CHERN CLASSES AND SEGRE CLASSES

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Contents

This is a note for my talks given at the Boston University Number Theory Expository Seminar (BUNTES) in Fall 2022. The main reference is Fulton's *Intersection Theory*.

1. Higher Chern Classes

People with background in algebraic topology would know the splitting principle for vector bundles. Here we introduce its scheme-theoretic version as an important technique.

Splitting Principle Let $\{E^j\}_j$ be a fintie collection of locally free sheaves over k-scheme X. Then there exists a flat morphism $f : X' \to X$ such that

(1) $f^* : A_*X \to A_*X'$ is injective and
(2) for every E in the collection, f^*E has a filtration by subbundles:

$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_0 = 0$$

with line bundle quotients $E_i/E_{i-1} = L_i$.

Indeed, we can construct X' in the following way: First take $p : \mathbf{P}(E) \to X$. Thus p^* is injective on the Chow group by Corollary 4.11. p^*E has a subbundle $\mathcal{O}_E(-1)$ of rank 1. Let $E' = p^*E/\mathcal{O}_E(-1)$, which has rank $r-1$. Repeat the construction for $p' : \mathbf{P}(E') \to \mathbf{P}(E)$. The process terminates after finitely many iterations. Then apply the same construction for the other locally free sheaves in the collection.

Lemma 1.1. *Assume E has a filtration as above. Let s be a section of E, and $Z = Z(s) \subset X$ be the zero scheme of s. Then for any $\alpha \in A_k X$, there exists $\beta \in A_{k-r} Z$ such that*

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = \beta$$

In particular, if s is nowhere vanishing, then $\beta = 0$.

Proof. The section s induces a section $\overline{s}$ of the quotient line bundle L_r. If $Y = Z(\overline{s})$ is the zero scheme of $\overline{s}$, then $D_r = (L_r, Y, \overline{s}|_{X-Y})$ is a pseudo-divisor on X. Intersection with D_r gives a class $D_r \cdot \alpha \in A_{k-1} Y$ such that

$$c_1(L_r) \cap \alpha = j_*(D_r \cdot \alpha)$$

where $j : Y \to X$ is the inclusion. By projection formula, one has

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = j_* \left(\prod_{i=1}^{j-1} c_1(j^* L_i) \cap (D_r \cdot \alpha) \right) \tag{1}$$

s also induces a section s' of $j^* E_{r-1}$, whose zero scheme is Z. By induction on r, one finds the cycle on the right hand side of (1) is represented by a cycle on Z. $\square$

Now we define higher Chern classes. Recall Segre class maps $s_i(E) \cap \bullet : A_k(X) \to A_i(X)$ (associated to a vector bundle E) are defined to be

$$s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}_1)^{e+i} \cap p^* \alpha),$$

where $p : \mathbf{P}(E) \to X$ is the projection. We define the *Segre series* $s_t(E)$ to be the formal power series

$$s_t(E) = \sum_{i \geq 0} s_i(E) t^t$$

The *Chern polynomial*[1] $c_t(E)$ is then defined to be the formal inverse of $s_t(E)$, i.e.

$$c_t(E) = \sum_{i \geq 0} c_i(E) t^i := s_t(E)^{-1}$$

The *total Chern class* is defined to be $c(E) := c_t(E)|_{t=1} = \sum_{i \geq 0} c_t(E)$. Here is a list of basic properties of Chern classes.

Theorem 1.2.

1. *Vanishing:* $\forall i > \text{rank } E, c_i(E) = 0$.
2. *Commutativity:* $c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha)$
3. *Projection formula:* $f : X' \to X$ *proper.*

$$f_*(c_i(f^* E) \cap \alpha) = c_i(E) \cap f_* \alpha$$

[1]We will see in a second that $c_t(E)$, which is a priori a power series, is a polynomial and thus justify its name.

(4) Pullback: $f : X' \to X$ *flat.*

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$$

(5) Whitney sum: $0 \to E' \to E \to E'' \to 0$ *exact.*

$$c_t(E) = c_t(E')c_t(E'')$$

(6) Normalization: $c_1(E)$ *agrees with the first Chern class defined before.*

Proof. (2),(3),(4),(6) follow from Theorem 4.6. Let $f : X' \to X$ be a flat morphism as in the Splitting Principle. Then f^* is injective and

$$f^*(c_i(E) \cap \alpha) = c_i(f^*E) \cap f^*\alpha = 0$$

for $i > \operatorname{rank} E$ provided that (a) has been proved for f^*E. Thus, it suffices to assume E has a filtration as in the Splitting Principle.

