# INTRODUCTION TO PRISMATIC COHOMOLOGY I

XINYU ZHOU

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## 1. Historical motivations

Recall the $A_{\mathrm{inf}}$-cohomology constructed in [BMS18] and extended to the semistable case in [ČK19]. Let $C$ be a complete algebraically nonarchimedean field over $\mathbb{Q}_p$ with residue field $k$, and let $\mathfrak{X}$ be a smooth proper scheme over $\operatorname{Spf} \mathscr{O}_C$. Write $X$ for the adic generic fiber of $\mathfrak{X}$. Then we have a complex $A\Omega_{\mathfrak{X}} \in D(\mathfrak{X}_{\mathrm{Zar}})$ given by

$$A\Omega_{\mathfrak{X}} \coloneqq L\eta_\mu(R\nu_* \mathbb{A}_{\mathrm{inf},X}),$$

where $\nu : X_{\text{proét}} \to X_{\text{ét}}$ is the canonical projection, $\mathbb{A}_{\mathrm{inf},X} \coloneqq W(\mathcal{O}_X^{\flat+})$ is the usual infinitesimal period sheaf, and $L\eta_\mu$ is the décalage functor with respect to the distinguished element $\mu \in A_{\mathrm{inf}} = \mathbb{A}_{\mathrm{inf}}(\mathscr{O}_{C^\flat})$. The $A_{\mathrm{inf}}$-cohomology $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ of $\mathfrak{X}$ is then defined as

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \coloneqq R\Gamma(\mathfrak{X}_{\mathrm{Zar}}, A\Omega_{\mathfrak{X}}).$$

There are a few comparison results between $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ and the other well-konw $p$-adic cohomology theories.

**Theorem 1.1** ([BMS18] Theorem 1.10)**.** *As above, assume $\mathfrak{X}$ is proper smooth.*

*(1) Crystalline comparison: Let $\mathfrak{X}_k$ denote the (reduced) special fiber of $\mathfrak{X}$. Then*

$$A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k) \simeq W\Omega^\bullet_{\mathfrak{X}_k/W(k)}.$$

(2) *de Rham comparison:*

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\inf}} \mathscr{O}_C \simeq \Omega^{\bullet,\mathrm{cont}}_{\mathfrak{X}/\mathscr{O}_C},$$

*where* $\Omega^{i,\mathrm{cont}}_{\mathfrak{X}/\mathscr{O}_C} = \varprojlim_n \Omega^i_{(\mathfrak{X}/p^n)/(\mathscr{O}_C/p^n)}$.

(3) *Etale comparison:*

$$A\Omega_{\mathfrak{X}} \otimes_{A_{\inf}} A_{\inf}[1/\mu] \simeq (R\nu_* \mathbb{A}_{\inf,X}) \otimes_{A_{\inf}} A_{\inf}[1/\mu].$$

**Remark 1.2.** To see why the étale comparison above compares with the étale cohomology (instead of some random proétale cohomology), we need one more comparison result (see [BMS18, Theorem 5.6]): Assume $Y$ is a smooth proper adic space over $C$. Then the inclusion $A_{\inf} \to \mathbb{A}_{\inf}$ induces an almost isomorphism

$$R\Gamma_{\text{ét}}(Y, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\inf} \to R\Gamma(Y, \mathbb{A}_{\inf})$$

The cohomology of the cone is killed by $W(\mathfrak{m}^\flat)$.

However, it seems very hard to extend the results to non-smooth or non-proper cases. The semistable proper case, which is the mildest non-smooth case, is treated in [ČK19], but the authors have to do many complicated local computations to get the results. The other drawback of the $A_{\inf}$-cohomology is that there is no hope to have analogous results over bases like $\mathbb{Z}_p$ if one works within the framework of the $A_{\inf}$-cohomology. This is because the base need to have all $p$-power roots of unity to have the distinguished element $\mu$.

The rescue of all these problems is the prismatic cohomology in [BS22]. For a $p$-adic formal scheme $X$ over a prism $A$ (to be defined later), the (relative) prismatic cohomology $\Delta_{X/A}$, roughly speaking, is the cohomology of the structure sheaf $\mathcal{O}_{\Delta}$ on the relative prismatic site $(X/A)_{\Delta}$. Modulo some technical conditions, we have comparisons between the prismatic cohomology $\Delta_{X/A}$ to all well-known $p$-adic cohomology theories, and the $p$-adic formal scheme is not required to be smooth or proper. The prismatic cohomology also allows a natural coefficient systems, namely, vector bundles (or perfect complexes) over the structure sheaf $\mathcal{O}_{\Delta}$. This flexibility in coefficients gives rise to the classification of crystalline representations and local systems in [BS23] and [GR24].

Let us summarize the main comparison results from [BS22] now.

**Theorem 1.3.** *Let $(A, I)$ be a bounded prism, and let $X$ be a smooth $p$-adic formal scheme over* Spf $A/I$*, where $A/I$ has the $p$-adic topology.*

(1) *Crystalline comparison: If $I = (p)$, then there is a canonical $\phi$-equivariant isomorphism*

$$R\Gamma_{\mathrm{crys}}(X/A) \simeq R\Gamma_{\Delta}(X/A)\widehat{\otimes}^L_{A,\phi_A} A = \phi_A^* R\Gamma_{\Delta}(X/A)$$

*of commutative algebras in $D(A)$.*

(2) *Hodge-Tate comparson: If $X = \operatorname{Spf} R$ is affine, there is a canonical $R$-module isomorphism*

$$\Omega^i_{R/(A/I)}\{-i\} \simeq H^i(R\Gamma_{\Delta}(X/A) \otimes^L_A A/I).$$

*Here for any $A/I$-module $M$, $M\{i\} := M \otimes_{A/I} (I/I^2)^{\otimes i}$ is the $i$-th Breuil-Kisin twist.*

(3) *de Rham comparison: There is a canonical isomorphism*

$$R\Gamma_{\mathrm{dR}}(X/(A/I)) \simeq R\Gamma_{\Delta}(X/A)\widehat{\otimes}^L_{A,\phi_A} A/I = \phi_A^* R\Gamma_{\Delta}(X/A) \otimes^L_A A/I$$

*of commutative differential graded algebras in* $D(A)$.

