# 1 The Time Series

We describe here fractional Gaussian noise (FGN) a type of time series that we use in describing the statistical methods for long-range dependence.

The best way to introduce fractional Gaussian noise is through its "parent" *fractional Brownian motion* $\{B_H(t),\ t \geq 0\}$. Fractional Brownian motion is a Gaussian process with mean 0, stationary increments, variance $EB_H^2(t) = t^{2H}$ and covariance

$$EB_H(s)B_H(t) = \frac{1}{2}\{s^{2H} + t^{2H} - |s-t|^{2H}\} \tag{1.1}$$

It is statistically self-similar in the sense that $\{B_H(at),\ t \geq 0\}$ has the same finite-dimensional distributions as $\{a^H B_H(t),\ t \geq 0\}$ for all $a > 0$. The crucial index is $H$, a parameter which takes values between 0 and 1, called the *self-similarity* parameter.

*Fractional Gaussian noise* $\{X_i,\ i \geq 1\}$ is the increment of fractional Brownian motion, namely

$$X_i = B_H(i+1) - B_H(i),\ i \geq 1.$$

It is a mean zero, stationary Gaussian time series whose autocovariance function $\gamma(h) = EX_iX_{i+h}$ is given by $\gamma(h) = 2^{-1}\{(h+1)^{2H} - 2h^{2H} + |h-1|^{2H}\}$, $h \geq 0$. An important point about $\gamma$ is that it satisfies

$$\gamma(h) \sim H(2H-1)h^{2H-2} \quad \text{as } h \to \infty \tag{1.2}$$

when $H \neq 1/2$ ( $\sim$ means "asymptotic to"). Since $\gamma(h) = 0$ for $h \geq 1$ when $H = 1/2$, the $X_i$'s are white noise in this case. The $X_i$'s, however, are positively correlated when $\frac{1}{2} < H < 1$, and we say that they display *long-range dependence* or *long-memory*. The index $H$, in this context, measures the *intensity* of long-range dependence. The spectral density (Fourier transform of $\gamma$) is

$$f(\lambda) = C_H\Big(2\sin\frac{\lambda}{2}\Big)^2 \sum_{k=-\infty}^{\infty} \frac{1}{|\lambda + 2\pi k|^{2H+1}} \sim C_H|\lambda|^{1-2H} \text{ as } \lambda \to 0, \tag{1.3}$$

where $C_H$ is a constant.

For more details, refer to Samorodnitsky and Taqqu [1], [2]. The second reference is a recent monograph about stochastic processes with infinite variance. Chapter 7 of that monograph presents Gaussian and infinite variance self-similar processes and discusses time series with long-range dependence.

# References


[1] G. Samorodnitsky and M. S. Taqqu. Linear models with long-range dependence and finite or infinite variance. In D. Brillinger, P. Caines, J. Geweke, E. Parzen, M. Rosenblatt, and M. S. Taqqu, editors, *New Directions in Time Series Analysis, Part II*, pages 325–340, New York, 1992. IMA Volumes in Mathematics and its Applications, Volume 46, Springer-Verlag.

[2] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, New York, London, 1994.