Let $p : \mathbf{P}(E) \to X$ be the associated projective bundle. From the canonical embedding $\mathcal{O}(-1) \to p^*E$, we get a surjective map $p^*E^\vee \to \mathcal{O}(1)$ or $s : \mathcal{O} \to p^*E \otimes \mathcal{O}(1)$, which is a nowhere vanishing section of $p^*E \otimes \mathcal{O}(1)$. The vector bundle $p^*E \otimes \mathcal{O}(1)$ has a filtration with line bundle quotients $p^*L_i \otimes \mathcal{O}(1)$ provided by the filtration on E. Then by Lemma 1.1,

$$\prod_{i=1}^{r} c_1(p^*L_i \otimes \mathcal{O}(1)) = 0$$

Let σ_i, τ_i be the i-th elementary symmetric polynomials in $c_1(L_1), \ldots, c_1(L_r)$ and $c_1(p^*L_1)$, $\ldots, c_1(p^*L_r)$, respectively. Denote $\zeta = c_1(O(1))$. Recall by Theorem 4.9, we have

$$c_1(p^*L_i \otimes \mathcal{O}(1)) = c_1(p^*L_i) + c_1(\mathcal{O}(1)) = c_1(p^*L_i) + \zeta$$

With $e = r - 1$, we have

$$\zeta^{e+i} + \tau_1\zeta^{e+i-1} + \ldots + \tau_r\zeta^{i-1} = 0$$

for all $i \geq 1$. Thus, for all $\alpha \in A_*X$, one has

$$p_*((\zeta^{e+i} + \tau_1\zeta^{e+i-1} + \ldots + \tau_r\zeta^{i-1}) \cap p^*\alpha) = 0$$

This means

$$(1 + \sigma_1 t + \ldots + \sigma_r t^r)s_t(E) = 1$$

which is equivalent to

$$c_t(E) = \prod_{i=1}^{r}(1 + c_1(L_i)t)$$

which apparently implies (1).

□

We can factor formally $c_t(E)$ as

$$c_t(E) = \prod_{i=1}^{r}(1 + \alpha_i t).$$

Here α_i are the *Chern roots* of E. If E admits a filtration as in the splitting principle, then $\alpha_i = c_1(L_i)$. Chern roots enable us to prove some additional properties of Chern classes.

Theorem 1.3.

(1) (Dual bundles) $c_i(E^\vee) = (-1)^i c_i(E)$
(2) (Tensor products) Let E *and* F *be two vector bundles of ranks* r *and* s*, respectively. Then Chern roots allow us to determine Chern classes of the tensor product* $E \otimes F$ *in terms of Chern classes of* E *and* F*. (The general formulae, unfortunately, are complicated.)*
(3) (Exterior products) $c_t(\bigwedge^p E) = \prod_{i_1 < \cdots < i_p}(1 + (\alpha_{i_1} + \cdots + \alpha_{i_p})t)$
(4) (Symmetric products) $c_t(\mathrm{Sym}^p E) = \prod_{i_1 \leq \cdots \leq i_p}(1 + (\alpha_{i_1} + \cdots + \alpha_{i_p})t)$

Proof. (1) If E has a filtration with line bundle quotients L_i, then $E^\vee$ has a filtration with line bundle quotients $L_i^\vee$. Thus, if E has Chern roots $\alpha_1, \ldots, \alpha_r$, then $E^\vee$ has Chern roots $-\alpha_1, \ldots, -\alpha_r$.

(2) If $\alpha_1, \ldots, \alpha_r$ are Chern roots of E and $\beta_1, \ldots, \beta_s$ are Chern roots of F, then $\alpha_i + \beta_j$ are Chern roots of $E \otimes F$. This is again shown by first considering the case that E and F have filtrations as in the Splitting Principle. Indeed, suppose they have filtrations

$$0 = E_0 \subset \ldots \subset E_r = E$$

with line bundle quotients L_i and

$$0 = F_0 \subset \ldots \subset F_s = F$$

with line bundle quotients N_j. Then the tensor product $E \otimes F$ has a filtration

$$0 = E_0 \otimes F_0 + 0 \otimes F \subset \ldots \subset E_i \otimes F_j + E_{i-1} \otimes F \subset \ldots \subset E \otimes F$$

where (i, j) is ordered lexicographically, which has successive quotient $L_i \otimes N_j$.

(3) and (4) are shown similarly. □

Let's introduce two more notions before moving to examples. The *Chern character* $\mathrm{ch}(E)$ is defined to be

$$\mathrm{ch}(E) = \sum_{i=1}^{r} \exp(\alpha_i).$$

The *Todd class* $\mathrm{td}(E)$ is defined to be

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - \exp(-\alpha_i)}$$

These two notions will appear in the several fundamental theorems in intersection theory, e.g., the Hirzebruch-Riemann-Roch theorem:

Theorem 1.4 (Hirzebruch-Riemann-Roch)**.** *Let E be a vector bundle on a smooth complete variety X. Then*

$$\chi(X, E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(T_X)$$

Now we turn to examples.

Example 1.5 (Affine space).

$$A_k(\mathbf{A}^n) = \begin{cases} 0 & k < n \\ \mathbb{Z} & k = n \end{cases}$$

By Proposition 4.7, the pullback map $A_k\, \mathbf{A}^n \to A_{k+m}\, \mathbf{A}^{m+n}$ is surjective for any k, m and n. Then the statement follows.

