BRAUER-MANIN OBSTRUCTION

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This is the note for the STAGE talk at MIT on the étale Brauer obstructions and insufficiencies. We make the following conventions. Throughout this note, base field k will always be a number field, although many of the results can be extended to global function fields. We will call a k-variety nice if it is smooth, projective, and geometrically integral. $\mathbb{A} = \mathbb{A}_k$ is the ring of adèles in k. For a k-variety X, $\overline{X} := X \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ \overline{k}$ is its base change to the algebraic closure $\overline{k}$ of k. A torsor $f : Y \to X$, if not stated otherwise, is an fppf torsor under a linear k-group G.

1. Etale Brauer-Manin and Comparisons

Recall the definition of the descent obstruction

$$X(\mathbb{A})^{\mathrm{desc}} := \bigcap_{\text{linear } G} \bigcap_{f \in H^1(X,G)} \bigcup_{\tau \in H^1(k,G)} f^\tau(Z^\tau(\mathbb{A})) \tag{1}$$

We can form new obstructions by applying obstructions to finite étale covers of the variety X. Let F be an obstruction. Then we define

$$X(\mathbb{A})^{\text{ét},F} := \bigcap_{\text{finite étale } G} \bigcap_{f \in H^1(X,G)} \bigcup_{\tau \in H^1(k,G)} f^\tau(Z^\tau(\mathbb{A})^F) \tag{2}$$

and call it *the étale F-obstruction.* A priori, these étale variants of obstructions look stronger. But the following theorem will show we do not actually obtain new obstructions from these constructions.

Theorem 1.1.

$$X(\mathbb{A})^{\acute{e}t,\mathrm{Br}} = X(\mathbb{A})^{\acute{e}t,\mathrm{desc}} = X(\mathbb{A})^{\mathrm{desc}} = X(\mathbb{A})^{\mathrm{desc},\mathrm{desc}} \tag{3}$$

The last inequality is the main theorem of [1], which has been briefly discussed last time. We will therefore focus on the rest of the theorem. In order to prove the theorem, we shall show the inclusions

$$X(\mathbb{A})^{\text{desc}} \subseteq X(\mathbb{A})^{\text{ét,desc}} \subseteq X(\mathbb{A})^{\text{ét,Br}} \subseteq X(\mathbb{A})^{\text{desc}} \tag{4}$$

The middle inclusion is easy: it is a direct consequence of the fact that the descent obstruction is stronger than the Brauer-Manin obstruction. We sketch the proofs for the first and the third inclusions. (Both of the two inclusions are based on arguments in [6].)

Theorem 1.2 ([5])**.** *Let G be a finite k-group. Then*

$$X(\mathbb{A})^{\text{desc}} = \bigcup_{\tau \in H^1(k,G)} f^\tau(Z^\tau(\mathbb{A})^{\text{desc}}) \tag{5}$$

In particular, $X(\mathbb{A})^{\text{desc}} \subseteq X(\mathbb{A})^{\text{ét,desc}}$.

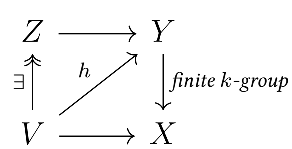
The key ingredients are the following propositions.

Proposition 1.3 (Stoll)**.** *Let X be proper over k and $Y \to X$ be a torsor. For any $(P_v) \in X(\mathbb{A})^{\text{desc}}$, there exists a twist $Y^\tau \to X$ of $Y \to X$ satisfying the following: for any surjective X-torsor morphism $Z \to Y^\tau$, there exists a twist $Z^\sigma \to Y^\tau$ such that (P_v) lies in the image of $Z^\sigma(\mathbb{A})$*

Proof. Recall the fact that there are only finitely many twists of a given torsor that contain adelic points. □

This proposition simply says that for any adelic point, we can start with any torsor and find a torsor from which the adelic point descend.

Proposition 1.4. *Let $Y \to X$ be a torsor under a finite k-group, $Z \to Y$ a torsor. Then there exists a torsor $V \to X$ and a surjective X-torsor morphism $h : V \to Y$ such that V admits a surjective Y-torsor morphism to Z.*



Sketch of proof. The construction of V is simple. Let $R_{\square/\square}(\square)$ be the Weil restriction. Then we can take $V := R_{Y/X}(Z) \times_X Y$. One can verify that $V \to Y$ is a torsor under the Y-group $R_{Y/X}(G_Y)$. One then carefully verifies that $V \to Z$ is a surjective Y-torsor morphism. □