(4) *Etale comparison: Assume* $A$ *is perfect. Let* $X_\eta$ *be the adic generic fiber of* $X$ *over* $\mathbb{Q}_p$ *treated as a diamond. For any* $n \geq 0$, *there is a canonical isomorphism*

$$R\Gamma_{\acute{e}t}(X_\eta, \mathbb{Z}/p^n\mathbb{Z}) \simeq (R\Gamma_{\Delta}(X/A) \otimes^L_A (A/p^n)[1/I])^{\phi=1}$$

*of commutative algebras in* $D(\mathbb{Z}/p^n)$

(5) $A_{\mathrm{inf}}$ *comparison: Let* $C, \mathscr{O}_C$ *be as above, and let* $A := A_{\mathrm{inf}} = A_{\mathrm{inf}}(\mathscr{O}_C)$ *with* $I := \ker(\theta : A_{\mathrm{inf}} \to \mathscr{O}_C)$. *Then there exists a canonical* $\phi$*-equivariant isomorphism*

$$R\Gamma_{A_{\mathrm{inf}}}(X) \simeq \phi_A^* R\Gamma_{\Delta}(X/A_{\mathrm{inf}}).$$

The generality of the results for the prismatic cohomology does come with some costs. One of the issues is one always needs to use homotopical algebra, such as derived completions, descent, and $\infty$-categories. But fortunately, as we will see, most of the issues can be addressed very elegantly. So there is no need to be scared by some unfamiliar words in this theory .

The plan of these notes is as follows: we will introduce the basics of the prismatic cohomology based on [BS22], [BL22a] and many other expository resources. In particular, we will go over the main comparison theorems in even non-smooth case. After that, we will discuss the proof of a very general form of Fontaine's $C_{\mathrm{crys}}$-conjecture in [GR24]. We hope this can give people some sense on what the prismatic cohomology can do.

## 2. $\delta$-rings

The prismatic cohomology is formulated based on $\delta$-rings, which are roughly rings equipped with lifts of the Frobenius. In fact, the central objects, prisms, are $\delta$-rings with additional structures and properties.

**Definition 2.1.** A $\delta$-ring is a pair $(A, \delta)$ consisting of a ring $A$ and a map of sets $\delta : A \to A$ satisfying: for any $x, y \in A$

$$\delta(0) = \delta(1) = 0$$
$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$
$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i} = \delta(x) + \delta(y) - \frac{x^p + y^p - (x+y)^p}{p}.$$

We will call such a map $\delta$ a $p$-derivation.

We also define map $\phi : A \to A$ by

$$\phi(x) = x^p + p\delta(x).$$

The requirements on the map $\delta$ are meant to make $\phi$ a lift of Frobenius.. This is made precise by the following lemma.

**Lemma 2.2.**

(1) *The map* $\phi : A \to A$ *is a ring homomorphism that induces the Frobenius map* $x \mapsto x^p$ *on* $A/pA$.

*(2) If $A$ is $p$-torsion-free, then this construction gives a bijection between $p$-derivations $\delta$ and Frobenius lifts on $A$.*

*Proof.* Proof of (1) is straightforward. For (2), as $A$ is $p$-torsion-free, the formula $\phi(f) = f^p + p\delta(f)$ uniquely defines $\delta(f)$ given $\phi$. So it suffices to show that $\delta$ so defined satisfies the condition in the definition of $\delta$-rings, which is easy. □

**Example 2.3.**

(1) Maybe the most important examples of $\delta$-rings are the rings of Witt vectors. So suppose $R$ is a perfect $\mathbb{F}_p$-algebra. Then the ring of Witt vectors $W(R)$ admits a unique lift of Frobenius, which makes $W(R)$ a $\delta$-ring. This also shows $W(R)$ has a unique $\delta$-structure.

(2) The ring $\mathbb{Z}$ with $\delta(n) = \frac{n-n^p}{p}$ is a $\delta$-ring. This makes $\phi$ the identity map. In fact $(\mathbb{Z}, \delta)$ is the initial object in the category of $\delta$-rings.

(3) If $A$ is a $\mathbb{Z}[1/p]$-algebra, then any endomorphism $\phi$ of $A$ is a $\delta$-structure since the condition that $\phi$ lifts the Frobenius on $A/p$ is vacuously true.

(4) The free $\delta$-ring $\mathbb{Z}\{x\}$ on a variable $x$ is the polynomial ring $\mathbb{Z}[x_0, x_1, x_2, \ldots]$ with $x = x_0$ and $\delta(x_i) = x_{i+1}$. In general, for any set $S$, we can form the free $\delta$-ring $\mathbb{Z}\{S\}$ on $S$.

**Definition 2.4.** A morphism of $\delta$-rings $f : (A, \delta) \to (A', \delta')$ is a ring homomorphism $f : A \to A'$ such that $f \circ \delta = \delta' \circ f$.

The category of $\delta$-rings is denoted as $\mathbf{Ring}_\delta$.

Next we demonstrate a connection between $\delta$-structures and the length-2 Witt vectors $W_2$. For any ring $A$, $W_2(A)$ can be defined explicitly as follows: $W_2(A) = A \times A$ as sets, and addition and multiplication are defined by

$$(x, y) + (z, w) := (x + z, y + w + \frac{x^p + z^p - (x + z)^p}{p})$$

and

$$(x, y) \cdot (z, w) = (xz, x^p w + z^p y + pyw).$$

We have a natural projection homomorphism $\epsilon : W_2(A) \to A, (x, y) \mapsto x$. It is immediate from the definition that a $\delta$-structure on $A$ is the same as a ring map $w : A \to W_2(A)$ such that $\epsilon \circ w = \mathrm{id}_A$. More precisely, the correspondence is given by the relation $w(x) = (x, \delta(x))$.