Example 1.6 (Projective space). [1] If $0 \leq k \leq n$, then $A_k\, \mathbf{P}^n = \mathbb{Z}$. This is clear for $k = n$ and $k = n-1$. For $0 \leq k \leq n-2$, consider the exact sequence

$$A_k(\mathbf{P}^{n-1}) \longrightarrow A_k(\mathbf{P}^n) \longrightarrow A_k(\mathbf{A}^n) \longrightarrow 0$$

By induction on n, we may assume $A_k(\mathbf{P}^{n-1}) = \mathbb{Z}$. Let $[L^k]$ be the class of k-dimensional linear space, i.e., a generator of $A_k(\mathbf{P}^n)$. Notice $c_1(\mathcal{O}(1)) \cap [L^k] = [L^{k-1}]$. If $d[L^k]$ is rationally equivalent to zero, then $(c_1(\mathcal{O}(1)))^k \cap d[L^k] = d[L^0] = 0$. This means

$$dL^0 = \sum_i n_i div(r_i)$$

with $r_i \in R(C_i)$ and C_i being curves in $\mathbf{P}^n$. However, $\sum_i n_i div(r_i)$ had total degree 0. Thus, d must be 0.
Thus, $A_*\, \mathbf{P}^n = \mathbb{Z}^{n+1} = \mathbb{Z}[H]/(H^{n+1})$[2]. Recall we also have the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(1) \to T_{\mathbf{P}^n} \to 0$$

Thus,

$$c(T_{\mathbf{P}^n}) = (1 + c_1(H))^{n+1}$$

where H is a hyperplane in $\mathbf{P}^n$.

Example 1.7 (Abelian variety). Suppose $i : X \to \mathbf{P}^m$ is a closed embedding of an n-dimensional abelian variety. One has the adjunction sequence

$$0 \to T_X \to i^* T_{\mathbf{P}^m} \to N \to 0,$$

where N is the normal bundle of X in $\mathbf{P}^m$. Thus, $c_1(i^*\, \mathcal{O}(1))^{m-n+1} = 0$. From the sequence, we have

$$c(N) = (1 + c_1(i^*\, \mathcal{O}(1)))^{m+1}.$$

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[1]It would be a useful strategy to lift intersection problems on a projective variety to problems on a projective space into which the variety is embedded as a closed subvariety.

[2]It is not clear at this point that the Chow group has this ring structure.

By higher dimensional Bézout's theorem,

$$\int_X c_1(i^* \mathcal{O}(1))^n = \deg(X),$$

which is nonzero. Thus, $m - n + 1 > n$, i.e., $m > 2n - 1$. In other words, this shows X cannot be embedded into $\mathbf{P}^m$ if $m < 2\dim(X)$.

Example 1.8 (Ramification). Let X, Y be two smooth n-dimensional varieties and $f : X \to Y$ be a morphism. Define $R(f)$ to be the subset of X where the induced map of tangent spaces is not an isomorphism. (This is the same as ramification in the algebraic sense.) $R(f)$ is endowed with a scheme structure by identifying $R(f)$ with the zero scheme of

$$\wedge^n df : \wedge^n T_X \to \wedge^n f^* T_Y$$

or, equivalently, the zero scheme of a section of the line bundle $\bigwedge^n f^* T_Y \otimes \bigwedge^n T_X^\vee$. If $R(f) \neq X$, then by Exanple 4.5, we have

$$[R(f)] = (c_1(f^* T_Y) - c_1(T_X)) \cap [X]$$

If $n = 1$, by taking degrees of both sides, the formula above is reduced to the Riemann-Hurwitz formula:

$$2g_X - 2 = \deg(f)(2g_Y - 2) + \deg R(f)$$

Example 1.9 (Blowing-up). Let X be a smooth n-dimensional variety and Y be a smooth m-dimensional subvariety of X. Suppose $\mathcal{I} = \mathcal{I}_Y$ is the ideal sheaf of Y. Let $\pi : \tilde{X} = \operatorname{Proj}\left(\bigoplus_{d \geq 0} \mathcal{I}^d\right) \to X$ be the blowing-up of X along Y. Then $\tilde{X} := \operatorname{Proj}\left(\bigoplus_{d \geq 0} \mathcal{I}^d / \mathcal{I}^{d+1}\right) = \mathbf{P}(N)$ is the exceptional divisor, where $N = \operatorname{Spec}\left(\bigoplus_{d \geq 0} \mathcal{I}^d / \mathcal{I}^{d+1}\right) \to Y$ is the normal cone of Y. Denote by $\eta : \tilde{Y} \to Y$ the projection. Define the k-th self-intersection of $\tilde{Y}$ to be

$$\tilde{Y}^k = \tilde{Y} \cdot \cdots \cdot \tilde{Y} \cdot [\tilde{X}] \in A_{n-k}(\tilde{Y})$$