Proof of Theorem 1.2. Notice that $f^\tau(Y^\tau(\mathbb{A})^{\text{desc}}) \subseteq X(\mathbb{A})^{\text{desc}}$. So the inclusion $X(\mathbb{A})^{\text{desc}} \supseteq \bigcup_{\tau \in H^1(k,G)} f^\tau(Y^\tau(\mathbb{A})^{\text{desc}})$ is obvious. To the the opposite inclusion, let $(P_v) \in X(\mathbb{A})^{\text{desc}}$, $X' \to X$ be a torsor under a finite k-group, $f : Y \to X$ a twist of $X' \to X$ satisfying Proposition 1.3. It suffices to show (P_v) lifts to a point in $Y(\mathbb{A})^{\text{desc}}$. Suppose (P_v) does not lift. Then $f^{-1}((P_v))$ is covered by $\{f^{-1}((P_v)) \cap (Y(\mathbb{A}) \setminus Y(\mathbb{A})^f)\}_{\text{all torsors } f: Z \to Y}$.

Since $f^{-1}((P_v))$ is compact (it is a product of finite sets), there exists $f_1, \dots, f_n$ such that $\{f^{-1}((P_v)) \cap (Y(\mathbb{A}) \setminus Y(\mathbb{A})^{f_i})\}_{i=1,\dots,n}$ cover $f^{-1}((P_v))$. Let $g : Z \to Y$ be the fiber product of all f_i, which is a torsor under some group G. Let $h : V \to Y$ be a surjective X-torsor morphism satisfying Proposition 1.4 under group H. Then by Proposition 1.3, there exists $\sigma \in H^1(k, H)$ such that (P_v) lifts to a point (M_v) in $V^\sigma(\mathbb{A})$. Let $\rho \in H^1(k, G)$ be the image of τ under the homomorphism $H \to G$. Then $V^\sigma \to Y$ factors through $Z^\rho \to Y$, and the image of (M_v) in $Z^\rho(\mathbb{A})$ has image in $f^{-1}((P_v)) \cap Y(\mathbb{A})^g$ by construction, contradiction. ☐

Now we show the third inclusion.

Theorem 1.5 ([2]). *Let X be a nice, projective k-variety. Then*

$$X(\mathbb{A})^{\acute{e}t,\mathrm{Br}} \subset X(\mathbb{A})^{\mathrm{desc}} \tag{6}$$

We need another version of Stoll's argument, Harari's theorem, and one more proposition.

Proposition 1.6 (Stoll). *Let X be a nice k-variety and $(P_v) \in X(\mathbb{A})^{\acute{e}t,\mathrm{Br}}$ be an adelic point. Let $g : Y \to X$ be a torsor under a finite k-group G. Then there exists a twist $Y^\tau \to X$, a torsor $V \to X$ under a finite k-group H, and an X-torsor morphism $V \to Y^\tau$ such that (P_v) lies in the image of $V(\mathbb{A})^{\mathrm{Br}}$.*

Recall we define the connected obstruction to be

$$X(\mathbb{A})^{\mathrm{conn}} := \bigcap_{\text{connected linear } G} \bigcap_{f \in H^1(X,G)} \bigcup_{\tau \in H^1(k,G)} f^\tau(Z^\tau(\mathbb{A})) \tag{7}$$

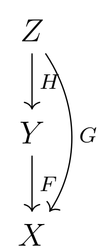
Theorem 1.7 (Harari 2002). $X(\mathbb{A})^{\mathrm{Br}} = X(\mathbb{A})^{\mathrm{conn}}$

Proposition 1.8. *Let $(P_v) \in X(\mathbb{A})^{\acute{e}t,\mathrm{Br}}$ and $f : Z \to X$ a torsor under a linear k-group G. Let*

$$1 \to H = G^0 \to G \to F \to 1 \tag{8}$$

be an exact squence of k-groups, where $H = G^0$ is the central connected component of G. Let $Y \to X$ be a torsor under F induced by $Z \to X$ and $Y^\tau \to X$ be a twist satisfying Stoll's argument. Then $\tau \in H^1(k, F)$ lifts to a 1-cocycle $\sigma \in H^1(k, G)$.

Proof of Theorem 1.5. Now take $(P_v) \in X(\mathbb{A})^{\text{ét},\mathrm{Br}}$ and G a linaer k-group. Then a torsor $f : Z \to X$ under G factors as



We want to show (P_v) lifts into $Z^\sigma(\mathbb{A})$ for some $\sigma \in H^1(k,G)$. By Proposition 1.8, one has $\tau \in H^1(k,F)$ satisfying Stoll's argument which lifts to $\sigma \in H^1(k,G)$. Explicitly, by Stoll's argument, one get the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad\psi\quad} & Y^\tau \\ & {}_{Q}\searrow \quad \swarrow_{F^\tau} & \\ & X & \end{array}$$

