A Prismatic Approach to Crystalline Local Systems

- K complete discretely valued p-adic field.
- $\mathcal{O}_K$ ring of integers
- k residue field
- $V_0 := W(k) \subseteq \mathcal{O}_K$
- $K_0 = V_0[p^{-1}] \subseteq K$
- X bounded p-adic formal scheme over $\mathrm{Spf}\, \mathcal{O}_K$
- X_η adic generic fiber viewed as a diamond.

§1. Preliminaries on period sheaves

Recall in classical p-adic Hodge theory, Fontaine defined period rings $\mathbb{A}_{\mathrm{inf}}$, $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{cris}}$, etc for the perfectoid field $\mathbb{C}_p$. This construction can be generalized to arbitrary perfectoid algebras over $\mathbb{Q}_p$.

Def. $\mathbb{A}_{\mathrm{inf}}$, $\mathbb{B}_{\mathrm{inf}}$, $\mathcal{O}\mathbb{B}_{\mathrm{inf}} \in \mathrm{Shv}(X_{\eta, \mathrm{pro\acute{e}t}})$ defined by for $\mathcal{U} \in \mathrm{Perf}/X_{\eta, \mathrm{pro\acute{e}t}}$, $\hat{\mathcal{U}} = \mathrm{Spa}(R, R^+)$

- $\mathbb{A}_{\mathrm{inf}}(R, R^+) := W(R^{+\flat})$
- $\mathbb{B}_{\mathrm{inf}}(R, R^+) := \mathbb{A}_{\mathrm{inf}}(R, R^+)[\frac{1}{p}]$
- $\mathcal{O}\mathbb{B}_{\mathrm{inf}}(R, R^+) := R \otimes_{V_0} \mathbb{B}_{\mathrm{inf}}(R, R^+)$

Def $\theta : \mathbb{A}_{\mathrm{inf}}(R, R^+) \longrightarrow R^+$ the usual map

$\tilde{\theta} := \theta \circ \varphi^{-1}$ "HT specialization map"

- $\mathbb{B}_{\mathrm{dR}}^+(R, R^+) := \mathbb{B}_{\mathrm{inf}}(R, R^+)^{\wedge}_{\ker(\tilde{\theta})}$
- $\mathbb{B}_{\mathrm{dR}}(R, R^+) := \mathbb{B}_{\mathrm{dR}}^+(R, R^+)[1/t]$ with filtration

$$\mathrm{Fil}^r \mathbb{B}_{dR}(R, R^+) = \sum_{i \in \mathbb{Z}} t^i \cdot \mathrm{Fil}^{i+r} \mathbb{B}_{dR}^+(R, R^+).$$

where $t = \log[\varepsilon] \in B_{dR}^+(\mathcal{O}_C)$ is the canonical element.

<u>Def</u> • $A_{cris}(R, R^+)$ p-completion of pd-envelope of $\ker\theta$ in $A_{inf}(R, R^+)$ with filtration

$\mathrm{Fil}^r \mathbb{A}_{inf}(R, R^+) :=$ p-completion of PD ideal generated by $x^{[i]}$ for $i \geq r$ and $i \in \ker(\vartheta)$.

• $\mathbb{B}_{cris}^+(R, R^+) = \mathbb{A}_{cris}(R, R^+)[1/p]$ with induced filtration

• $\mathbb{B}_{cris}(R, R^+) = \mathbb{B}_{cris}^+(R, R^+)[1/t] = \mathbb{B}_{cris}^+(R, R^+)[1/\mu]$ with filtration

$$\mathrm{Fil}^r \mathbb{B}_{cris}(R, R^+) = \sum_{i \in \mathbb{Z}} t^{-i} \mathrm{Fil}^{i+r} \mathbb{B}_{cris}^+(R, R^+).$$

Thus, we get a natural map

$$\varphi^*_{A_{inf}} \mathbb{B}_{cris} \longrightarrow \mathbb{B}_{dR}$$

compatible with the filtrations on both sides.

<u>Def</u> $\mathbb{B}_{cris, K}(R, R^+) := \mathbb{B}_{cris}(R, R^+) \otimes_{K_0} K$

<u>Lemma</u>.

1) natural map

$$\mathbb{B}_{cris, K} \hookrightarrow \varphi^*_{A_{inf}} \mathbb{B}_{cris, K} \hookrightarrow \mathbb{B}_{dR}.$$

which are injective and compatible with $\varphi^*_{A_{inf}} \mathbb{B}_{cris} \longrightarrow \mathbb{B}_{dR}$.