The restriction of the line bundle $\mathcal{O}_{\tilde{X}}(\tilde{Y})$ to $\tilde{X}$ is

$$\mathcal{O}_{\tilde{X}}(\tilde{Y})|_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-1)$$

Therefore, one has

$$\tilde{Y}^k = (-1)^{k-1} c_1(\mathcal{O}_{\tilde{Y}}(1))^{k-1} \cap [\tilde{Y}]$$

The intersection of total Segre class of N with $[Y]$ is then

$$\begin{aligned} s(N) \cap [Y] &= \sum_{i \geq 0} \eta_*(c_1(\mathcal{O}(1))^{e+i}) \cap \eta^*[Y] \\ &= \sum_{i \geq 0} \eta_*(c_1(\mathcal{O}(1))^{e+i} \cap [Y]) \\ &= \sum_k (-1)^{k-1} \eta_*(\tilde{Y}^k) \end{aligned}$$

2. Rational Equivalence on Cones

Now we consider pulling back cycles on scheme X to a vector bundle. Let E be a vector bundle of rank $r = e + 1$ over a scheme X with projection $\pi : E \to X$ and $p : \mathbf{P}(E) \to X$ the associated projective bundle. Denote by $\mathcal{O}(1)$ the canonical line bundle on $\mathbf{P}(E)$.

Theorem 2.1. *(1) The flat pullback*

$$\pi^* : A_{k-r}X \to A_k E$$

is an isomorphism for all k.

(2) Any $\beta \in A_k \mathbf{P}(E)$ can be uniquely written as

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i$$

for $\alpha_i \in A_{k-e+i}X$. Thus, there are canonical isomorphisms

$$\bigoplus_{i=0}^{e} A_{k-e+i}X \cong A_k \mathbf{P}(E)$$

Proof. Surjectivity of π^* has been shown in Theorem 4.7. Let

$$\theta_E : \sum_{i=0}^{e} A_{k-e+i}X \to A_k \mathbf{P}(E), \oplus\alpha_i \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i$$

We want to show θ_E is surjective. By Noetherian induction, it suffices to consider the case where E is trivial. By induction on the rank, it suffices to show surjectivity of $\theta_{E\oplus 1}$, assuming the surjectivity of θ_E. Let $P = \mathbf{P}(E), Q = \mathbf{P}(F) = \mathbf{P}(E \oplus 1)$, and $q : Q \to X$ be the projection. Consider the following commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{i} & Q & \xleftarrow{j} & E \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle q} & \swarrow{\scriptstyle \pi} & \\ & & X & & \end{array}$$

By Proposition 4.10, the row of the following commutative diagram is exact.

$$\begin{array}{ccccccc} A_kP & \xrightarrow{i_*} & A_kQ & \xrightarrow{j^*} & A_kE & \longrightarrow & 0 \\ & & {\scriptstyle q^*}\uparrow & \nearrow{\scriptstyle \pi^*} & & & \\ & & A_{k-r}X & & & & \end{array}$$

Before proceeding, we show a lemma.

Lemma 2.2. *For $\forall\alpha \in A_*X$,*

$$c_1(\mathcal{O}_F(1)) \cap q^*\alpha = i_*p^*\alpha$$

Proof of the lemma. It suffices to prove for the case $\alpha = [V]$, where V is a subvariety of X. Denote $j : V \to X$ the inclusion. We may identify $\mathcal{O}_F(1)$ as $\mathcal{O}_F(i_*P)$. Therefore, $c_1(\mathcal{O}_F(1)) \cap q^*\alpha = i_*[j^*i_*P] = i_*p^*[V]$ by the commutativity of intersection with Cartier divisor. □

Now if $\beta \in A_*Q$, write $j^*\beta = \pi^*\alpha = j^*q^*\alpha$ for some $\alpha \in A_*X$. Then $\beta - q^*\alpha \in \ker j^* = \operatorname{im} i_*$. Then

$$\beta - q^*\alpha = i_* \left(\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*\alpha_i \right) \tag{2}$$

(using the surjectivity of θ_E) for some $\alpha_i \in A_*X$. Since $i^*\mathcal{O}_F(1) = \mathcal{O}_E(1)$, by projection formula, we have

$$(2) = \sum_{i=0}^{e} c_1(\mathcal{O}_F(1))^i \cap i_*p^*\alpha_i$$

Now applying the lemma gives

$$\beta = q^*\alpha + \sum_{i=0}^{e} c_1(\mathcal{O}_F(1))^{i+1} \cap q^*\alpha_i = \sum_{i=0}^{e+1} c_1(\mathcal{O}_F(1))^i \cap q^*\alpha_i$$

which shows the surjectivity of θ_F.
To show the uniqueness, suppose there is a nontrivial relation

