Math. Ann. 296, 403–429 (1993)

Mathematische Annalen

Formal and rigid geometry

II. Flattening techniques

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Received April 2, 1992; in revised form October 12, 1992

Mathematics Subject Classification (1991): 11Gxx, 14Axx, 14Gxx

The relationship between formal and rigid geometry was discussed in [FI]. The objective of the present paper is to provide a proof of the basic result announced by Raynaud [R] that a flat morphism between quasi-compact rigid spaces is induced by a flat morphism between suitable formal models. Thereby we establish techniques which are of interest for further developments in this field and which have already been applied with great success. The method in the case of schemes of finite presentation is documented in [RG]. In the situation of formal schemes of topologically finite presentation over a valuation ring of height 1, the method was worked out by Mehlmann in his thesis [M]. In this article we elaborate a more conceptual program and give rigorous proofs in a more general situation which will be important for the foundations of rigid geometry via formal schemes. For example we will treat the relative case where the base is an arbitrary complete noetherian ring.

In Sect. 1 we provide lifting techniques in formal geometry. They allow to deduce a kind of preparation theorem for a formal morphism. Such a statement corresponds to Weierstrass' theorem in complex analysis and is the basic lemma for many proofs in this article. In the complex case, the good relative spaces are the n-dimensional balls over a complex space. In formal geometry, the good relative spaces have to be more general, since their topology is somehow coarser. For many purposes, the smooth affine relative schemes with geometrically connected fibres are sufficiently nice. For such a formal scheme, the global ring admits a topological basis under fairly mild conditions. As a consequence, such schemes admit ideals of coefficients, as we explain in Sect. 2. In Sect. 3 we recall the notion of a dévissage of a module. The latter provides a technique to reduce the study of modules over a T-scheme to a question about modules over T-schemes which are smooth with geometrically connected fibres. Furthermore, in the latter situation it is easy to relate flatness at the rigid points to flatness at the formal level. The flattening technique is worked out in Sect. 4. It is a means to produce a suitable admissible formal blowing-up of a formal base scheme T such that the strict transform of a rig-flat module on a formal T-scheme becomes flat. The applications of this theory are discussed in Sect. 5. For example it follows that a rigid morphism with fibre dimension $\leq n$ admits a formal model with fibre dimension $\leq n$.

We assume that the reader is familiar with the results of [FI]. Notations are as in [FI].

1 Lifting techniques

In the following we fix a formal base scheme S which is assumed to be noetherian or the formal spectrum of a complete valuation ring R of height 1. We denote by $\mathscr{I} \subset \mathcal{O}_S$ the ideal of definition of S; in the affine case $S = \mathrm{Spf}(A)$ the ideal of definition of A is denoted by $\mathfrak{I}$. Formal S-schemes are always assumed to be topologically of finite presentation over S. If $f: X \to Y$ is a formal morphism, the levels of f are denoted by $f_\lambda: X_\lambda \to Y_\lambda$ for $\lambda \in \mathbb{N}$; so f_0 is the level zero of f.

We recall some definitions on smoothness. For a formal S-scheme Y, we denote by $\mathbb{D}_Y^n$ the formal spectrum of the restricted power series ring $\mathcal{O}_Y\langle\xi_1, \ldots, \xi_n\rangle$; it corresponds to the affine n-space in classical algebraic geometry. The $\mathcal{O}_Y$-module of *continuous* differential forms of degree 1 is denoted by $\Omega^1_{Y/S}$. For example $\Omega^1_{\mathbb{D}^n_S/S}$ is the free $\mathcal{O}_{\mathbb{D}^n_S}$-module generated by $d\xi_1, \ldots, d\xi_n$. For a formal S-morphism $f: X \to Y$ we write $\Omega^1_{X/Y}$ for the $\mathcal{O}_X$-module of relative continuous differential forms of degree 1. The module $\Omega^1_{X/Y}$ is a coherent $\mathcal{O}_X$-module on X.

Definition 1.1. A formal S-morphism $f: X \to Y$ is *smooth* at $x \in X_0$ of relative dimension r if there exists an open neighborhood U of x and a Y-immersion $j: U \to \mathbb{D}_Y^n$ such that the following conditions are satisfied:

(a) locally at $y = j(x)$, the sheaf of ideals defining $j(U)$ as a formal subscheme of $\mathbb{D}_Y^n$ is generated by $(n-r)$ sections $g_{r+1}, \ldots, g_n$,
(b) the differentials $dg_{r+1}, \ldots, dg_n$ are linearly independent in $\Omega^1_{\mathbb{D}^n_Y/Y} \otimes k(y)$.

