function,

$$E[\exp(i\theta\epsilon)] = \exp(-\sigma^\alpha |\theta|^\alpha).$$

where σ is the scale parameter. We will concentrate on the case $1 < \alpha \le 2$, with the understanding that for $\alpha = 2$ the process is Gaussian. If $\alpha < 2$, a stable random variable with scale parameter σ, skeweness parameter $\beta \in [-1, 1]$ and centering parameter μ has the characteristic function,

$$E[\exp(i\theta\epsilon)] = \exp\left[-\sigma^\alpha |\theta|^\alpha (1 - i\beta(\text{sign } \theta) \tan\frac{\pi\alpha}{2} + i\mu\theta)\right],$$

where $i = \sqrt{-1}$, and sign $\theta = -1, 1, 0$ if $\theta < 0, > 0, = 0$ respectively. In addition to the symmetric α-stable series ($\beta = \mu = 0$), one can use series that are totally skewed to the right, ($\beta = 1$, $\mu = 0$).

A Pareto random variable ϵ has the cumulative distribution function,

$$F(x) = 1 - \frac{1}{x^\alpha}, \quad x \ge 1. \tag{3.2}$$

Pareto random variables also have infinite variance, and in fact satisfy (3.1) exactly for all $x \ge 1$.

When generating series with infinite variance, we use independent symmetric α-stable variables as innovations in FARIMA(p, d, q) series (see (1.4)), and skewed stable and *Pareto* distributions as the innovations in a FARIMA$(0, d, 0)$ series. The parameter d is restricted to the interval $[0, 1 - 1/\alpha)$, which ensures that the time series are well-defined [3], Th. 7.13.1. The resulting process X will be long-range dependent and α-stable if the innovations are α-stable, and asymptotically will be in the domain of attraction of an α-stable distribution if the innovations are Pareto.

For more information on Gaussian FARIMA series, see Brockwell and Davis [2] or Beran [1] and, for stable processes, Samorodnitsky and Taqqu [3]. The FARIMA is a useful family of models because it allows for long-range dependence and offers a lot of flexibility in modelling short-range dependence structure, by choosing appropriate autoregressive and moving average coefficients in (1.4). Also see [4].

References

[1] J. Beran. *Statistics for Long-Memory Processes.* Chapman & Hall, New York, 1994.

[2] P. J. Brockwell and R. A. Davis. *Time Series: Theory and Methods.* Springer-Verlag, New York, 2nd edition, 1991.

[3] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, New York, London, 1994.

[4] M. S. Taqqu and V. Teverovsky. Estimating long-range dependence in finite and infinite variance series. In R. Adler, R. Feldman, and M. S. Taqqu, editors, *A Practical Guide to Heavy Tails: Statistical Techniques for Analyzing Heavy-Tailed Distributions*, Boston, 1996. Birkhäuser. To appear in 1997.