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i = 0$$

Let ℓ be the largest integer with $\alpha_\ell \neq 0$. Then

$$p_*(c_1(\mathcal{O}(1))^{e-\ell} \cap \beta) = \sum_{j=-\ell}^{0} s_j(E) \cap \alpha_{j+l} = \alpha_\ell$$

which is a contradition.
Finally, we show the injectivity of π^*. Let $F = E \oplus 1, Q = \mathbf{P}(F)$. If $\pi^*\alpha = 0, \alpha \neq 0$, then $j^*q^*\alpha = 0$. So

$$q^*\alpha = i_* \left(\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*\alpha_i \right) = \sum_{i=0}^{e} c_1(\mathcal{O}_F(1))^{i+1} \cap q^*\alpha_i$$

by the lemma. But this contradicts the uniqueness for Q. □

With the theorem, we are able to define Gysin homomorphisms. Consider the *zero section* $s = s_E : X \to E$ of the vector bundle E, i.e., the homomrophism defined by the augmentation morphism $\operatorname{Sym} E \to \mathcal{O}_X$.

3. Segre Classes of Cones

So far we have only regarded Segre classes $s_i(E)$ of a vector bundle E over scheme X as endomorphisms of the Chow group A_*X. In this section, we show the total Segre class $s(E)$ is really a cycle class on the scheme X.

A morphism $C \to X$ of schemes is a cone over X if C is isomorphic to $\mathrm{Spec}\ S^\bullet$ for some graded $\mathcal{O}_X$-algebra $S^\bullet$. For any cone C, we define its projective completion $p : \mathbf{P}(C \oplus 1) \to X$ to be $\mathrm{Proj}\,(S^\bullet \otimes_{\mathcal{O}_X} \mathrm{Sym}\,\mathcal{O}_X) = \mathrm{Proj}\,(S^\bullet[z]) \to X$. We define the Segre class $s(C)$ of a cone by

$$s(C) = p_*\left(\sum_{i\geq 0} c_1(\mathcal{O}_P(1))^i \cap [\mathbf{P}(C \oplus 1)]\right)$$

Now suppose C is given by a vector bundle, i.e. $C = E = \mathrm{Spec}\,(\mathrm{Sym}\,\mathcal{E})$ with $\mathcal{E}$ being a locally free sheaf on X.

Proposition 3.1. $s(E) = s(E) \cap [X]$

Proof.

$$\begin{aligned} s(E) \cap [X] = s(E \oplus 1) \cap [X] &= \sum_i s_i(E \oplus 1) \cap [X] \\ &= \sum_i p_*\left(c_1(\mathcal{O}(1))^i \cap p^*[X]\right) \\ &= p_*\left(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i \cap [\mathbf{P}(C \oplus 1)]\right) \end{aligned}$$

□

Proposition 3.2. *If C has irreducible components $C_1, \ldots, C_r$ with geometric multiplicities $m_1, \ldots, m_r$, then*

$$s(C) = \sum_{i=1}^r m_i s(C_i)$$

Proof. Notice that $\mathbf{P}(C \oplus 1)$ has irreducible components $\mathbf{P}(C_i \oplus 1)$ with multiplicities m_i. Denote by $q_i : \mathbf{P}(C_i \oplus 1) \to X$ the projections. Then

$$\begin{aligned} s(C) &= q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\mathbf{P}(C \oplus 1)] \right) \\ &= q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \sum_{j=1}^{r} m_j [\mathbf{P}(C_i \oplus 1)] \right) \\ &= \sum_{j=1}^{r} q_{j*} \left(\sum_{i \geq 0} c_1(\mathcal{O}_{\mathbf{P}(C_j \oplus 1)}(1))^i \cap [\mathbf{P}(C_i \oplus 1)] \right) \\ &= \sum_{j=1}^{r} m_j s(C_i) \end{aligned}$$

□

4. Segre Classes of Subschemes

Let X be a closed subscheme of a scheme Y defined by ideal sheaf $\mathcal{I}$, and $C = C_X Y$ be the normal cone of X in Y. Then the Segre class of X in Y is defined to be $s(X, Y) := s(C_X Y) \in A_* X$.

Remark 4.1. It may be helpful to clarify a use of notations. Let Y be a closed subscheme of X defined by ideal sheaf $\mathscr{I}$. The *normal cone* $C_Y X$ over Y is the cone associated to the *conormal sheaf* $\mathscr{I} / \mathscr{I}^2$, that is, $C_Y X := \operatorname{Spec} \bigoplus_{d \geq 0} \mathscr{I}^d / \mathscr{I}^{d+1} \to Y$. If X and Y are smooth varieties, then $\mathscr{I} / \mathscr{I}^2$ is a locally free sheaf (of rank $\operatorname{codim}(Y, X)$), In this case, we denote $N_Y X = C_Y X$ and call $N_Y X$ the *normal bundle* of Y in X.