A formal morphism is *smooth* if it is smooth at all points of X_0.

A formal morphism is *étale* (resp. *étale at* x) if it is smooth (resp. smooth at x) of relative dimension 0.

Lemma 1.2. *Let $f: X \to Y$ be a morphism of formal S-schemes and let x be a point of X_0. The following conditions are equivalent:*

(a) *f is smooth at x of relative dimension r.*
(b) *f is flat at x and $f_0: X_0 \to Y_0$ is smooth at x of dimension r.*
(c) *$f_\lambda: X_\lambda \to Y_\lambda$ is smooth at x of relative dimension r for all λ.*

The proof is quite easy due to the criterion on flatness [FI, 1.6] and we leave it to the reader. The condition (c) allows to generalize various properties of smooth morphisms to smooth formal ones. Again we leave it to the reader to make precise statements.

A pointed formal S-morphism $f: (X, x) \to (Y, y)$ consists of a formal S-morphism $f: X \to Y$ and of points x of X_0 and y of Y_0 with $f_0(x) = y$. An étale neighborhood of x is an étale pointed formal S-morphism $(X', x') \to (X, x)$; it is called *elementary étale* if, in addition, the residue extension $k(x')/k(x)$ is trivial. Let us decribe two types of lifting techniques which we will use frequently in this article.

The first one is quite clear by working on the powers of the ideal of definition and using Nakayama's lemma (for nilpotent ideals).

Lemma 1.3. *Let $f: X \to Y$ be an affine formal S-morphism and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Let E be a system of elements of $\Gamma(X, \mathcal{M})$.*
(a) *If E_0 generates $f_*\mathcal{M}_0$ over $\mathcal{O}_{Y_0}$, then $f_*\mathcal{M}$ is topologically generated by E over $\mathcal{O}_Y$.*
(b) *If E induces a basis of $f_*\mathcal{M}_0$ over $\mathcal{O}_{Y_0}$ and if $\mathcal{M}$ is flat over $\mathcal{O}_Y$, then E is a topological basis of $f_*\mathcal{M}$ over $\mathcal{O}_Y$.*

Proof. (a) One can assume that Y and, hence, X are affine. Then one proceeds as in the proof of [FI, 1.5].
(b) Due to (a) we obtain a surjective map $\varphi: \hat{\mathcal{O}}_Y^E \to f_*\mathcal{M}$. Since $\mathcal{M}$ is flat over Y, we get

$$\mathrm{Ker}(\varphi) \subset \mathcal{I}^\lambda \hat{\mathcal{O}}_Y^E$$

for all $\lambda \in \mathbb{N}$. Since $\hat{\mathcal{O}}_Y^E$ is separated, we see $\mathrm{Ker}(\varphi) = 0$. □

Lemma 1.4. *Let (T, t) be a pointed formal S-scheme.*
(a) *Let $h_0: (Y_0, y) \to (T_0, t)$ be a smooth pointed S_0-morphism. Then, after replacing Y_0 by an open neighborhood of y, the morphism h_0 lifts to a smooth pointed formal S-morphism $h: (Y, y) \to (T, t)$.*
(b) *Let $f: (X, x) \to (T, t)$, be a pointed morphism of formal S-schemes. Consider a commutative diagram*

$$\begin{tikzcd} (X_0, x) \arrow[rr] \arrow[dr, "f_0"'] & & (Y_0, y) \arrow[dl, "h_0"] \\ & (T_0, t) . & \end{tikzcd}$$

If h_0 is smooth, then after shrinking to open neighborhoods it lifts to a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[rr] \arrow[dr, "f"'] & & (Y, y) \arrow[dl, "h"] \\ & (T, t) . & \end{tikzcd}$$

where h is a smooth formal morphism. If h_0 is étale (not necessarily affine), the diagram lifts uniquely and it is not necessary to shrink.