2) Fundamental exact sequence

$$0 \longrightarrow \hat{\mathbb{Q}}_p \longrightarrow (\mathbb{B}_{cris})^{\varphi=1} \longrightarrow \mathbb{B}_{dR}/\mathbb{B}_{dR}^+ \longrightarrow 0$$

Def $(X_s/V_0)_{cris}$ big crystalline site.

(a) Crystal in coherent sheaves on X_s is a sheaf of $\mathcal{O}_{X_s/V_0}$-modules on $(X_s/V_0)_{cris}$ st.

1) for each PD-thickening $(U, T, \delta) \in (X_s/V_0)_{cris}$ $\mathcal{E}|_T$ is a coherent $\mathcal{O}_T$-module

2) $\forall\, \alpha : (U, T, \delta) \longrightarrow (U', T', \delta')$ the induced map $\alpha^*(\mathcal{E}|_{T'}) \longrightarrow \mathcal{E}|_T$ is an $\mathcal{O}_T$-linear isomorphism.

(b) $\mathrm{Isoc}(X_s/V_0)$ isocrystals over X_s

- objects : crystals on X_s
- $\mathrm{Hom}_{\mathrm{Isoc}(X_s/V_0)}(\mathcal{E}, \mathcal{E}') := \mathrm{Hom}_{\mathcal{O}_{X_s/V_0}}(\mathcal{E}, \mathcal{E}') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$

The absolute Frobenius F on X_s induces a map

$$F : (X_s/V_0)_{cris} \longrightarrow (X_s/V_0)_{cris}$$

Def An F-isocrystal on X_s is a pair $(\mathcal{E}, \varphi)$

- $\mathcal{E}$ an isocrystal on X_s
- $\varphi : F^*\mathcal{E} \longrightarrow \mathcal{E}$ isom

category of F-isocrystal $\quad \mathrm{Isoc}^{\varphi}(X_s/V_0)$

We can consider F-isocrystals over $X_{p=0}$.

__Lemma__ (Ogus) The inclusions of sites

$$(X_s/V_0)_{cris} \subseteq (X_{p=0}/V_0)_{cris}$$

induces an equivalence

$$\mathrm{Isoc}^{\varphi}(X_s/V_0) \simeq \mathrm{Isoc}^{\varphi}(X_{p=0}/V_0)$$

§2. Crystalline Local systems

We give a Faltings style definition of crystalline local systems on X_η.

X smooth

$X_s = X_{red}$ reduced special fiber of X

$X_{p=0} = X \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.

Def. $\mathcal{E} \in \mathrm{Isoc}^{\varphi}(X_s/V_0)$ F-crystal on $(X_s/V_0)_{cris}$

$\mathbb{B}^+_{cris}(\mathcal{E})$ $\mathbb{B}_{cris}(\mathcal{E}) \in \mathrm{Shv}(X_{\eta,proet})$.

$$\mathcal{U} = \mathrm{Spa}(S, S^+) \in X_{\eta, proet}$$

The map $\mathbb{A}_{crys}(S, S^+) \longrightarrow S^+/p$ is a pro-pd-thickening on $(X_{p=0}/V_0)_{cris}$

Set

- $\mathbb{B}^+_{cris}(\mathcal{E})(\mathcal{U}) := \mathcal{E}(\mathbb{A}_{crys}(S, S^+))[1/p]$
$$= \left(\varprojlim_r \mathcal{E}(\mathbb{A}_{cris}(S, S^+)/p, \mathbb{A}_{cris}(S, S^+)/p^r, r)\right)[1/p]$$
where r is the canonical pd-structure.
- $\mathbb{B}_{cris}(\mathcal{E})(\mathcal{U}) := \mathbb{B}^+_{cris}(\mathcal{E})(\mathcal{U})[1/\mu]$
- $\mathbb{B}^+_{cris, K}(\mathcal{E}) := \mathbb{B}^+_{cris}(\mathcal{E}) \otimes_{K_0} K$

Note $\mathbb{B}_{cris}(\mathcal{E})$ is a module over the sheaf $\mathbb{B}_{cris}$ which is equipped with a Frobenius endomorphism

$$F : \mathbb{B}_{cris} \longrightarrow \mathbb{B}_{cris}$$

There is a canonical isomorphism

$$F^*\mathbb{B}_{cris}(\mathcal{E}) \xrightarrow{\sim} \mathbb{B}_{cris}(F^*\mathcal{E})$$

where $F^*\mathcal{E}$ is the pullback by the Frobenius over the special fiber.