**Lemma 2.5.** *The category* $\mathrm{Ring}_\delta$ *admits arbitary limits and colimits. The formations of limits and colimits commute with the forgetful functor to* **Ring***.*

*Proof.* The limit of a diagram $\{A_i\}$ of $\delta$-rings is given by the limit $\lim_i A_i$ of the underlying rings; one checkes easily this gives a natural $\delta$-structure on $\lim_i A_i$. For colimits, we use the $W_2$-contruction of $\delta$-structures discussed above. For a diagram $\{A_i\}$ of $\delta$-ring, let $w_i : A_i \to W_2(A_i)$ be the ring maps correponding to the $\delta$-structures. Taking colimits gives $\mathrm{colim}\, A_i \to \mathrm{colim}\, W_2(A_i)$. The functoriality of $W_2$ gives a natural map $\mathrm{colim}_i W_2(A_i) \to W_2(\mathrm{colim}_i A_i)$. Composing the two maps gives a map $w : \mathrm{colim}_i A_i \to W_2(\mathrm{colim}_i A_i)$ whose composition with the projection to the first factor $W_2(\mathrm{colim}_i A_i) \to \mathrm{colim}_i A_i$ is the identity. This gives a $\delta$-structure on $\mathrm{colim}_i A_i$, which completes the construction of the colimit of $\{A_i\}$ as $\delta$-rings. □

By general category theory, the forgetful functor $\mathbf{Ring}_\delta \to \mathbf{Ring}$ admits both a left adjoint and a right adjoint. The right adjoint is the Witt vectors. For any set $S$, applying the left adjoint functor to the polynomial ring $\mathbb{Z}[S]$ gives the free $\delta$-ring $\mathbb{Z}\{S\}$.

## 3. Derived completions

Before we introduce the central concept, prisms, we need to address some concepts in derived algebra that are necessary to the prismatic cohomology.

**Definition 3.1.** Let $A$ be a ring, and $I$ a finitely generated ideal of $A$. An $A$-module $M$ is classically $I$-complete if the natural map $M \to \lim_n M/I^nM$ is an isomorphism.

Note this in particular means $M$ is $I$-adically separated, i.e., $\bigcap_n I^nM = 0$.
We mention a few issues with the notion of (classical) completion.

(1) Classically complete $A$-modules do not form an abelian category.
(2) completion is neither right nor left exact in general. In fact, completion can be viewed as the composition of tensor product, which is right exact, and inverse limit, which is left exact.
(3) The completion of a flat module may not be flat.
(4) If $I$ is not required to be finitely generated, then the $I$-adic completion $\lim_n M/I^nM$ of a module $M$ may not be complete.

The point is that classical completion is not robust enough. We need a weaker but more flexible notion of completion.

**Definition 3.2.** Let $A$ be a ring, and $I$ a finitely generated ideal of $A$. An $A$-module $M$ is derived $I$-complete if for $f \in I$,

$$\mathrm{Hom}_A(A_f, M) = 0 \quad \text{and} \quad \mathrm{Ext}^1_A(A_f, M) = 0.$$

**Remark 3.3.** Note $A_f$ admits a free resolution as an $A$-module:

$$0 \to A[T] \xrightarrow{\times(1-Tf)} A[T] \xrightarrow{T\mapsto f^{-1}} A_f \to 0.$$

So for any $A$-module $M$, $\mathrm{Ext}^n_A(A_f, M) = 0, \forall n \geq 2$. The condition in the definition above can be then reformulated as

$$\mathrm{Ext}^n_A(A_f, M) = 0, \forall n \geq 0.$$

**Lemma 3.4.**

*(1) If $M$ is classically $I$-complete, then it is derived $I$-complete.*
*(2) If $M$ is derived $I$-complete, then the natural map $M \to \lim_n M/I^nM$ is surjective.*

*Proof of (1).* Suppose $M$ is classically $I$-complete. We have

$$\mathrm{Hom}_A(A_f, M) = \mathrm{Hom}_A(A_f, \lim_n M/I^nM) = 0.$$

To show that $\mathrm{Ext}^1_A(A_f, M) = 0$, consider an extension

$$0 \to M \to E \to A_f \to 0.$$

For each $n \geq 0$, pick $e_n \in E$ mapping to $f^{-n} \in A_f$, and set $\delta_n = fe_{n+1} - e_n \in M$. Since $M$ is complete, we may define the elements

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \cdots$$

which satisfy $fe'_{n+1} = e'_n$. We thus obtain a map $A_f \to E$ splitting the sequence by mapping $f^{-n}$ to $e'_n$. □

**Corollary 3.5.** *Assumptions as above. $M$ is classically $I$-complete if and only if $M$ is $I$-adically separated and derived $I$-complete.*

*Proof.* Directly by Lemma 3.4. □

Next, we introduce one of the most important results for derived complete modules, the derived Nakayama.

**Lemma 3.6** (Derived Nakayama)**.** *Let $M$ be a derived $I$-complete $A$-module. Then $M = 0$ if and only if $M/IM = 0$.*

*Proof.* [Sta25, Tag 0G1U] □

The inclusion of the category of derived $I$-complete modules to the category of $A$-modules admits a left adjoint $M \mapsto \widehat{M}$, called derived $I$-completion. The full subcategory of $A$-modules consisting of derived $I$-complete modules form an abelian category.

We conclude this section by showing an explicit construction of derived $I$-completion in a simple case.

**Example 3.7.** Let $f \in A$ be a non-zerodivisor. The derived $f$-adic completion of $M$ is given by

$$\widehat{M} = \mathrm{Ext}^1_A(A_f/A, M)[1].$$

So if $A = \mathbb{Z}, I = (p)$, and $M = \mathbb{Q}/\mathbb{Z}$, then $\widehat{M} := \mathbb{Z}_p[1]$

## 4. Prisms

Fix a prime $p$.

**Definition 4.1.** A prism is a pair $(A, I)$ of a $\delta$-ring $A$ and an invertible ideal $I$ of $A$ such that

(1) $A$ is derived $(p, I)$-complete.
(2) $p \in (I, \phi(I))$ (equivalent to that after some localization of $A$, we have $I = (d)$ with $\delta(d)$ invertible in $A$).

We now define morphisms between prisms. But we first recall the following definition.

**Definition 4.2.** For a ring $A$ with a finitely generated ideal $I$, an $A$-module $M$ is $I$-completely flat (resp. $I$-completely faithfully flat) if $M/IM$ is a flat (resp. faithfully flat) $A/I$-module and $\mathrm{Tor}^A_i(A/I, M) = 0$ for $i > 0$. The last condition is equivalent to that for any $A/I$-module $N$, $\mathrm{Tor}^A_i(N, M) = 0$ for $i > 0$. If $M$ is (faithfully) flat, then it is $I$-completely (faithfully) flat.

**Definition 4.3.** A map $f : (A, I) \to (B, J)$ of prisms is a map of $\delta$-rings $f : A \to B$ such that $f(I) \subseteq J$. A map $(A, I) \to (B, J)$ of prisms is called (faithfully) flat if the underlying ring map $A \to B$ is $(p, I)$-completely (faithfully) flat.