Consider then the twist $f^\sigma : Z^\sigma \to X$ (under G^σ). One then has

$$\begin{array}{ccc} R & \longrightarrow & Z^\sigma \\ \downarrow{\scriptstyle H^\sigma} & & {\scriptstyle H^\sigma}\downarrow \\ V & \xrightarrow[\psi]{} & Y^\tau \\ \downarrow{\scriptstyle Q} & & {\scriptstyle F^\tau}\downarrow \\ X & \xrightarrow{=} & X \end{array} \qquad (Z^\sigma \xrightarrow{G^\sigma} X)$$

We then apply the fact that the connected obstruction is equal to the Brauer-Manin obstruction (Harari) to the connected linear group H^σ: $V(\mathbb{A})^{\mathrm{Br}}$ is contained in $V(\mathbb{A})^{g:R\to V}$. In particular, if $(Q_v) \in V(\mathbb{A})^{\mathrm{Br}}$ is an adelic point above (P_v), then it can be lifted to $(R'_v) \in R^\mu(\mathbb{A})$ for some cocycle $\mu \in H^1(k,H^\sigma)$. Then we pass this adelic point to $(R_v) \in Z^{\sigma\mu}(\mathbb{A}) = Z^\rho(\mathbb{A})$. We conclude that $(P_v) \in X(\mathbb{A})^f$. □

2. Insufficiency

For simplicity, we assume the base field k is a number field (although many of the constructions also work for k being a global function field). Recall we have seen for several times that Châtelet surfaces can provide examples of insufficiency of obstructions. With this idea, we are now going to construct a nice variety based on Châtelet surfaces, on which the etale Brauer-Manin obstruction fails.

Fix $a \in k^*$ and fix coprime seprable degree-4 polynomials $P_\infty(x), P_0(x) \in k[x]$ such that the Châtelet surface V_∞ given by

$$y^2 - az^2 = P_\infty(x) \tag{9}$$

over k satisfies $V_\infty(\mathbb{A}) \neq \emptyset$ but $V_\infty(k) = \emptyset$.

Proposition 2.1. *There exists a nice Châtelet surface V_∞ given by*

$$y^2 - az^2 = P_\infty(x) \tag{10}$$

over k violating the Hasse principle.

Let (u,v) and (w,x) be coordinates on two $\mathbf{P}^1_k$. Let $\tilde{P}_\infty(w,x), \tilde{P}_0(w,x)$ the homogenizations of P_∞ and P_0. Define a section

$$s_1 := u^2\tilde{P}_\infty(w,x) + v^2\tilde{P}_0(w,x) \in \Gamma(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(2,4)) \tag{11}$$

Let $Z_1 \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the zero locus of s_1 and $F \subset \mathbf{P}^1$ be the branch locus of $Z_1 \xrightarrow{pr_1} \mathbf{P}^1$. Let $\alpha_1 : V \to \mathbf{P}^1 \times \mathbf{P}^1$ be the conic bundle given by

$$y^2 - az^2 = s_1 \tag{12}$$

Choose C to be a nice curve over k such that $C(k)$ is finite and nonempty and a surjective morphism $\gamma : C \to \mathbf{P}^1$ etale over F such that $\gamma(C(k)) = \{\infty\}$. Now let $X := V \times_{\mathbf{P}^1} C$, where $\mathbf{P}^1$ is the first component. X is the vareity that we need. Relations between these vareities are summarized in the following diagram.

$$\begin{array}{ccc} X & \longrightarrow & V \\ {\scriptstyle\alpha}\big\downarrow \ \ {\scriptstyle\beta} & & {\scriptstyle\alpha_1}\big\downarrow \ \ {\scriptstyle\beta_1} \\ C \times \mathbf{P}^1 & \xrightarrow{(\gamma, id)} & \mathbf{P}^1 \times \mathbf{P}^1 \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & {\scriptstyle \mathrm{pr}_1}\big\downarrow \\ C & \xrightarrow{\gamma} & \mathbf{P}^1 \end{array}$$

Now we show

Theorem 2.2. $X(k) = \emptyset$ *but* $X(\mathbb{A})^{et,\mathrm{Br}} \neq \emptyset$.

We show the theorem in steps.

Lemma 2.3. $X(k) = \emptyset$.

Proof. Since $\gamma(C(k)) = \{\infty\}$ but $V_\infty(k) = \emptyset$, one has $X(k) = \emptyset$. □

Lemma 2.4. $X(\mathbb{A})^{\mathrm{Br}} \supseteq V_\infty(\mathbb{A}) \times C(k)$

Proof. Denote $\overline{\square}$ the base change to algebraic closure. We first notice that $\mathrm{Br}(\overline{C \times \mathbf{P}^1}) \cong \mathrm{Br}(\overline{C}) = 0$. Recall the Hochschild-Serre spectral sequence induces

$$\mathrm{Br}\, k \to \mathrm{Br}^1(X) \to H^1(k, \mathrm{Pic}\ \overline{X}) \to H^3(k, \mathbf{G}_m) \tag{13}$$

Since X is a $\mathbf{P}^1$-bundle over $B = C \times \mathbf{P}^1$, $\mathrm{Br}(\overline{X}) = 0$. Together with the fact that $H^1(k, \mathbf{G}_m) = 0$, one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}\, k & \longrightarrow & \mathrm{Br}\, B & \longrightarrow & H^1(k, \mathrm{Pic}\ \overline{B}) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathrm{Br}\, k & \longrightarrow & \mathrm{Br}\, X & \longrightarrow & H^1(k, \mathrm{Pic}\ \overline{X}) & \longrightarrow & 0 \end{array}$$

with exact rows.