If $\mathcal{E} \in \mathrm{Isoc}^{\varphi}(X_s/V_0)$ then

$$\varphi : F^* \mathbb{B}_{cris}(\mathcal{E}) \xrightarrow{\sim} \mathbb{B}_{cris}(F^*\mathcal{E}) \xrightarrow{\sim} \mathbb{B}_{cris}(\mathcal{E}).$$

We claim if $\mathcal{E}$ is a filtered F-crystal, then we can also construct a filtration on $\mathbb{B}_{cris}(\mathcal{E})$

Def. A sheaf of $\hat{\mathbb{Z}}_p$-modules L on X_η is <u>weakly crystalline</u> if

1) L is lisse, i.e., locally on $X_{\eta, \text{proét}}$ of the form $\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} M$ for some finitely generated $\mathbb{Z}_p$-module M
2) $\exists$ $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \mathrm{Isoc}^{\varphi}(X_s/V_0)$ and isom

$$\vartheta : \mathbb{B}_{crys}(\mathcal{E}) \xrightarrow{\sim} \mathbb{B}_{crys} \otimes_{\hat{\mathbb{Z}}_p} L.$$

s.t ϑ commutes with the Frobenius isomorphisms on both sides.

L is <u>crystalline</u> if $(\mathcal{E}, \varphi)$ underlie a filtered F-isocrystal $(\mathcal{E}, \varphi, \mathrm{Fil}^{\bullet}(E))$ on X and ϑ satisfies: $\vartheta_K = \vartheta \otimes_{K_0} K : \mathbb{B}_{cris,K}(\mathcal{E}) \xrightarrow{\sim} \mathbb{B}_{cris,K} \otimes_{\hat{\mathbb{Z}}_p} L$ is compatible with the filtrations on both sides.

L is a <u>local system</u> if L is <u>torsion-free</u>

We won't discuss filtrations for two reasons.

1) studying filtrations ultimately requires significant inputs from (derived) de Rham cohomology and infinitesimal cohomology of X_η.

2) **Prop** Any weakly crystalline local sheaf L on $X_{\eta,\text{proét}}$ is crystalline for a uniquely determined filtration on the associated F-isocrystal.

§3. Analytic prismatic F-crystals

Idea: $(R, R^+) \in \text{Perf}$ $\quad I = (p, \varpi)$ $\quad \varpi \in R$ pseudo-uniformizer

$\mathcal{Y}_{[0,\infty]} = \text{Spa}\, A_{\inf}(R, R^+) \setminus V(I)$

analytic locus of $\text{Spa}\, A_{\inf}(R, R^+)$.

Lemma (Kedlaya)

$$\text{Vect}(\text{Spa}\, A_{\inf}(R, R^+) \setminus V(I)) \simeq \text{Vect}(\text{Spec}\, A_{\inf}(R, R^+) \setminus V(I))$$

Def analytic prismatic crystals (in vector bundles) over X

$$\text{Vect}^{an}(X_{\Delta}) := \lim_{(A, I) \in X_{\Delta}} \text{Vect}(\text{Spec}\, A \setminus V(p, I))$$

(Drinfeld–Mathew) $\text{Vect}^{an}(X_{\Delta})$ satisfies (p, I)-completely flat descent.

<u>Etale realization functor</u>

X bounded p-adic formal scheme.

Here "bounded" means locally X is of the form $\mathrm{Spf}\, R$, where $R[p^\infty]$ is of bounded exponent.

<u>Def</u> (Laurent F-crystals) denoted with $\varphi = 1$ in Bhatt – Scholze

$$D_{perf}(X_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I_{\Delta}}]^{\wedge}_{p}) := \lim_{(A,I) \in X_{\Delta}} D_{perf}(A[1/I]^{\wedge}_{p})$$

Concretely, an object in $D_{perf}(X_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge}_{p})$ is a pair (E, φ_E), where

- E is a crystal of perfect complexes on $(X_{\Delta}, \mathcal{O}_{\Delta}[\frac{1}{I_{\Delta}}]^{\wedge}_{p})$, and
- $\varphi_E : \varphi^* E \xrightarrow{\sim} E$ is an isomorphism

We define also $\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge}_{p})$

We show how to realize Laurent F-crystals as $\mathbb{Z}_p$-local systems.