Morphisms of prisms have the following rigidity property.

**Lemma 4.4.** *If $f : (A, I) \to (B, J)$ is a map of prisms, then $f$ induces an isomorphism $I \otimes_A B \simeq J$. In particular, $IB = J$.*

*Conversely, if $(A, I)$ is a prism and $A \to B$ is a map of $\delta$-rings with $B$ being derived $(p, I)$-complete, then $(B, IB)$ is a prism exactly when $B[I] = 0$.*

**Example 4.5.**

(1) (Crystalline prism) Let $A$ be a (classically) $p$-complete $p$-torsion-free $\delta$-ring with $I = (p)$. Then $(A, (p))$ forms a prism, usually called a *crystalline prism*. Note in this case, $\delta(p) = \frac{p-p^p}{p} = 1 - p^{p-1}$ is invertible in $A$.

(2) (Breuil-Kisin prism) Suppose $k$ is a perfect field of characteristic $p$. Let $A = W(k)[[u]]$ with $\delta(u) = 0$ (equivalent to $\phi(u) = u^p$) and $I = (E(u))$, where $E(u)$ is an Eisenstein polynomial. Note $A/I \simeq \mathscr{O}_K$ via map $u \mapsto \pi$, where $\mathscr{O}_K$ is the ring of integers of a $p$-adic field $K$ totally ramified over $W(k)[1/p]$, and $\pi$ is a uniformizer in $K$. A prism of the form $(W(k)[[u]], (E(u)))$ is called a *Breuil-Kisin prism*.

(3) ($q$-crystalline prism) Let $A = \mathbb{Z}_p[[q-1]]$, which is the $(p, [p]_q)$-adic completion of $\mathbb{Z}[q]$. The $\delta$-structure on $A$ is given by $\delta(x) = \frac{x-x^p}{p}$ Let $I = ([p]_q)$, where

$$[p]_q = \frac{q^p - 1}{q - 1} = 1 + q + \cdots + q^{p-1}$$

is the $q$-analog of $p$. Then $(A, I)$ is a prism.

(4) ($\delta$-ring with no prism structure) A $\delta$-ring can have many prism structures, i.e., many choices of $I$. But there are also $\delta$-rings with no prism structure. Consider the local ring $W(\mathbb{F}_p[x]/(x^2))$ complete with respect to the $p$-adic topology. Its $\delta$-structure comes from the $W_2$-construction discussed above. The ring $W(\mathbb{F}_p[x]/(x^2))$ does not have any prism structure: any non-zerodivisor $d$ of $W(\mathbb{F}_p[x]/(x^2))$ is a unit.

The names of the prisms in the example come from their appearance in the classical theory or comparison theorems with the prismatic cohomology. For example, crystalline prisms are those over which we can formulate the crystalline comparison, and the Breuil-Kisin prisms already appear in the classical theory of Breuil-Kisin modules.

**Definition 4.6.** A prism $(A, I)$ is

(1) bounded if $A/I$ has bounded $p^\infty$-torsions. This implies $A$ is classically $(p, I)$-complete.
(2) orientable if $I$ is principal. A choice of the generator $d$ of $I$ is called an orientation.
(3) perfect if $\phi : A \to A$ is an isomorphism.
(4) transversal if $A/I$ is $p$-torsion free.

The notions "bounded", "orientable", and "transversal" are mainly for technical reasons. We will always try to reduce the problems to these three cases so we have better algebra results to use. As shown below, perfect prisms are equivalent to perfectoid ring. We will also often try to reduce to this case so that we can use perfectoid geometry, such as the tilting equivalence. Note also the definition of perfect prisms is independent of the ideal $I$. Here we take the defintion of perfectoid rings as in [BMS18, §3]

**Definition 4.7.** A ring $S$ is perfectoid if and only if it is $\pi$-adically complete for some element $\pi \in S$ such that $\pi^p$ divides $p$, the Frobenius map $\phi : S/pS \to S/pS$ is surjective, and the kernel of $\theta : \mathbb{A}_{\mathrm{inf}}(S) \to S$ is principal.

**Example 4.8.** The following rings are examples of perfectoid algebras. First, any perfect $\mathbb{F}_p$-algebra is perfectoid (where we take $\pi = 0$); here, perfect means that the Frobenius map is an isomorphism. Moreover, the $p$-adic completion $\mathbb{Z}_p^{\mathrm{cycl}}$ of $\mathbb{Z}_p[\zeta_{p^\infty}]$ is perfectoid; one may also take the $p$-adic completion of the ring of integers of any other algebraic extension of $\mathbb{Q}_p$ containing the cyclotomic extension. Another example is $\mathbb{Z}_p^{\mathrm{cycl}}\langle T^{1/p^\infty}\rangle$, and there are many obvious variants.

**Remark 4.9.** Note this definition is slightly more general than the one in [SW20], which has a stronger restriction on the topology on $S$. For example, the discrete ring $\mathbb{F}_p$ is not considered as a perfectoid algebra in [SW20].

**Lemma 4.10.** *There is an equivalence of categories:*

- *The category of perfectoid rings $R$.*
- *The category of perfect prisms $(A, I)$.*

*The functors are $R \mapsto (A_{\mathrm{inf}}(R), \ker\theta)$ and $(A, I) \mapsto A/I$, respectively.*

*Proof.* Let $R$ be a perfectoid ring. Since $A_{\mathrm{inf}}(R)$ is just the Witt vector construction on the tilt $R^\flat$, which is perfect and in characteristic $p$, $A_{\mathrm{inf}}(R)$ is a perfect $\delta$-ring, i.e., with a natural $\delta$-structure and $\phi$ being an isomorphism. By the general results in $p$-adic Hodge theory, $A_{\mathrm{inf}}(R)$ is also classically $(p, \ker\theta)$-complete, and hence also derived $(p, \ker\theta)$-complete. Thus, we get a prism.

Conversely, we claim the following properties of perfect prisms $(A, I)$:

(1) $I$ is principal and generated by a distinguished element $d$, i.e., $\delta(d)$ is a unit in $A$.

(2) Any perfect prism $(A, I)$ is bounded, and thus classically $(p, I)$-complete.