Proof. (a) Since h_0 is smooth, we may assume that Y_0 is given as a closed subscheme

$$Y_0 = V(\bar{f}_{r+1}, \ldots, \bar{f}_n) \subset \mathbb{A}^n_{T_0}$$

such that there exists an $(n-r) \times (n-r)$ minor of the Jacobi matrix $(\partial\bar{f}_i/\partial\xi_j)$ which is invertible on Y_0. Then choose liftings $f_{r+1}, \ldots, f_n$ in $\mathcal{O}_T\langle\xi_1, \ldots, \xi_n\rangle$ of $\bar{f}_{r+1}, \ldots, \bar{f}_n$ and set

$$Y = V(f_{r+1}, \ldots, f_n) \subset \mathbb{D}^n_T$$

and let $h: Y \to T$ be the morphism induced by the projection $\mathbb{D}^n_T \to T$.
(b) follows from (a) and the lifting property of smooth (resp. étale) morphisms. □

Now we want to use these lemmata to carry over results of [RG], which are proved there in the context of schemes of finite type, to the situation of formal schemes. The preceding lemma applied to [RG, 1.1.1] yields a useful corollary of Zariski's Main Theorem which will serve as a preparation theorem like Weierstrass' Preparation Theorem in complex analysis.

Theorem 1.5. *Let $f:(X,x)\to(T,t)$ be a pointed formal S-morphism. Let $d=\dim_x(X\otimes k(t))$. Then there exists a commutative diagram of formal S-morphisms*

$$\begin{tikzcd}
(X,x) \arrow[ddd, "f"'] & (X',x') \arrow[l, "u"'] \arrow[d, "g"] \\
& (Z',z') \arrow[d, "h"] \\
& (T',t') \arrow[d, phantom, ""] \\
(T,t) & \arrow[l, phantom, ""]
\end{tikzcd}$$

$$\begin{array}{ccc}
(X,x) & \xleftarrow{u} & (X',x') \\
\Big\downarrow f & & \downarrow g \\
 & & (Z',z') \\
 & & \downarrow h \\
(T,t) & \xleftarrow{v} & (T',t')
\end{array}$$

where
(a) *T', Z', X' are affine and u, v are elementary étale,*
(b) *h is smooth with irreducible geometric fibres of dimension d,*
(c) *g is finite and x' is the only point above z'.*

Next we remind the reader that there is the notion of a pure morphism in [RG, 3.3.3]; it is needed to characterize projective modules among the flat ones; cf. [RG, 3.3.5]. We omit the precise definition of a pure morphism of finite type, since on the one hand the definition is heavy and on the other hand we only need to know that smooth morphisms with irreducible geometric fibres of constant dimension are pure; cf. 1.7(c) below. In particular we will make use of the fact that, for a smooth morphism of schemes $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ with geometrically irreducible fibres of constant dimension, B is a projective A-module; cf. [RG, 3.3.1].

Definition 1.6. A formal S-morphism $f: X\to Y$ is *pure* if $f_0: X_0\to Y_0$ is pure; cf. [RG, 3.3.3]. A coherent $\mathcal{O}_X$-Module $\mathcal{M}$ is *f-pure* if $\mathcal{M}_0$ is f_0-pure. The formal morphism f is *pure* if $\mathcal{O}_X$ is f-pure.

Examples 1.7 [RG, 3.3.4]. (a) For a proper formal morphism $f: X\to Y$, any coherent $\mathcal{O}_X$-module $\mathcal{M}$ is f-pure.
(b) Let f be quasi-compact and separated such that f_0 is quasi-finite. Then f is pure if and only if f is finite.
(c) A faithfully flat formal morphism with irreducible geometric fibres of constant dimension without embedded components is pure. In particular, a smooth surjective formal morphism with irreducible geometric fibres is pure.

Proposition 1.8. *Let $f: X\to Y$ be an affine formal S-morphism. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module which is pure and flat over Y.*
(a) *$f_*\mathcal{M}_0$ is a projective $\mathcal{O}_{Y_0}$-module.*
(b) *Locally on Y, the direct image $f_*\mathcal{M}$ has a topological $\mathcal{O}_Y$-basis.*
(c) *Assume that Y is affine and connected and that f is flat and pure. If all fibres of f_0 are at least of dimension 1, then $f_*\mathcal{O}_X$ has a topological $\mathcal{O}_Y$-basis.*

Proof. (a) follows from [RG, 3.3.5].
(b) follows from (a) and 1.3. Recall that a projective module is locally free if it is finitely generated. For the general case we may assume that Y_0 is connected and noetherian. So, if $f_*\mathscr{M}_0$ is not finitely generated, it is free due to a result of Bass [Ba].
(c) follows from [RG, 3.3.11] and 1.3. □

2 Ideals of coefficients

As in Sect. 1 we fix a formal base scheme S. Formal S-schemes are always assumed to be topologically of finite presentation.