Proposition

$\exists$ natural identifications.

$$D_{perf}(X_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}[1/\mathcal{I}_{\mathbb{A}}]^{\wedge}_p)^{\varphi=1} \simeq D_{perf}(X_{\mathbb{A}}, \mathcal{O}_{\mathbb{A},perf}[1/\mathcal{I}_{\mathbb{A}}]^{\wedge}_p)^{\varphi=1}$$
$$\simeq D^{(b)}_{lisse}(X_\eta, \mathbb{Z}_p)$$

proof sketch.

two tasks

① show the invariance under completed perfection

② show the equivalence to $D^{(b)}_{lisse}(X_\eta, \mathbb{Z}_p)$

Both rely on the mod-p Riemann-Hilbert correspondence (essentially) by Katz

Lemma. S $\mathbb{F}_p$-algebra. Extension of scalars along $\mathbb{F}_p \longrightarrow \mathcal{O}_{Spec(S)_{et}}$ and taking Frobenius-fixed points give mutually inverse equivalences

$$D^b_{lisse}(Spec(S), \mathbb{F}_p) \simeq D_{perf}(S)^{\varphi=1}$$

For ①. since

$$D_{perf}(X_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}[1/\mathcal{I}_{\mathbb{A}}]^{\wedge}_p) = \lim_{(A,I)\in X_{\mathbb{A}}} D_{perf}(A[1/I]^{\wedge}_p)$$

$$D_{perf}(X_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}[1/\mathcal{I}_{\mathbb{A}}]^{\wedge}_p) = \lim_{(A,I)\in X_{\mathbb{A}}} D_{perf}(A_{perf}[1/I]^{\wedge}_p)$$

It is enough to show

$$D_{perf}(A[1/I]^{\wedge}_p) \xrightarrow{\sim} D_{perf}(B[1/J]^{\wedge}_p)$$

where (B, J) is the perfection of (A, I)
It is then enough to show this mod p (flatness + Nakayama)
This then follows from some Elkik approximation + Katz's lemma.
So it suffices to show

<u>Lemma</u> R char $p > 0$ $t \in R$ s.t. R is derived t-complete

$S = (R_{perf})^\wedge_t$

$$D_{perf}(R[1/t])^{\varphi=1} \xrightarrow{a} D_{perf}(R_{perf}[1/t])^{\varphi=1} \xrightarrow{b} D_{perf}(S[1/t])^{\varphi=1}$$

are equivalences.

<u>proof</u>.

a is an equivalence by Katz's lemma and top. invariance of etale sites.

We claim $b \circ a$ (and thus b) is fully faithful.

Using Katz's lemma for S and the full faithfulness of $b \circ a$ it suffices to check the pull-back functor

$$D^b_{cons}(\mathrm{Spec}\, S[1/t], \mathbb{F}_p) \longrightarrow D^b_{cons}(\mathrm{Spec}\, R[1/t], \mathbb{F}_p)$$

is essentially surjective. It follows from

$$\pi_1^{et}(\mathrm{Spec}\, S[1/t]) \simeq \pi_1^{et}(\mathrm{Spec}\, R[1/t])$$

which is in turn proved with Elkik's approximation theorem

Now for ② by descent, we may assume $X = \mathrm{Spf}\, R$
R qrsp. Thus X_Δ has an initial object (Δ_R, I)

Thus, $$D_{perf}(X_\Delta, \mathcal{O}_\Delta[1/I]^\wedge_p)^{\varphi=1} \simeq D_{perf}(\Delta_R[1/I]^\wedge_p)^{\varphi=1}$$

If R is furthermore perfectoid, then

$$\Delta_R[1/I]^\wedge_p \simeq W(R^\flat)[1/I]^\wedge_p$$

$$\simeq W(R^\flat[1/I]) \ ?$$

<u>Claim</u> (Bhatt – Scholze)

Katz's lemma + tilting equivalence implies

$$D_{perf}(\Delta_R[1/I]^\wedge_p)^{\varphi=1} \simeq D_{perf}(W(R^\flat)[1/I])^{\varphi=1}$$

$$\simeq D^b_{lisse}(\mathrm{Spa}(R[1/p], R), \mathbb{Z}_p)$$

$$\simeq D^b_{lisse}(\mathrm{Spec}\, R[1/p], \mathbb{Z}_p)$$

$$\simeq D^b_{lisse}(\mathrm{Spec}\, R^\flat[1/I], \mathbb{Z}_p)$$

mod p $\quad D_{perf}(R^\flat[1/I]) \simeq D_{perf}(R[1/p])$

AS

Realization functors.