For a proof, see [BS22, Lemma 3.8] Since $A$ is a perfect $\delta$-ring, the Frobenius on $R/p$ with $R \coloneqq A/d = A/I$ is surjective. Since $A$ is perfect, one can show $A \simeq W(S)$ for some perfect $\mathbb{F}_p$-algebra $S$. Via this isomorphism, we can write $d = [a_0] + pu$ for some unit $u \in A$. Let $\pi \in R$ be the image of $[a_0^{1/p}]$. Then $\pi^p | p$ in $R$. This shows $R$ is perfectoid if $R$ is also classically $p$-adically complete. □

## 5. The prismatic site and cohomology

Fix a bounded prism $(A, I)$ as "the base". All formal schemes over $A$ are assumed to have the $(p, I)$-adic topology, and thus formal schemes over $A/I$ have the $p$-adic topology. Fix a smooth $p$-adic formal $A/I$-scheme $X$; in future talks, we will remove the smoothness condition in many cases by using nonabelian derived functors, i.e., left Kan extensions.

**Definition 5.1.** Let $R$ be a $\delta$-ring. An element $d \in R$ is called distinguished if $\delta(d)$ is a unit

An important property of prisms is the rigidity of the ideal $I$

**Lemma 5.2** ([BS22] Lemma 3.5)**.** *Any map $(A, I) \to (B, J)$ of prisms induces $IB = J$. Conversely, if $A \to B$ is a map of $\delta$-rings with $B$ being derived $(p, I)$-complete, then $(B, IB)$ is a prism exactly when $B[I] = 0$.*

*Sketch of Proof.* It suffices to assume $(A, I)$ and $(B, J)$ are orientable, i.e., $I = (d)$ and $J = (e)$ are principal ideals generated by non-zerodivisors. In the ring $B$, we then have $d = ef$ for some element $f \in B$. □

**Definition 5.3.** Let $(X/A)_{\Delta}$ be the category of maps $(A, I) \to (B, IB)$ of bounded prisms together with a map $\mathrm{Spf}(B/IB) \to X$ over $A/I$. Such an object is usually denoted as

$$(\mathrm{Spf}\, B \leftarrow \mathrm{Spf}(B/IB) \to X) \in (X/A)_{\Delta}.$$

We endow $(X/A)_{\Delta}$ with the flat topology. That is, a map

$$\begin{pmatrix} \mathrm{Spf}\, C/IC & \longrightarrow & \mathrm{Spf}\, C \\ \downarrow & & \\ X & & \end{pmatrix} \to \begin{pmatrix} \mathrm{Spf}\, B/IB & \longrightarrow & \mathrm{Spf}\, B \\ \downarrow & & \\ X & & \end{pmatrix}$$

is a flat cover if $(B.IB) \to (C, IC)$ is a faithfully flat map of prisms, i.e., $C$ is $(p, IB)$-completely flat over $B$. The category $(X/A)_{\Delta}$ with the flat topology so defined is called the prismatic site of $X/A$.

**Remark 5.4.**

(1) In literature, it is also common to consider only affine $X = \mathrm{Spf}\, R$. In this case one can define the prismatic site without using formal spectrum. But then one needs to take the opposite category to make all arrows go the correct directions.

(2) It is sometimes also useful to endow $(X/A)_{\Delta}$ with different topologies: étale, Nisnevich, or Zariski.

**Definition 5.5.** Define the following presheaves on $(X/A)_{\Delta}$:

(1) $\mathcal{O}_{\Delta} : (B, I) \mapsto B$; this will be called the structure sheaf.

(2) $\overline{\mathcal{O}}_{\Delta} : (B, I) \mapsto B/I$; this will be called the reduced structure sheaf.

(3) $\mathcal{I}_{\Delta} : (B, I) \mapsto I$.

Similar to the usual faithfully flat descent, we have the following.

**Proposition 5.6.** *These presheaves are sheaves on the prismatic site* $(X/A)_{\Delta}$.

*Proof.* [BS22, Corollary 3.12]. □

**Definition 5.7.** The prismatic cohomology of $X$ is $R\Gamma_{\Delta}(X/A) \coloneqq R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta})$.

To study prismatic cohomology, we need to introduce a functor that relates the prismatic topos and the étale topos of $X$. The prismatic cohomology will then be computed on the étale site of $X$.

There is a natural functor

$$\nu = \nu_X : \mathrm{Shv}((X/A)_{\Delta}) \to \mathrm{Shv}(X_{\text{ét}})$$

For any $U \to X$ in $X_{\text{ét}}$, the prismatic site $(U/A)_{\Delta}$ is a slice of $(X/A)_{\Delta}$. Pushforward along this map can be explicitly give as

$$(\nu_* \mathcal{F})(U \to X) = H^0((U/A)_{\Delta}, \mathcal{F}|_{(U/A)_{\Delta}})$$

**Definition 5.8.** The prismatic complex of $X$ is defined to be

$$\Delta_{X/A} \coloneqq R\nu_* \mathcal{O}_{\Delta} \in D(X_{\text{ét}}, A).$$

The Hodge-Tate complex of $X$ is defined to be

$$\overline{\Delta}_{X/A} \coloneqq R\nu_* \overline{\mathcal{O}}_{\Delta} \in D(X_{\text{ét}}, A).$$

Note that

$$\overline{\Delta}_{X/A} \simeq \Delta_{X/A} \otimes^L_A A/I.$$

It is often useful to reduce to the affine case, i.e., $X = \operatorname{Spf} R$. In this case,

$$R\Gamma_{\Delta}(\operatorname{Spf} R/A) = R\Gamma((\operatorname{Spf} R/A)_{\Delta}. \mathcal{O}_{\Delta}) = R\Gamma(\operatorname{Spf} R, \Delta_{\operatorname{Spf} R/A}).$$

We also write $\Delta_{R/A} \coloneqq R\Gamma_{\Delta}(\operatorname{Spf} R/A)$. Let us take a look at the simplest case of prismatic cohomology.

**Example 5.9.** Let $(R/A)_{\Delta}$ be the opposite category of $(\operatorname{Spf} R/A)_{\Delta}$. If $R = A/I$, $(R/A)_{\Delta}$ can be identified with the category of prisms over $(A, I)$ and $(R/A)_{\Delta}$ has an initial object $(R \simeq A/I \leftarrow A)$ corresponding to $(A, I)$ in the category of all prisms over $(A, I)$. For this reason, we often refer to $(R \simeq A/I \leftarrow A)$ the initial object even if in the actual prismatic site $(\operatorname{Spf} R/A)_{\Delta}$, it is the final object.

