A Prismatic Approach to Crystalline Local Systems

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Goal:

Thm A X smooth p-adic formal scheme over $\mathcal{O}_K$

$\exists$ natural

$$T: \mathrm{Vect}^{an,\varphi}(X_{\Delta}) \longrightarrow \mathrm{Loc}^{cris}_{\mathbb{Z}_p}(X_\eta)$$

Rmk. X_η is usually treated as a diamond.

§1. Period sheaves.

classical p-adic Hodge: Fontaine has

A_{inf}, B_{dR}, B_{cris}, etc.

constructed for $\mathbb{C}_p$ perfectoid ring.

Def. $\mathbb{A}_{inf} \in \mathrm{Shv}(X_{\eta,\mathrm{proet}})$ defined by: for $U = \mathrm{Spa}(R, R^+)$ (in $X_{\eta,\mathrm{proet}}$)

- $\mathbb{A}_{inf}(R, R^+) := W(R^{+\flat})$
- $\mathbb{B}_{inf}(R, R^+) = \mathbb{A}_{inf}(R, R^+)[1/p]$

Def. $\theta: \mathbb{A}_{inf}(R, R^+) \longrightarrow R^+$ natural map.

$\tilde{\theta} := \theta \circ \varphi^{-1}$ "Hodge - Tate specialization map"

(Note in literature usually we just consider θ)

- $\mathbb{B}^+_{dR}(R, R^+) := \mathbb{B}_{inf}(R, R^+)^{\wedge}_{\ker(\tilde{\theta})}$.

- $\mathbb{B}_{dR}(R, R^+) := \mathbb{B}^+_{dR}(R, R^+)[\frac{1}{t}]$

$t = \log[\varepsilon] \in B^+_{dR}(\mathcal{O}_C)$)

filtration.

Def $\mathbb{A}_{cris}(R, R^+)$ p-completion of PD-envelope of $\ker\theta$ in $\mathbb{A}_{inf}(R, R^+)$.

(this has a natural filtration).

$\mathbb{B}^+_{\mathrm{cris}}(R, R^+) = \mathbb{A}_{\mathrm{cris}}(R, R^+)[1/p]$

$\mathbb{B}_{\mathrm{cris}}(R, R^+) = \mathbb{B}^+_{\mathrm{cris}}(R, R^+)[1/t]$.

$\hookrightarrow$ induced filtration.

Def. $\mathbb{B}_{\mathrm{cris}, K}(R, R^+) = \mathbb{B}_{\mathrm{cris}}(R, R^+) \otimes_{K_0} K$.

K is p-adic field. $\mathcal{O}_K$ the ring of integers

$X / \mathcal{O}_K$ $\quad k$ residue field. $K_0 := W(k)[\frac{1}{p}]$.

Lemma

1). $\mathbb{A}_{\mathrm{inf}}$, $\mathbb{B}_{\mathrm{inf}}$, $\mathbb{B}^+_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{A}_{\mathrm{cris}}$, $\mathbb{B}^+_{\mathrm{cris}}$, $\mathbb{B}_{\mathrm{cris}}$ are sheaves over $X_{\eta, \mathrm{proet}}$.

2). Fundamental exact sequence

$$0 \longrightarrow \hat{\mathbb{Q}}_p \longrightarrow (\mathbb{B}_{\mathrm{cris}})^{\varphi=1} \longrightarrow \mathbb{B}_{\mathrm{dR}} / \mathbb{B}^+_{\mathrm{dR}} \longrightarrow 0$$

Correct def of $\mathbb{Z}_p$-constant sheaf over $X_{\eta, \mathrm{proet}}$.

$$\hat{\mathbb{Z}}_p := \varprojlim_n \nu^* \mathbb{Z}/p^n\mathbb{Z}$$

$$\nu : X_{\eta, \mathrm{proet}} \longrightarrow X_{\eta, \mathrm{et}}.$$

$$\hat{\mathbb{Q}}_p := \text{"} \hat{\mathbb{Z}}_p \otimes \mathbb{Q}_p \text{"} = \mathrm{colim}\,(\hat{\mathbb{Z}}_p \xrightarrow{p} \hat{\mathbb{Z}}_p \xrightarrow{p} \hat{\mathbb{Z}}_p \to \cdots).$$

Def. V_0 = ring of integers of K_0 = $W(k)$

X_s reduced special fiber of X.

$(X_s / V_0)_{\mathrm{cris}}$ big crystalline site.

$\mathrm{Isoc}(X_s / V_0)$ cat of isocrystals over X_s.

- objects : crystals over X_s.
- $\mathrm{Hom}_{\mathrm{Isoc}(X_s/V_0)}(\mathcal{E}, \mathcal{E}') = \mathrm{Hom}_{\mathcal{O}_{X_s/V_0}}(\mathcal{E}, \mathcal{E}') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

$F : X_s \to X_s$ absolute Frobenius on X_s.