- Etale realization.

$$T : \mathrm{Vect}^{an,\varphi}(X_{\Delta}) \longrightarrow \mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge}_{p}) \simeq \mathrm{Loc}_{\mathbb{Z}_p}(X_{\eta})$$

$$(\mathcal{E}, \varphi_{\mathcal{E}}) \longmapsto \mathcal{E} \otimes_{\mathcal{O}_{\Delta}} \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge}_{p}.$$

induced by the open immersion $\mathrm{Spec}\, A[1/I] \hookrightarrow \mathrm{Spec}\, A$ and the equivalence

- crystalline realization

$$D_{crys} : \mathrm{Vect}^{an,\varphi}(X_{\Delta}) \longrightarrow \mathrm{Vect}^{an,\varphi}(X_{p=0,\Delta})$$

$$\simeq \mathrm{Isoc}^{\varphi}(X_{p=0,\mathrm{cris}}) \simeq \mathrm{Isoc}^{\varphi}(X_{s,\mathrm{crys}})$$

Theorem 1 $\forall\, \mathcal{E} \in \mathrm{Vect}^{an,\varphi}(X_{\Delta})$, $T(\mathcal{E})$ is a crystalline local system.

proof sketch.

Claim. $T(\mathcal{E}) \otimes_{\mathbb{Z}_p} A_{inf}(S^+)[1/\mu] \simeq A_{inf}(\mathcal{E})[1/\mu](S, S^+)$

We can ignore the filtration and it suffices to construct an F-isocrystal $(\mathcal{E}_s, \varphi)$ over $(X_s/V_0)_{\mathrm{cris}}$ s.t.

$$B_{cris}(\mathcal{E}_s) \simeq A_{inf}(\mathcal{E})[1/\mu] \otimes_{A_{inf}[1/\mu]} B_{cris}$$

$(\mathcal{E}_{cris}, \varphi_{\mathcal{E}_{cris}}) := D_{cris}(\mathcal{E}, \varphi)$ over $(X_{p=0}/\mathbb{Z}_p)_{\mathrm{cris}}$ satisfies: for any perfectoid algebra S^+ over X Frobenius-equivariant isomorphism

$$\mathcal{E}(A_{cris}(S^+), c_p))[1/p] \simeq \mathcal{E}_{cris}(A_{cris}(S^+), S^+/p)[1/p]$$

If (S, S^+) is perfectoid algebra over X_η, then this isom induces Frob-equivariant isomorphism of $B_{cris}(S, S^+)$-modules.

$$\underbrace{\mathcal{E}(A_{cris}(S^+), c_p))[1/\mu]}_{\begin{matrix}\cong \\ A_{inf}(\mathcal{E}) \otimes_{A_{inf}} B_{cris}(S, S^+)\end{matrix}} \simeq \mathcal{E}_{cris}(A_{cris}(S^+), S^+/p)[1/\mu]$$

For RHS, restriction along $(X_s/V_0)_{cris} \subseteq (X_{p=0}/V_0)_{cris}$ $\subseteq (X_{p=0}/\mathbb{Z}_p)_{cris}$ gives $(\mathcal{E}_s, \varphi_{\mathcal{E}_s})$ on X_s.

Under this operation

$$\mathcal{E}_{cris}(A_{cris}(S^+), S^+/p)[1/\mu] \simeq B_{cris}(\mathcal{E}_s).$$

§4. Prismatic classification of crystalline local systems.

We now state the main theorem.

<u>Theorem A</u> The étale realization functor

$$T : \mathrm{Vect}^{an.\varphi}(X_{\Delta}) \longrightarrow \mathrm{Loc}^{crys}_{\mathbb{Z}_p}(X_{\eta, \text{proét}})$$

is an equivalence of categories.

§4.1 Full faithfulness.

To show <u>faithfulness</u>, we show this for

$$\mathrm{Vect}^{an.\varphi}(X_{\Delta}) \longrightarrow \mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge}_{p})$$

Suffices to show

<u>Lemma</u>. $\mathcal{E}, \mathcal{E}' \in \mathrm{Vect}^{an.\varphi}(X_{\Delta})$ $f : \mathcal{E} \longrightarrow \mathcal{E}'$ a map. if the base change

$$f \otimes id : \mathcal{E} \otimes_{\mathcal{O}_{\Delta}} \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge}_{p} \longrightarrow \mathcal{E}' \otimes_{\mathcal{O}_{\Delta}} \mathcal{O}_{\Delta}[1/I_{\Delta}]^{\wedge}_{p}$$

is 0, then so is f.

<u>Proof sketch</u>. Check on X Zariski-locally.

Since X is smooth. we can assume $X = \mathrm{Spf}\, R$ and it admits a p-completely flat cover $Y = \mathrm{Spf}\, S \longrightarrow X$ for some integral perfectoid ring S (by using framing) (This should be a lemma by Gabber?) By descent. It suffices to check f over Y_{Δ} is 0.