In this case, we have $\Delta_{R/A} \simeq A$ and $\overline{\Delta}_{R/A} \simeq A/I$.

We end this section with a few words on the absolute prismatic site. For any smooth $p$-adic formal scheme $X$, the absolute prismatic site $(X)_{\Delta}$ has objects being bounded prisms $(B, J)$ equipped with a map $\operatorname{Spf}(B/J) \to X$ of $p$-adic formal schemes. If $X = \mathbb{Z}_p$, then the absolute prismatic site $(\mathbb{Z}_p)_{\Delta}$ is just another name for the category of all prisms (with flat topology). Using the notation of absolute prismatic site $(R)_{\Delta}$, for a $p$-adic formal scheme $X$ over $R$, one can consider for example the functor

$$(A, I) \mapsto R\Gamma((X_A/A)_{\Delta}, \mathcal{O}_{\Delta})$$

where $(A, I) \in (R)_{\Delta}$. This defines a sheaf on the site $(R)_{\Delta}$. One usefulness of absolute prismatic sites is illustrated by the following lemma.

**Lemma 5.10.** *Let $(A, I)$ be a perfect prism corresponding to a perfectoid ring $R = A/I$. Then for any prism $(B, J)$, any map $A/I \to B/J$ of commutative rings lifts uniquely to a map $(A, I) \to (B, J)$ of prisms.*

This means $A \in (R)_{\Delta}$ is the initial object. So $(R)_{\Delta} = (R/A)_{\Delta}$.

## 6. Hodge-Tate and Crystalline comparisons

In this section, we discuss the proof of the Hodge-Tate comparison.

**Theorem 6.1** (Hodge-Tate Comparison)**.** *If $X = \operatorname{Spf} R$ is affine, there is a canonical $R$-module isomorphism*

$$\Omega^i_{R/(A/I)}\{-i\} \simeq H^i(R\Gamma_{\Delta}(X/A) \otimes^L_A A/I).$$

*Here for any $A/I$-module $M$, $M\{i\} := M \otimes_{A/I} (I/I^2)^{\otimes i}$ is the $i$-th Breuil-Kisin twist.*

In fact, following the stacky approach in [BL22b], it won't be more difficult give a natural proof of a more general comparison theorem.

**Definition 6.2.** A Hodge-Tate crystal is a $\overline{\mathcal{O}}_{\Delta}$-linear prismatic crystal over $(X/A)_{\Delta}$, i.e., a projective module over $\overline{\mathcal{O}}_{\Delta}$.

**Theorem 6.3** ([GR24] Proposition 5.11)**.** *Let $(A, I)$ be a bounded prism, and let $X = \operatorname{Spf} R$ be a smooth affine formal scheme over $A/I$. Let $\mathcal{E}$ be a Hodge-Tate crystal in perfect complexes over $(X/A)_{\Delta}$. Then the cohomology $R\Gamma((X/A)_{\Delta}, \mathcal{E})$ admits a finite filtration whose $i$-th graded piece is*

$$\mathcal{E}(R) \otimes_{\mathcal{O}_X} \Omega^i_{X/\overline{A}}\{-i\}[-i]$$

*Here, $\mathcal{E}(R)$ is a perfect $R$-complex and both $\mathcal{E}(R)$ and the filtration are functorial with respect to $\mathcal{E}$. In particular, $R\Gamma((X/A)_{\Delta}, \mathcal{E})$ is a perfect $R$-complex.*

The special case $\mathcal{E} = \overline{\mathcal{O}}_{X/A}$ is the Hodge-Tate comparison theorem in [BS22].

**Remark 6.4.** We will follow the proof in [GR24] building on the ideas in [BL22b]. The original proof in [BS22] for the Hodge-Tate comparison is based on reduction to crystalline cohomology and the Cartier isomorphism. The use of these techniques is in some sense only artificial. The method in [GR24] based on the Hodge-Tate stack gives a more natural explanation as we will see soon.

Unfortunately, we won't have time and space for the full theory of prismatization here. We will only briefly mention the constructions and main properties; we will come back to this part in future talks.

6.1. **A first glimpse at prismatization.** We are looking for a ring stack, the prismatic stack, whose coherent cohomology computes the prismatic cohomology. This has many advantages compared to the site-theoretic approach $(X/A)_{\Delta}$ of prismatic cohomology. For example, this often gives a natural coefficient system for cohomology and can also often reduce the proof of comparison theorems to the case of $\mathbf{G}_a$ via transmutation.

The starting point of the story is to find the hypothetical initial prism. Of course, such an initial object does not exist. Nevertheless, denote by $\mathcal{Z}$ for this hypothetical object. Recall we have the initial oriented prism $(A_0, I_0)$, where

- $A_0 = \mathbb{Z}\{x, \delta(x)^{-1}\}^{\widehat{}}_{(p,x)}$
- $I_0 = (x)$

There should be a unique map $\operatorname{Spf} A_0 \to \mathcal{Z}$. For any orientable prism $(A, I)$, a map $\operatorname{Spf} A \to \operatorname{Spf} A_0 \times_{\mathcal{Z}} \operatorname{Spf} A_0$ is determined by two generators $d$ and $d'$ of $I$. We have $d = ud'$ for some unit $u \in A^\times$. Thus, we get the identification

$$\operatorname{Spf} A_0 \times_{\mathcal{Z}} \operatorname{Spf} A_0 \simeq \operatorname{Spf} A_0 \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spf}(\mathbb{Z}_p\{u^\pm\}^{\widehat{}}_p)$$

**Lemma 6.5.**

*(1) As functors on rings,* $\operatorname{Spec} \mathbb{Z}\{u\} \simeq W$ *and* $\operatorname{Spec} \mathbb{Z}\{u^\pm\} \simeq W^\times$

*(2) As functors on* $\mathrm{Nilp} := \{R$ *p-nilpotent*$\}$,

$$\operatorname{Spf}(A_0)(R) = \{\sum_{i=0}^{\infty} [a_i] p^i\}$$

*where* $a_0 \in R$ *is nilpotent and* $a_1 \in R^\times$.