$F : (X_s / V_0)_{cris} \longrightarrow (X_s / V_0)_{cris}$

Def. F-isocrystal on X_s $(\mathcal{E}, \varphi)$

- $\mathcal{E}$ isocrystal on X_s
- $\varphi : F^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ an isom

$\mathrm{Isoc}^{\varphi}(X_s / V_0)$ cat of F-isoc

Often useful to consider $X_{p=0} = X \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.

Lemma. (Ogus) natural inclusion

$(X_s / V_0)_{cris} \subseteq (X_{p=0} / V_0)_{cris}$

induces an equivalence.

$\mathrm{Isoc}^{\varphi}(X_s / V_0) \simeq \mathrm{Isoc}^{\varphi}(X_{p=0} / V_0)$.

§2. Crystalline Local Systems

mimic Faltings' definition of crystalline local systems

X smooth

Basic idea: isocrystal on $X_s \rightsquigarrow$ sheaf over X_η.

Def. $\mathcal{E} \in \mathrm{Isoc}^{\varphi}(X_s / V_0)$.

$\mathrm{Spa}(S, S^+) = U$

$\mathbb{B}^+_{cris}(\mathcal{E})$ $\quad$ $\mathbb{B}_{cris}(\mathcal{E}) \in \mathrm{Shv}(X_{\eta, proet})$

$U = \mathrm{Spa}(S, S^+) \in X_{\eta, proet}$ $\quad$ U perfectoid.

$\mathbb{A}_{cris}(S, S^+) \longrightarrow S^+/p$ is a pro-PD-thickening

(pro-object in $(X_{p=0} / V_0)_{cris}$)

- $\mathbb{B}^+_{cris}(\mathcal{E})(U) := \mathcal{E}(\mathbb{A}_{cris}(S, S^+))[\frac{1}{p}]$

$:= \left(\varprojlim_r \mathcal{E}\left(\underbrace{\mathbb{A}_{cris}(S, S^+)/p, \ \mathbb{A}_{cris}(S, S^+)/p^r, \ \gamma}_{\in (X_{p=0}/V_0)_{cris}} \right) \right) [\frac{1}{p}]$

& natural PD structure.

- $\mathbb{B}_{cris}(\mathcal{E})(U) := \mathbb{B}^+_{cris}(\mathcal{E})(U)[\frac{1}{\mu}]$

C completed alg closure of k.

$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_C^b$.

$[\varepsilon] \in A_{inf}(\mathcal{O}_C) = W(\mathcal{O}_C^b)$

$\mu = [\varepsilon] - 1$

- $\mathbb{B}^+_{cris,K}(\mathcal{E}) = \mathbb{B}^+_{cris}(\mathcal{E}) \otimes_{K_0} K$.

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With this proetale sheaf $\mathbb{B}_{cris}(\mathcal{E})$, we can define

<u>Def.</u> (Crystalline local systems)

A sheaf of $\hat{\mathbb{Z}}_p$-module $\mathbb{L}$ on $X_{\eta,\text{proet}}$ is <u>weakly crystalline</u> if

1). $\mathbb{L}$ is lisse. (locally $\mathbb{L} \simeq \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} M$ for some f.g. $\mathbb{Z}_p$-module M)

2). ("satisfying "crystalline comparison")

$\exists\ (\mathcal{E}, \varphi) \in \mathrm{Isoc}^{\varphi}(X_s/W)$ + an isom

$$\theta : \mathbb{B}_{cris}(\mathcal{E}) \xrightarrow{\sim} \mathbb{B}_{cris} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{L}.$$

s.t. θ commutes with Frobenius isomorphisms

$\mathbb{L}$ crystalline

$\quad$ ||

$\quad$ unramified

$\quad$ ||

$\quad$ good reduction.

$\boxed{\mathbb{L} \text{ family of rep}}$

$\downarrow$

$\boxed{X \text{ base.}}$

if X is only semistable (X_η smooth)

then we can define notion of <u>semistable local systems</u>

Rmk It is not clear how to define crystalline local systems for non smooth X.

weakly crys. $\searrow$

$\mathbb{L}$ crystalline if $(\mathcal{E}, \varphi)$ underlies a filtered F-isocrystal $(\mathcal{E}, \varphi, \mathrm{Fil}^\bullet(\mathcal{E}))$ on X and θ satisfies

$$\theta_K = \theta \otimes_{K_0} K : B_{\mathrm{crys},K}(\mathcal{E}) \xrightarrow{\sim} B_{\mathrm{crys},K} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{L}$$

is compatible with filtrations on both sides.

- $\mathbb{L}$ is a local system if it is torsion-free.

Primary example. family of ab varieties $A \longrightarrow X_\eta$ w/ good reductions.