Definition 2.1. Let $f: Z \to T$ be an S-morphism and $\mathscr{J}$ be a coherent ideal of $\mathcal{O}_Z$. A coherent ideal $\mathscr{C}$ of $\mathcal{O}_T$ is called an *ideal of coefficients of* $\mathscr{J}$ if it satisfies the following properties:

(i) the projection $Z \times_T V(\mathscr{C}) \to Z$ factors through $V(\mathscr{J})$,
(ii) if $T' \to T$ is a morphism such that the projection $Z' = Z \times_T T' \to Z$ factors through $V(\mathscr{J})$, then $T' \to T$ factors through $V(\mathscr{C})$.

Due to the universal property the ideal of coefficients of a given ideal $\mathscr{J}$ is uniquely determined if it exists. Moreover it is compatible with base change in the sense that $\mathscr{C}\mathcal{O}_{T'}$ is the ideal of coefficients of $\mathscr{J}\mathcal{O}_{Z'}$ for any base change $T' \to T$. The set-theoretical locus $V(\mathscr{C})$ is just the union of all points t of T such that the fibre $Z \times_T t$ of t is contained in $V(\mathscr{J})$. In other terms a coherent sheaf of ideals $\mathscr{C}$ of $\mathcal{O}_T$ is an ideal of coefficients of a coherent ideal $\mathscr{J}$ of $\mathcal{O}_Z$ if $\mathscr{J} \subset \mathscr{C}\mathcal{O}_Z$, if $\mathscr{C}$ is the smallest coherent ideal of $\mathcal{O}_T$ with this property, and if the latter remains valid after base change.

Remark 2.2. If $\mathscr{C}$ is the ideal of coefficients of an open ideal $\mathscr{J}$, then $\mathscr{C}$ is open, too.

Proposition 2.3. *Let $f: Z \to T$ be a quasi-compact formal S-morphism. Assume either that f is affine, pure and flat or that f is flat with geometrically irreducible fibres of constant dimension without embedded components (for example, smooth with geometrically irreducible fibres of constant dimension).*
(a) *In the noetherian case, any coherent ideal $\mathscr{J}$ of $\mathcal{O}_Z$ admits an ideal of coefficients $\mathscr{C}$ in $\mathcal{O}_T$.*
(b) *In the classical rigid case, any open coherent ideal of $\mathcal{O}_Z$ admits an ideal of coefficients in $\mathcal{O}_T$ and any coherent ideal of $\mathcal{O}_{Z_{\mathrm{rig}}}$ admits a coherent ideal of coefficients in $\mathcal{O}_{T_{\mathrm{rig}}}$.*
Furthermore, if $\mathscr{J} \subset \mathcal{O}_Z$ is a (not necessarily open) coherent ideal such that $\mathcal{O}_{T_{\mathrm{rig}}}$ is the ideal of coefficients of $\mathscr{J}_{\mathrm{rig}}$, then $\mathscr{J}$ has an ideal of coefficients which is open.
(c) *For $\lambda \in \mathbb{N}$, any coherent ideal $\mathscr{J}$ of $\mathcal{O}_{Z_\lambda}$ admits an ideal of coefficients $\mathscr{C}$ in $\mathcal{O}_{T_\lambda}$.*

Proof. (a) Consider first an affine situation $Z = \mathrm{Spf}(B) \to T = \mathrm{Spf}(A)$ and assume that there exists a topological basis $(e_i;\ i \in I)$ of B over A. Let $(f_1, \ldots, f_r)$ be a generating system of an ideal $\mathfrak{b}$ of B. Using the basis of B over A we can write

$$f_j = \hat{\Sigma}_{i \in I}\, a_{ij} e_i \quad \text{for} \quad j = 1, \ldots, r$$

as a converging sum. Now set

$$\mathfrak{a} = (a_{ij};\, i \in I, j \in J)\,.$$

Since any ideal of B is closed, we obtain $\mathfrak{b} \subset B\mathfrak{a}$. Since $(e_i;\, i \in I)$ is a basis of B over A, the ideal $\mathfrak{a}$ is the smallest ideal $\mathfrak{c}$ of A with $\mathfrak{b} \subset B\mathfrak{c}$. So $\mathfrak{a}$ is the ideal of coefficients of $\mathfrak{b}$.