Claim 2.5. *We claim that since the degeneracy locus Z of $X \to B$ is nice, one has* $H^1(k, \mathrm{Pic}\ \overline{B}) \cong H^1(k, \mathrm{Pic}\ \overline{X})$.

Thus, $\operatorname{Br} B \cong \operatorname{Br} X$. Similarly, one has $\operatorname{Br} C \cong \operatorname{Br} B$. Now if $\beta_{\mathbb{A}} : X(\mathbb{A}) \to C(\mathbb{A})$ is the map induced by β, then

$$X(\mathbb{A})^{\mathrm{Br}} = \beta_{\mathbb{A}}^{-1}(C(\mathbb{A})^{\mathrm{Br}}) \supseteq \beta_{\mathbb{A}}^{-1}(C(k)) = V_\infty(\mathbb{A}) \times C(k). \tag{14}$$

□

Now we apply the same arguments to étale covers of X.

Theorem 2.6. $X(\mathbb{A})^{\text{\'et},\mathrm{Br}} \supseteq V_\infty \times C(k)$.

Proof. Let G be a finite étale k-group and $f : Y \to X$ be a G-torsor. Notice since $X \to C \times \mathbf{P}^1$ has geometrically simply connected fibers, one has an equivalence between $\mathrm{FEt}(X)$ and $\mathrm{FEt}(C)$ induced by the projection $X \to C$. Thus, $f : Y \to X$ is induced from a G-torsor $D \to C$, i.e.

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ D & \longrightarrow & C \end{array}$$

Now $Y \to D$ is just like $X \to C$. So we may apply all arguments previously on $X \to C$ to $Y \to D$ and obtain

$$Y^\sigma(\mathbb{A})^{\mathrm{Br}} \supseteq V_\infty(\mathbb{A}) \times D^\sigma(k) \tag{15}$$

Thus, we see that

$$\bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A})^{\mathrm{Br}}) \supseteq V_\infty(\mathbb{A}) \times C(k) \tag{16}$$

In conclusion, we see that $X(\mathbb{A})^{\text{\'et},\mathrm{Br}} \supseteq V_\infty \times C(k)$. □

3. Appendix

Proposition 3.1 (Birational Invariance of $\mathrm{Br}(X)$)**.** *Let X and X' be nice varieties over a number field k. If X and X' are birational, then $\mathrm{Br}(X)$ and $\mathrm{Br}(X')$ are isomorphic.*

Proof. See [4] Corollary 6.8.7. □

Proposition 3.2 (SGA 1 IX 6.8)**.** *Soit $f : X \to S$ un morphisme propre surjectif de présentation finie, à fibres géom'etriquement connexes. Alors f est un morphisme de descente effective pour la catégorie fibrée des préschémas étales finis sur d'autres. Le foncteur $S' \mapsto X \times_S S'$ induit une équivalence de la catégorie des préschémas étales et finis sur S avec la catégorie des préschémas étales et finis sur X qui induisent sur chaque fibre X_s un revêtement géométriquement trivial.*

Now in our example, $X \to C \times \mathbf{P}^1$ has geometric fibers isomorphic to $\mathbf{P}^1$ or two copies of $\mathbf{P}^1$ intersecting at double points. Thus, every finite étale covering of X satisfies the condition in the proposition and therefore, one has an equivalence

$$\mathrm{FEt}(X) \xrightarrow{\sim} \mathrm{FEt}(C \times \mathbf{P}^1) \xrightarrow{\sim} \mathrm{FEt}(C) \tag{17}$$

We also summarize the arguments in [6]. A weaker version, as used in [2], reads as follows.

Proposition 3.3. *Let X be a nice k-variety and $(P_v) \in X(\mathbb{A})^{\text{ét},\mathrm{Br}}$ be an adelic point. Let $g : Y \to X$ be a torsor under a finite k-group G. Then there exists a twist $Y^\tau \to X$, a torsor $Z \to X$ under a finite k-group H, and an X-torsor morphism $Z \to Y^\tau$ such that (P_v) lies in the image of $Z(\mathbb{A})^{\mathrm{Br}}$.*

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