Lemma 4.2. *Assume Y is of pure-dimension, and $Y_1, \ldots, Y_r$ are irreducible components with geometric multiplicities $m_1, \ldots, m_r$. Let X be a closed subscheme of Y and define $X_i = X \cap Y_i$. Then*

$$s(X, Y) = \sum_{i=1}^{r} m_i s(X_i, Y_i)$$

in $A_* X$.

Proof. Denote by $M_X Y$ the blowing-up of $Y \times \mathbf{A}^1$ along $X \times \{0\}$. Then $M_{X_i} Y_i$ are the irreducible components of $M_X Y$ with multiplicities m_i. Then

$$[M_X Y] = \sum m_i [M_{X_i} Y_i]$$

Thus,

$$[\mathbf{P}(C_X Y \oplus 1)] = \sum_{i=1}^{r} m_i [\mathbf{P}(C_{X_i} Y_i \oplus 1)]$$

□

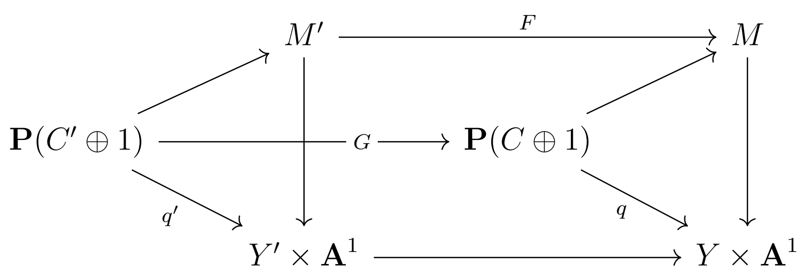
The lemma would not be true if one does not require Y to be of pure dimension. For example, consider $Y = V(xz, yz) \subset \mathbf{A}^3$ and $X \subset Y$ cut out by $x - z = 0$. Then $Y = \mathbf{A}^2 \cup \mathbf{A}^1 = Y_1 \cup Y_2$. Denote by P the intersection of the two irreducible components of Y and $X_i = X \cap Y_i, i = 1, 2$. Then an easy computation shows $s(X_1, Y_1) + s(X_2, Y_2) = 2[P]$ whereas $s(X, Y) = 3[P]$. The problem with non-pure dimensional case is components of $\mathbf{P}(C \oplus 1)$ would not be annihilated at once by a power of $c_1(\mathcal{O}(1))$. Thus, there would be "extra" cycles in $s(X, Y)$.

Proposition 4.3. *Let $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ be a closed subscheme, and $X' := f^{-1}(X)$. Let $g : X' \to X$ be the morphism induced by f.*

(1) If f is proper, Y is irreducible, and f maps each irreducible component of Y' onto Y, then
$$g_*(s(X', Y')) = \deg(Y'/Y)s(X, Y)$$
Here $\deg(Y'/Y)$ is defined to be $\deg(Y'/Y) = \sum_{i=1}^r \deg(Y_i'/Y)$ where $Y_1', \dots, Y_r'$ are irreducible components of Y' with multiplicities $m_1, \dots, m_r$.

(2) If f is flat, then
$$g^*(s(X, Y)) = s(X', Y')$$

Proof. We may assume Y' is irreducible. We have the following commutative diagram



where $M = \mathrm{Bl}_{X\times\{0\}} Y \times \mathbf{A}^1$ and $M' = \mathrm{Bl}_{X'\times\{0\}} Y' \times \mathbf{A}^1$. $\mathbf{P}(C' \oplus 1)$ and $\mathbf{P}(C \oplus 1)$ are the exceptional divisors in M' and M, respectively. Let $\mathcal{O}(1)$ be the canonical line bundle on $\mathbf{P}(C \oplus 1)$. Then $G^*\mathcal{O}(1)$ is the canonical line bundle on $\mathbf{P}(C' \oplus 1)$. Notice $F_*[M'] = d[M]$, where $d := \deg(Y'/Y)$. By projection formula, we have
$$\begin{aligned} g_*s(X', Y') &= g_*q'_*(\sum_{i\geq 0} c_1(G^*\mathcal{O}(1))^i \cap [\mathbf{P}(C' \oplus 1)]) \\ &= q_*G_*(\sum_{i\geq 0} c_1(G^*\mathcal{O}(1))^i \cap [\mathbf{P}(C' \oplus 1)]) \\ &= q_*(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i \cap G_*[\mathbf{P}(C' \oplus 1)]) \\ &= q_*(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i \cap d[\mathbf{P}(C' \oplus 1)]) \\ &= ds(X, Y) \end{aligned}$$

Similarly for (b),

$$\begin{aligned} g^*s(X,Y) &= g^*q_*(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i \cap [\mathbf{P}(C\oplus 1)]) \\ &= q'_*G^*(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i \cap [\mathbf{P}(C\oplus 1)]) \\ &= q'_*(\sum_{i\geq 0} c_1(G^*\,\mathcal{O}(1))^i \cap G^*[\mathbf{P}(C\oplus 1)]) \\ &= s(X',Y') \end{aligned}$$

□

Remark 4.4. I don't think it has been shown in earlier chapters that $g_*q'_* = q_*G_*$ if all four morphism here are proper. But this is easy. One just needs to show $(g\circ f)_* = f_*g_*$ provided that f and g are proper, which is in fact clear from the definition of proper push-forward.