*Proof.* For 1), $R$ any ring.

$$\operatorname{Hom}_{\mathrm{Ring}}(\mathbb{Z}\{u\}, R) \simeq \operatorname{Hom}_{\mathrm{Ring}_\delta}(\mathbb{Z}\{u\}, W(R)) \simeq \operatorname{Hom}_{\mathrm{Ring}}(\mathbb{Z}[u], W(R)) = W(R).$$

For 2) □

It makes sense then to define the following approximation of the initial prism $\mathcal{Z}$.

**Definition 6.6.** We define the Cartier-Witt stack to be $\mathrm{WCart} := [\operatorname{Spf} A_0 / W^\times]$

The stack has the following moduli interpretation.

**Lemma 6.7.** *For an adic ring $R$ over $\mathbb{Z}_p$,* $\mathrm{WCart}(R)$ *is equivalent to the groupoid of maps* $I \xrightarrow{\alpha} W(R)$ *where*

- *$I$ is an invertible module of $W(R)$*
- *locally on* $\operatorname{Spec} R$, *$I$ is generated by $\alpha = [a_0] + [a_1]p + \cdots$, where $a_0$ is topologically nilpotent and $a_1 \in R^\times$.*

A map $I \xrightarrow{\alpha} W(R)$ is called a Cartier-Witt divisor. Note $\alpha$ is not required to be injective. So this may not be an ideal in general, only a quasi-ideal.

**Theorem 6.8.** $R\Gamma(\mathrm{WCart}, \mathcal{O}) \simeq R\Gamma((\mathbb{Z}_p)_{\Delta}, \mathcal{O}_{\Delta})$.

For a Cartier-Witt divisor $I \xrightarrow{\alpha} W(R)$, we write $\overline{W(R)} = \mathrm{Cone}(\alpha)$. Since $\alpha$ may not even be injective, this cone is understood as a 1-truncated animated ring and $\operatorname{Spec} \overline{W(R)}$ as a derived scheme. But we should not need to worry about the derived geometry here.

**Definition 6.9.** In general, for any bounded $p$-adic formal scheme $X$, we define the presheaf $X^{\Delta}$ over $\operatorname{Spf}(\mathbb{Z}_p)$ as follows: for a $p$-nilpotent ring $R$, the groupoid $X^{\Delta}(R)$ consists of Cartier-Witt divisors $I \xrightarrow{\alpha} W(R)$ together with a map $\operatorname{Spec}(\overline{W(R)}) \to X$ of derived $\operatorname{Spf} \mathbb{Z}_p$-schemes.

Unwinding the definition, we see $\mathbb{Z}_p^{\Delta}$ is just WCart.

We claim there is a natural map

$$u : \mathbb{Z}_p^{\Delta} \to \widehat{\mathbf{A}^1}/\mathbf{G}_m$$

over $\operatorname{Spf}\mathbb{Z}_p$, where $\widehat{\mathbf{A}^1}$ is the formal completion of $\mathbf{A}^1$ at 0. Write $\mathbb{Z}_p^{\mathrm{HT}} \subset \mathbb{Z}_p^{\Delta}$ for the preimage of $B\,\mathbf{G}_m \subset \widehat{\mathbf{A}^1}/\mathbf{G}_m$.

**Definition 6.10.** For a bounded $p$-adic formal scheme $X$, the Hodge-Tate stack $X^{\mathrm{HT}} \subset X^{\Delta}$ is defined as the pullback of $\mathbb{Z}_p^{\mathrm{HT}} \subset \mathbb{Z}_p^{\Delta}$.

In general, we can define a relative version of the Hodge-Tate stack $(X/A)^{\mathrm{HT}}$. If $A$ is perfect, then we have $(X/A)^{\mathrm{HT}} = X^{\mathrm{HT}}$.

**Theorem 6.11.** *There are natural isomorphisms*

$$R\Gamma((X/A)^{\Delta}), \mathcal{O}) \simeq R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta})$$

*and similarly,*

$$R\Gamma((X/A)^{\mathrm{HT}}), \mathcal{O}) \simeq R\Gamma((X/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$$

We now introduce some of the basic properties of the Hodge-Tate stacks.

**Proposition 6.12.** *Fix a bounded prism $(A, I)$ and a smooth $p$-adic formal $\overline{A}$-scheme $X$. Then the Hodge-Tate structure map $\pi^{\mathrm{HT}} : (X/A)^{\mathrm{HT}} \to X$ is naturally a torsor under the group $B\mathbf{V}(T_{X/\overline{A}}\{1\})^\sharp$. Moreover, this torsor splits if $X$ is affine. If $X = \operatorname{Spf} R$, with a choice of the lift $R$ to $\widetilde{R}$ over $A$, we have*

$$(X/A)^{\mathrm{HT}} \simeq X \times_{\operatorname{Spf}\overline{A}} B\mathbf{V}(T_{X/\overline{A}}\{1\})^\sharp.$$

6.2. **Sketch of proof.**

**Definition 6.13.** A derived $I$-complete $A$-algebra $R$ is called $I$-completely étale (resp. $I$-completely smooth, $I$-completely ind-smooth) if $R \otimes_A^L A/I$ is étale (resp. smooth, ind-smooth) over $A/I$, i.e., that $R \otimes_A^L A/I$ is concentrated in degree 0, where it is given by an étale (resp. smooth, ind-smooth) $A/I$-algebra. We note that by Elkik's algebraization results, $R$ is $I$-completely étale (resp. smooth) if and only if it is the derived $I$-completion of an étale (resp. smooth) $A$-algebra.

*Sketch of proof.* As $R$ is $p$-adic smooth over $\overline{A} \coloneqq A/I$, we can find a bounded prism $\widetilde{R}$ over $A$ with $\widetilde{R}/I\widetilde{R} \simeq R$. In particular, by evaluating the crystal $\mathcal{E}$ at $(\widetilde{R}, I\widetilde{R})$, we get a perfect complex over $R$ that is functorial with respect to $\mathcal{E}$. We denote this $R$-linear perfect complex by $\mathcal{E}(R)$.