$R^i f_* \hat{\mathbb{Z}}_p$ defines a crystalline local system.

We usually won't consider the filtrations

1). filtrations would need strong inputs from (derived) de Rham & infinitesimal cohomology of X_η.

2) Prop. Any weakly crystalline lisse sheaf $\mathbb{L}$ on $X_{\eta,\text{proét}}$ is crystalline for a uniquely determined filtration (on $(\mathcal{E}, \varphi)$)

§3. Analytic prismatic F-crystals.

Idea $(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_p}$, $\varpi \in R$ pseudo-uniformizer

$$\mathcal{Y} = \mathcal{Y}_{[0,\infty]} = \mathrm{Spa}\, A_{\mathrm{inf}}(R, R^+) \setminus V(p, \varpi),$$

$\|$

analytic locus of $\mathrm{Spa}\, A_{\mathrm{inf}}(R, R^+)$

Lemma (Kedlaya)

$$\mathrm{Vect}(\mathrm{Spa}\, A_{\mathrm{inf}}(R, R^+) \setminus V(p, \varpi))$$
$$\simeq \mathrm{Vect}(\mathrm{Spec}\, A_{\mathrm{inf}}(R, R^+) \setminus V(p, \varpi))$$

Def. analytic prismatic crystals (in vector bundles) on X.

$$\mathrm{Vect}^{an}(X_{\Delta}) := \lim_{(A,I)\in X_{\Delta}} \mathrm{Vect}(\mathrm{Spec}\, A \setminus V(p, I))$$

(Drinfeld – Mathew) $\mathrm{Vect}^{an}(X_{\Delta})$ satisfies (p, I)-completely flat descent

Rmk. ① $\exists$ natural inclusion

$$\mathrm{Vect}(X_{\Delta}) \hookrightarrow \mathrm{Vect}^{an}(X_{\Delta})$$

Def. analytic F-crystal.

$$\mathrm{Vect}^{\varphi}(\mathrm{Spec}(A) \setminus V(p, I))$$

$= (\mathcal{M}, \varphi_{\mathcal{M}})$ • $\mathcal{M}$ is v.b / $\mathrm{Spec}(A) \setminus V(p, I)$

• $\varphi_{\mathcal{M}} : \varphi_A^* \mathcal{M}[1/I] \longrightarrow \mathcal{M}[1/I]$

$$\mathrm{Vect}^{an,\varphi}(X_{\Delta}) = \lim_{(A,I)\in X_{\Delta}} \mathrm{Vect}^{\varphi}(\mathrm{Spec}(A)\setminus V(p, I))$$

Recall in Goal

$$T : \underline{\mathrm{Vect}^{an,\varphi}(X_{\Delta})} \longrightarrow \underline{\mathrm{Loc}^{crys}_{\mathbb{Z}_p}(X_{\eta})}$$

$\downarrow$

Étale realization functor.

$$T: \mathrm{Vect}^{an,\varphi}(X_{\Delta}) \longrightarrow \boxed{\begin{matrix}\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I_{\Delta}}]^{\wedge}_p) \\ \wr\wr \quad v\text{-} \\ \mathrm{Loc}_{\mathbb{Z}_p}(X_{\eta}).\end{matrix}}$$

$\hookrightarrow \mathrm{Loc}^{crys}_{\mathbb{Z}_p}(X_{\eta})$

induced by $\mathrm{Spec}\, A[1/I] \hookrightarrow \mathrm{Spec}\, A$

Construction of T need the mod-p RH correspondence by Katz

$S/\mathbb{F}_p$

$$D^b_{lisse}(\mathrm{Spec}(S), \mathbb{F}_p) \simeq D^{\varphi}_{perf}(S).$$

Thm 1. $\forall\ \mathcal{E} \in \mathrm{Vect}^{an,\varphi}(X_{\Delta})$. $T(\mathcal{E})$ is crystalline

$$\underbrace{\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I_{\Delta}}]^{\wedge}_{p})}_{\text{Laurent F-crystals}} = \lim_{(A,I)\in X_{\Delta}} \mathrm{Vect}^{\varphi}(\mathrm{Spec}(A[\tfrac{1}{I}]^{\wedge}_{p})).$$

(Bhatt – Scholze on crystalline representation)

$$\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I_{\Delta}}]^{\wedge}_{p})$$
$$\cong$$
$$\mathrm{Loc}_{\mathbb{Z}_p}(X_{\eta}).$$

Prop. $D_{perf}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge}_{p}) \overset{①}{\simeq} D_{perf}(X_{\Delta}, \mathcal{O}_{\Delta,perf}[1/I_{\Delta}]^{\wedge}_{p})^{\varphi=1}$

$$\underset{②}{\simeq} D^{(b)}_{lisse}(X_{\eta}, \mathbb{Z}_p).$$

proof sketch.

Both ① & ② need.