Although one cannot expect any birational invariance in intersection theory in general, this proposition does show Segre classes of closed subschemes is birationally invariant (that is, the case $\deg(Y'/Y) = 1$) if the morphism g is proper.

Appendix: Zero Scheme

In this section, we give a brief review on the zero scheme of a section of a vector bundle. First, we give a sheaf-theoretic description. Let $\mathcal{E}$ be a locally free sheaf on a scheme X. A global section $s\in H^0(X,\mathcal{E})$ is a sheaf morphism $s:\mathcal{O}_X\to\mathcal{E}$. Taking dual gives $s^\vee:\mathcal{E}^\vee\to\mathcal{O}_X$. The zero scheme $Z(s)$ of the section s is then defined to be the closed subscheme of X determined by the ideal sheaf $\operatorname{im}(s^\vee)$.
Now we consider the geometric case. Let $\mathcal{E}$ be a locally free sheaf of rank n on a k-scheme X and $E = \operatorname{Spec}(\operatorname{Sym}(\mathcal{E}))\to X$ be the vector bundle asscoiated to $\mathcal{E}$. Let $s: X\to E$ be a section. Then locally this is given by $s:U\to\mathbf{A}^n_U = \mathbf{A}^n\times_k U$, which is determined by $(s_1,\dots,s_n)\in A^{\oplus n}$ if $U = \operatorname{Spec} A$. Then on U, the zero scheme $Z(s)$ of s is given by the vanishing locaus $V(s_1,\dots,s_n)$. Alternatively, thanks to the sheaf-theoretic description, we can also consider s as a sheaf morphism $s:\operatorname{Sym}(\mathcal{E})\to\mathcal{O}_X$ which corresponds to $s:\mathcal{E}\to\mathcal{O}_X$, whose image is an ideal sheaf $\mathcal{I}$. Then $Z(s)$ is the closed subscheme assocaited to $\mathcal{I}$.

Example 4.5 (Example 3.2.16). Let E be a vector bundle of rank r on X, s a section of E, $Z = Z(s)$ the zero scheme of s. Then one has

(1) For any $\alpha\in A_kX$, there is a class β in $A_{k-r}Z$ whose image in $A_{k-r}(A)$ is $c_r(E)\cap\alpha$. In particular, if $Z=\emptyset$, then $c_r(E)=0$.
(2) If X is purely n-dimensional, and s is a regular section of E, then Z is purely $(n-r)$-dimensional, and

$$c_r(E)\cap[X] = [Z]$$

Appendix: Relative Proj

In this section, we give a brief review of relative Proj, projective bundle, and blowing-up. Let X be a k-scheme and S be a sheaf of graded $\mathcal{O}_X$-algebras over X. Then one has a morphism of scheme $\pi : \mathrm{Proj}\,(S) \to X$, which is locally given by $\mathrm{Proj}\,(S(U)) \to U = \mathrm{Spec}\,(A) \subset X$. If E is a locally free sheaf on X, then we denote $\mathbf{P}(E) = \mathrm{Proj}\,(\mathrm{Sym}(E))$. $\mathrm{Proj}\,(S)$ comes with a canonical line bundle $\mathcal{O}(1)$, which is locally given by $\mathcal{O}(1)$ on $\mathrm{Proj}\,(S(U))$. Therefore, there is a natural surjection $\pi^*E \to O(1)$ on $\mathbf{P}(E)$.
Now suppose Y is a closed subscheme of X defined by ideal sheaf $\mathcal{I} = \mathcal{I}_Y$. The blowing-up of X along Y is defined to be $\pi : \tilde{X} := \mathrm{Proj}\,(\sum_{d\geq 0} \mathcal{I}^d) \to X$ where $\mathcal{I}^0 := \mathcal{O}_X$. Assume X and Y are smooth varieties. Let $\mathcal{I}\,/\,\mathcal{I}^2$ be the conormal sheaf of Y. Then the exceptional divisor Y' is isomorphic to $\mathbf{P}(\mathcal{I}\,/\,\mathcal{I}^2)$ and the normal sheaf N of Y' in $\tilde{X}$ corresponds to $\mathcal{O}_{\mathbf{P}(\mathcal{I}\,/\,\mathcal{I}^2)}(-1)$.