If $f: Z \to T$ is an affine formal morphism which is pure and flat, locally on T_0 there exists a basis of $f_*\mathcal{O}_Z$ over $\mathcal{O}_T$ due to 1.8. So, by what we have already said, locally on T_0 there exists an ideal of coefficients. Since these ideals are compatible with base change, they fit together to build a coherent ideal of $\mathcal{O}_T$.

Now let $f: Z \to T$ be flat with geometrically irreducible fibres of constant dimension without embedded components. It suffices to work locally around a point $t \in T_0$. Then we may assume that Z is a union of finitely many open subschemes Z_i, $i \in I$, such that each $Z_i \to T$ is affine and has geometrically irreducible fibres of constant dimension. In particular $Z_i \to T$ is affine, pure and flat by 1.7(c). By the case settled above $\mathcal{J}\mathcal{O}_{Z_i}$ has an ideal of coefficients $\mathcal{C}_i$ and, hence, the sum $\mathcal{C} = \Sigma_i \mathcal{C}_i$ is an ideal of coefficients of $\mathcal{J}$.

(b) We can do the same reasoning as in the noetherian case. We explain only the affine case where $Z = \mathrm{Spf}(B) \to T = \mathrm{Spf}(A)$ and where there exists a topological basis $(e_i;\, i \in I)$ of B over A. For a finitely generated ideal $\mathfrak{b} \subset B$ we can define the ideal $\mathfrak{a} \subset A$ as above. Then we claim that $\mathfrak{a}$ is an ideal of coefficients of $\mathfrak{b}$. Since $\mathfrak{b}$ is open, there exists a power $\mathcal{J}^\lambda$ with $\mathcal{J}^\lambda \subset \mathfrak{b}$. Then $\mathcal{J}^\lambda$ is contained in $\mathfrak{a}$. Since $\mathfrak{b}$ is finitely generated and since the sets $\{a_{ij},\, i \in I\}$ are finite modulo $\mathcal{J}^\lambda$ and since $\mathcal{J}^\lambda$ is finitely generated, $\mathfrak{a}$ is finitely generated. Then it is clear that $\mathfrak{a}$ is an ideal of coefficients of $\mathfrak{b}$.

Now consider an ideal $\mathfrak{b}_K \subset B_{\mathrm{rig}}$. Such an ideal is finitely generated and, hence, it is of type $\mathfrak{b}_{\mathrm{rig}}$ for some finitely generated ideal $\mathfrak{b} \subset B$; for example one can take $\mathfrak{b} = \mathfrak{b}_K \cap B$, cf. [FI, 1.2(c)]. So we can define an ideal $\mathfrak{a} \subset A$ as above, but $\mathfrak{a}$ is not necessarily of finite type. Nevertheless $\mathfrak{a}_{\mathrm{rig}} \subset A_{\mathrm{rig}}$ is finitely generated and $\mathfrak{a}B_{\mathrm{rig}}$ is closed in B_{rig}, so we can conclude as in the noetherian case that $\mathfrak{a}_{\mathrm{rig}}$ is the ideal of coefficients of $\mathfrak{b}_K$.

Next consider an ideal $\mathfrak{b} \subset B$ such that $\mathfrak{b}_{\mathrm{rig}}$ has A_{rig} as an ideal of coefficients. In this case the ideal $\mathfrak{a}$ defined as above satisfies $\mathfrak{a}_{\mathrm{rig}} = A_{\mathrm{rig}}$. Henceforth $\mathfrak{a}$ is an open ideal. As in the first case of (b) one concludes that $\mathfrak{a}$ is finitely generated and that $\mathfrak{a}$ is the ideal of coefficients of $\mathfrak{b}$.

(c) follows similarly as (a) where one has to work with finite linear combinations instead of infinite converging ones. □

Now we want to use the existence of ideals of coefficients to show that certain properties of rings are transmitted from A to B if $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is a smooth formal morphism with geometrically irreducible fibres of constant dimension.