Lemma (Katz).

S $\mathbb{F}_p$-alg. Then $\mathbb{F}_p \to \mathcal{O}_{\mathrm{Spec}(S),et}$ defines an ~~one~~ natural equivalence $\xleftarrow{\text{taking Frobe-fix pts}}$

$$\underline{D^{b}_{lisse}(\mathrm{Spec}(S), \mathbb{F}_p) \xrightarrow{\sim} D^{\varphi}_{perf}(S)}$$

For ①. since both sides are limits over categories for (A,I). it is enough to show

$$D_{perf}(A[1/I]^{\wedge}_{p}) \xrightarrow{\sim} D_{perf}(B[1/J]^{\wedge}_{p})$$

(B,J) perfection of (A,I)

Flatness + Nakayama $\rightsquigarrow$ enough to show after mod p.

$\rightsquigarrow$ Lemma. R, char $p>0$, $t\in R$, s.t. R derived t-complete

$S = (R_{perf})^{\wedge}_{t}$.

$$D^{\varphi}_{perf}(R[1/t]) \underset{a}{\xrightarrow{\sim}} D^{\varphi}_{perf}(R_{perf}[1/t]) \underset{b}{\xrightarrow{\sim}} D^{\varphi}_{perf}(S[1/t]).$$

proof. a is an equivalence by Katz's lemma

both sides $\rightsquigarrow$ into Cat of $\mathbb{F}_p$-sheaves

WTS $D^b_{lisse}(\mathrm{Spec}(R[1/t]), \mathbb{F}_p) \simeq D^b_{lisse}(\mathrm{Spec}\, R_{perf}[1/t], \mathbb{F}_p)$

This follows from the top invariance of etale site.

In order to show $b\circ a$ is an equiv. it suffices to check the pullback functor

$$D^b_{lisse}(\mathrm{Spec}\, S[\tfrac{1}{t}], \mathbb{F}_p) \xrightarrow{\sim} D^b_{lisse}(\mathrm{Spec}\, R[\tfrac{1}{t}], \mathbb{F}_p)$$

Claim this is fully-faithful (technical).

To show essential surjectivity, it is enough to show

$$\pi_1^{et}(\mathrm{Spec}\, S[\tfrac{1}{t}]) \simeq \pi_1^{et}(\mathrm{Spec}\, R[\tfrac{1}{t}]).$$

which is in turn proved by using Elkik's approximation theorem

(2) $$D^{\varphi}_{perf}(X_{\Delta}, \mathcal{O}_{\Delta,perf}[1/\mathcal{I}_{\Delta}]^{\wedge}_{p})^{\varphi=1}$$

$$\underset{(3)}{\simeq} D^{(b)}_{lisse}(X_\eta, \mathbb{Z}_p).$$

by flat descent, may assume $X = \mathrm{Spf}\, R$, R perfectoid.

Thus X_{Δ} has initial object

$$(\Delta_R, I) = (W(R^b), I)$$

$$\Delta_R[\tfrac{1}{I}]^{\wedge}_{p} \simeq W(R^b)[\tfrac{1}{I}]^{\wedge}_{p}$$

Hence

$$D^{\varphi}_{perf}(X_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_{p}) \simeq \boxed{D^{\varphi}_{perf}(\Delta_R[\tfrac{1}{I}]^{\wedge}_{p})}$$

Claim. Katz's lemma + tilting equivalent to show

$$\boxed{D^{\varphi}_{\mathrm{perf}}(\Delta_{R[\frac{1}{2}]^{\wedge}_{p}})} \xrightarrow{\sim} D^{b}_{\mathrm{lisse}}(\mathrm{Spa}(R[1/p], R), \mathbb{Z}_p).$$

$\Rightarrow$ gives the ~~cos~~ functor T.

$$T : \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) \longrightarrow$$

L weakly crystalline $\longrightarrow$ $L[1/p]$ is de Rham.

$$\begin{array}{ccc} \uparrow & & \updownarrow \\ (\mathcal{E}, \varphi) & & (\mathcal{E}, \varphi) \end{array}$$

$\Downarrow$ Liu–Zhu

natural filtration

$$H^n_{\mathrm{proet}}(X_{\eta}, \mathbb{Q}_p) \overset{?}{\simeq} H^n_{\mathrm{et}}(X_{\eta}, \mathbb{Q}_p)$$

In general not equal for X coherent maybe equal?

X coherent. $\varinjlim_{x_p} \varprojlim_{n} \nu^*(\mathbb{Z}/p^n\mathbb{Z})$

$$H^n_{\mathrm{et}}(X_{\eta}, \mathbb{Q}_p) \otimes B_{\mathrm{cris}} \simeq H^n_{\mathrm{cris}}(X_s/\mathbb{Z}_p) \otimes B_{\mathrm{cris}}$$

as filtered modules