Appendix: Theorems from Previous Chapters

Theorem 4.6 (Proposition 3.1)**.**

(1) For $\forall \alpha \in A_k X$,
  (a) $s_i(E) \cap \alpha = 0$ for $i < 0$;
  (b) $s_0(E) \cap \alpha = \alpha$

(2) If E, F are vector bundles on X, $\alpha \in A_k X$, then $\forall i, j$,

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha)$$

(3) If $f : X' \to X$ is proper, E is a vector bundle over X, $\alpha \in A_* X'$, then $\forall i$,

$$f_*(s_i(f^*F) \cap \alpha) = s_i(E) \cap f_*(\alpha)$$

(4) If $f : X' \to X$ is flat, E is a vector bundle over X, $\alpha \in A_* X$, then $\forall i$,

$$s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha)$$

(5) If E is a line bundle on X, $\alpha \in A_* X$, then

$$s_1(E) \cap \alpha = -c_1(E) \cap \alpha$$

Theorem 4.7 (Proposition 1.9)**.** *Let $p : E \to X$ be a vector bundle of rank n. Then the flat pull-back*

$$p^* : A_k X \to A_{k+n} E$$

is surjective for all k.

Proof. Choose a closed subscheme Y of X so that $U = X - Y$ is affine open on which E is trivial. We then have a commutative diagram

$$\begin{array}{ccccccc} A_*Y & \longrightarrow & A_*X & \longrightarrow & A_*U & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_*(p^{-1}Y) & \longrightarrow & A_*E & \longrightarrow & A_*(p^{-1}U) & \longrightarrow & 0 \end{array}$$

where the vertical maps are flat pullbacks, and the rows are exact by Proposition 4.10. By a diagram chase it suffices to prove the assertion for the restrictions of E to U and to Y. By Noetherian induction, i.e., repeating the process on Y, it suffices to prove it for $X = U$. Thus we may assume E is trivialized, i.e. $E = X \times \mathbf{A}^n$. The projection factors

$$X \times \mathbf{A}^n \to A \times \mathbf{A}^{n-1} \to X$$

So we may assume $n = 1$. We must show that $[V]$ is in $p * A_k X$ for any $(k+1)$-dimensional subvariety V of E. We may replace X by the closure of $p(V)$ (cf. Proposition 4.12), so we may assume X is a variety and p maps V dominantly to X. Let A be the coordinate ring of X, $K = R(X)$ the quotient field of A, and let q be the prime ideal in $A[t]$ corresponding to V. If $\dim X = k$, then $V = E$, so $V = P^*[X]$. So we may assume dim $X = k+1$. Since V dominates X and $V \neq E$, the prime ideal $q \cdot K[t]$ is non-trivial; let $r \in K[t]$ generate $q \cdot K[t]$. Then

$$[V] - [div(r)] = \sum_i n_i [V_i],$$

for some $(k+1)$-dimensional subvarieties V_i of E whose projections to X are not dominant. Therefore $V_i = p^{-1}(W_i)$, with $W_i = p(V_i)$, so

$$[V] = [div(r)] + \sum_i n_i p * [W_i],$$

as required. □

Remark 4.8 (Noetherian induction). We recall the concept of Noetherian induction. Let X be a Noetherian topological space and P be a property of closed subsets of X. Assume that for any closed subset Y of X, if P holds for every proper closed subset of Y, then P holds for Y. (In particular, P must hold for the empty set.) Then P holds for X.

Theorem 4.9 (Proposition 2.5).

(1) *If α is rationally equivalent to 0 on X, then $c_1(L) \cap \alpha = 0$. Therefore $c_1(L)$ induces a well-defined homomorphism*

$$c_1(L) \cap \bullet : A_k X \to A_{k-1} X$$

(2) *(Commutativity) If L, L' are line bundles on X, $\alpha \in A_k X$, then*

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$$

in $A_{k-2} X$.

(3) *(Projection formula) If $f : X' \to X$ is a proper morphism, L a line bundle on X, $\alpha \in A_k X'$, then*

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*\alpha$$

(4) *(Flat pullback) If $f : X' \to X$ is flat of relative dimension n, L a line bundle on X, $\alpha \in A_k X$, then*

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$$

in $A_{k-1+n}X'$

(5) *(Additivity) If L, L' are line bundles on X, $\alpha \in A_k X$, then*

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$$

and

$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha$$

in $A_{k-1}X$.

Proposition 4.10 (Proposition 1.8)**.** *Let Y be a closed subscheme of a scheme X and $U = X - Y$. Let $i : Y \to X, j : U \to X$ be the inclusions. Then the sequence*

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j^*} A_k U \to 0$$

is exact for all k.

Corollary 4.11 (Corollary 3.1)**.** *The flat pullback*

$$p^* : A_k X \to A_{k+e}\,\mathbf{P}(E)$$

is a split monomorphism.

Proof. An inverse is $\beta \mapsto p_*(c_1(\mathcal{O}_E(1))^e \cap \beta)$. □

Proposition 4.12 (Proposition 1.7)**.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

be a cartesian square with g flat and f proper. Then g' is flat, f' is proper, and for all $\alpha \in Z_*X$

$$f'_*g'^*\alpha = g^*f_*\alpha$$

in Z_*Y'.