The lifting $\widetilde{R}$ defines a splitting of the $B\mathbf{V}(T_{X/\overline{A}}\{1\})^\sharp$-torsor $(X/A)^{\mathrm{HT}} \to X$, and thus defines an equivalence of categories

$$D_{\mathrm{perf}}((X/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \simeq D_{\mathrm{perf}}(B\mathbf{V}(T_X\{1\})^\sharp).$$

The latter category $D_{\mathrm{perf}}(B\mathbf{V}(T_X)\{1\}^\sharp)$ is further equivalent to the category of perfect complexes over the dual geometric vector bundle $\mathbf{B}(T_X\{1\}^\sharp)^\vee$ that are set-theoretically supported at the zero section. To see this, just notice a projective module on the classifying stack $B(\mathbf{V}(T_X\{1\})^\sharp)$ is equivalent to a representation of the commutative group scheme $\mathbf{V}(T_X\{1\})^\sharp$. But representations of a commutative group scheme $G$ are equivalent to coherent sheaves on the Cartier dual $G^\vee$.

The action of $S = \mathrm{Sym}^\bullet_R(T_R\{1\})$ on $M = \mathcal{E}(R)$ factors through the quotient $R = S/(T_R\{1\} \cdot S)$. So we have the isomorphisms

$$\begin{aligned} R\Gamma((X/A)_{\mathbb{\Delta}}, \mathcal{E}) &\simeq R\operatorname{Hom}_{B\mathbf{V}(T_X\{1\})^\sharp}(\mathcal{O}_X, \mathcal{E}(R)) \\ &\simeq R\operatorname{Hom}_S(R, \mathcal{E}(R)) \\ &\simeq R\operatorname{Hom}_S(\mathrm{Kos}_S(T_R\{1\} \otimes_R S), \mathcal{E}(R)) \end{aligned}$$

□

## 7. Non-abelian derived functors

**Definition 7.1.** Let $\mathcal{C}_i (i = 1, 2, 3)$ be $\infty$-categories. Given covariant functors $F : \mathcal{C}_1 \to \mathcal{C}_2, G : \mathcal{C}_1 \to \mathcal{C}_3$, the left Kan extension of $G$ along $F$ consists of a covariant functor $L : \mathcal{C}_2 \to \mathcal{C}_3$ and a natural transformation $\alpha : G \to L \circ F$ which are universal for this property; that is, if $M : \mathcal{C}_2 \to \mathcal{C}_3$ is another covariant functor and $\beta : G \to M \circ F$ is a natural transformation, then there is a unique natural transformation $\sigma : L \to M$ making the diagram commute.

The following example shows that left Kan extensions are generalizations of the usual left derived functors.

**Example 7.2.** Let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories, where $\mathcal{A}$ has enough projectives. Let $\mathrm{Perf}(\mathcal{A})$ be the subcategory of $D^-(\mathcal{A})$ consisting of perfect complexes. The functor $G$ can be extended to a functor $G : \mathrm{Perf}(\mathcal{A}) \to D^-(B)$. Let $F : \mathrm{Perf}(\mathcal{A}) \to D^-(A)$ be the inclusion. Then the left Kan extension of $G$ along $F$ is given by the usual left derived functor $LG$ of $G$.

**Definition 7.3.** For a ring $A$, let $\mathrm{Poly}_A$ be the full subcategory of $\mathrm{Ring}_A$ consisting of polynomial algebras over $A$ in finitely many variables. If we write $F : \mathrm{Ring}_A \to \mathrm{Set}$ for the forgetful functor and $G : \mathrm{Set} \to \mathrm{Ring}_A$ for the left adjoint of $F$, then $\mathrm{Poly}_A$ can also be understood as the essential image of the restriction of the left adjoint $G$ to finite sets.

The following proposition shows left Kan extensions exist in a very general case.

**Proposition 7.4.** *For $A \in \mathrm{Ring}$ and $F : \mathrm{Poly}_A \to D(\mathrm{Ab})$ a convariant functor, the functor $F$ admits a left Kan extension $LF : \mathrm{Ring}_A \to D(\mathrm{Ab})$ along the inclusion $\mathrm{Poly}_A \to \mathrm{Ring}_A$. Moreover, the functor $LF$ has the following properties:*

(1) *The restriction of $LF$ to $\mathrm{Poly}_A$ is the original functor $F$.*
(2) *$LF$ commutes with filtered colimits. In particular, if $A[S]$ is a polynomial algebra on a possibly infinite set $S$, we can compute $LF(A[S])$ as the colimit of $F(A[T])$ over all finite subsets $T$ of $S$.*
(3) *Given a simplicial resolution $P_\bullet \to B$ of an object $B \in \mathrm{Ring}_A$, $LF(B)$ is the colimit of $LF(P_\bullet)$.*

Next, we introduce the cotangent complex, which is of the utmost importance for the study of prismatic cohomology.

**Definition 7.5.** Let $A$ be a ring. The cotangent complex functor $L_{\bullet/A} : \mathrm{Ring}_A \to D(A)$ is the left Kan extension of the functor $\mathrm{Poly}_A \to D(A), B \mapsto \Omega^1_{B/A}[0]$. In fact, $L_{\bullet/A}$ can be defined for simplicial $A$-algebras.

**Proposition 7.6.** *Let $B \in \mathrm{Ring}_A$. The cotangent complex $L_{B/A} \in D(A)$ has the following properties.*

(1) *$L_{B/A} \in D^{\leq 0}(B)$.*
(2) *There is a natural (in $B$) isomorphism $H^0(L_{B/A}) \simeq \Omega^1_{B/A}$.*

*(3) If $A \to B$ is smooth, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$. In particular, if $A \to B$ is étale, then $L_{B/A} \simeq 0$. (The converse is not true; see the example below.)*

*(4) For any morphisms $A \to B \to C$, we have a distinguised triangle in $D(C)$*

$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B}.$$

A particular interesting class of ring (or scheme) morphisms is the local complete intersection morphism.

**Definition 7.7.** A ring map $A \to B$ is called a local complete intersection if it is of finite type and for some (equivalently, any) presentation $B = A[x_1, \dots, x_n]/I$, the ideal $I$ is Koszul-regular.

**Theorem 7.8.** [Qui70, Theorems 5.5, 5.6] *If a ring map $A \to B$ of finite type is a local complete intersection, then the cotangent complex $L_{B/A}$ has Tor-amplitude $[-1, 0]$. If $A$ is Noetherian, then the converse is also true.*

So at least when the base ring $A$ is Noetherian, we have a complete characterization of local complete intersections in terms of cotangent complexes.

## 8.

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