1 Aggregated Variance Method

Consider the aggregated series (1.1), obtained by dividing a given series of length N into blocks of length m, and averaging the series over each block.

$$X^{(m)}(k) := \frac{1}{m} \sum_{i=(k-1)m+1}^{km} X_i, \quad k = 1, 2, ..., [N/m]. \tag{1.1}$$

We compute its sample variance,

$$\widehat{\mathrm{Var}} X^{(m)} = \frac{1}{N/m} \sum_{k=1}^{N/m} (X^{(m)}(k) - \overline{X})^2. \tag{1.2}$$

The series $X^{(m)}$ scales like m^{H-1}, thus if the series is Gaussian, or at least finite variance, the sample variance will be asymptotically proportional to m^{2H-2} for large N/m and m.

For successive values of m, the sample variance of the aggregated series is plotted versus m on a log-log plot. The result should be a straight line with a slope of $2H - 2$. In practice, the slope is estimated by fitting a least-squares line to the points of the plot. We assume that both N and N/m are large. This ensures that both the length of each block and the number of blocks is large. If the series has no long-range dependence and finite variance, then $H = 0.5$ and the slope of the fitted line should be -1. In practice, the points at the very low and high ends of the plots are not used to fit the least-squares line. Indeed, short-range effects can distort the estimates of H if the low end of the plot is used, and at the very high end of the plot there are too few blocks to get reliable estimates of the variance.

2 Differenced Variance

Jumps in the mean and slowly decaying trends are two types of non-stationarity which can occur. To distinguish these from long-range dependence, we can difference the variance (see Teverovsky and Taqqu [3] for details), and study

$$\widehat{\mathrm{Var}} X^{(m_i+1)} - \widehat{\mathrm{Var}} X^{(m_i)} \tag{2.1}$$

where the m_i's are the successive values of m as defined above. Differencing introduces more fluctuation, but it can provide a way of detecting the presence of the two types of non-stationarity mentioned above. The method should be used together with the original aggregated variance method.

3 Another Modification

Another solution for the non-stationarity problem is to fit the function

$$C_1 + C_2 m^{(2H-2)}$$

to the $\widehat{\mathrm{Var}} X^{(m)}$. In practice the uncertainty of fitting three parameters instead of just one seems to make the estimates quite scattered.

The first two methods are discussed in Taqqu, Teverovsky and Willinger [2] and Taqqu and Teverovsky [1].

References

[1] M. S. Taqqu and V. Teverovsky. Estimating long-range dependence in finite and infinite variance series. In R. Adler, R. Feldman, and M. S. Taqqu, editors, *A Practical Guide to Heavy Tails: Statistical Techniques for Analyzing Heavy-Tailed Distributions*, Boston, 1996. Birkhäuser. To appear in 1997.

[2] M. S. Taqqu, V. Teverovsky, and W. Willinger. Estimators for long-range dependence: an empirical study. *Fractals*, 3(4):785–798, 1995. Reprinted in *Fractal Geometry and Analysis*, C.J.G. Evertsz, H-O Peitgen and R.F. Voss, editors. World Scientific Publishing Co., Singapore, 1996.

[3] V. Teverovsky and M. S. Taqqu. Testing for long-range dependence in the presence of shifting means or a slowly declining trend using a variance-type estimator. Preprint, 1995.