Proposition 2.4. *Let* $Z = \mathrm{Spf}(B) \to T = \mathrm{Spf}(A)$ *be a smooth surjective formal morphism with geometrically irreducible fibres of constant dimension.*

(a) *The fibres of* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *are irreducible and reduced.*

(b) *Assume that A has no $\mathfrak{I}$-torsion where $\mathfrak{I}$ is the ideal of definition. Then any associated prime ideal of B is the generic point ζ of a fibre over a point* $t \in \mathrm{Spec}(A)$ *belonging to the complement of the special fibre. In particular B has no $\mathfrak{I}$-torsion.*

(c) *If z is a point of Z_0 and if ζ is the generic point of the fibre of z under the morphism $Z_0 \to T_0$, the canonical map $B_z \to B_\zeta$ is injective.*

For the proof we need a lemma.

Lemma 2.5. *Keep the situation of 2.4. Let $f \in B$ be an element which does not vanish on any fibre of $Z_0 \to T_0$. Then*
(a) *f gives rise to a non-zero-divisor on B_λ for all $\lambda \in \mathbb{N}$.*
(b) *f is a non-zero-divisor on B.*
(c) *If Y_0 is a closed subscheme of Z_0 with* $\dim(Y_0/T_0) < \dim(Z_0/T_0)$ *then*

$$\Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \xrightarrow{\sim} \Gamma(\operatorname{Spec} B - Y_0, \mathcal{O}_{\operatorname{Spec} B})$$

is bijective if A has no $\mathfrak{J}$-torsion where $\mathfrak{J}$ is the ideal of definition.

Proof. (a) Let $Z_\lambda = \operatorname{Spec}(B_\lambda) \to T_\lambda = \operatorname{Spec}(A_\lambda)$ be the associated morphism of $A_\lambda \to B_\lambda$. Since we may perform an étale surjective base change, we may assume that $(Z_\lambda - V(f)) \to S_\lambda$ has a section. Then, by a standard argument on direct limits, we reduce to a noetherian base keeping the smoothness and the section. Furthermore it is possible to keep the connectedness of the fibres by using [EGA, IV$_3$, 15.6.5]. Thus we may assume that A_λ is noetherian. Since $A_\lambda \to B_\lambda$ is smooth with geometrically irreducible fibres, the associated prime ideals of B_λ are the generic points of certain fibres of $Z_0 \to T_0$ due to [AC, IV, par. 2, n°6, théorème 2]. Thus f is not a zero-divisor on B_λ.
(b) Consider an element $g \in B$ with $fg = 0$. From (a) we obtain $g \in \mathfrak{J}^\lambda B$ for any $\lambda \in \mathbb{N}$ where $\mathfrak{J}$ is the ideal of definition. Since B is separated, we see $g = 0$.
(c) First consider the noetherian case. For any $z \in Y_0$ we have $\operatorname{prof}(B_z) \geqq 2$. Indeed, let t be the image of z, then $\operatorname{prof}(A_t) \geqq 1$ since $(\operatorname{Spec}(A) - T_0)$ is schematically dense in $\operatorname{Spec}(A)$. Since z is not the generic point of $Z \otimes k(t)$, we have $\operatorname{prof}(B_z \otimes k(t)) \geqq 1$ and, hence, $\operatorname{prof}(B_z) \geqq 2$. Then the assertion follows from [EGA, IV$_2$, 5.10.5].

Consider the classical rigid case. The restriction map is injective, since B has no $\mathfrak{J}$-torsion. Let f be a section of $\mathcal{O}_{\operatorname{Spec} B}$ over $(\operatorname{Spec} B - Y_0)$. We can view f as an element of B_{rig}. So there exists a non-zero element π in the ideal of definition with $\pi f \in B$. Furthermore there exists an element $g \in B$ which does not vanish on the fibre of t such that $gf \in B$. Since the assertion is local on $\operatorname{Spec}(A)$, we may assume that g does not vanish on any fibre over T_0. Since π is a non-zero divisor on B, it is enough to show that (πf) is zero in $B/\pi B$. Since $g(\pi f)$ is zero in $B/\pi B$ and g is a non-zero-divisor on $B/\pi B$ due to (b), we see $(\pi f) = 0$ on $B/\pi B$. □

Lemma 2.6. *Keep the situation of 2.4 and assume that A is noetherian or that A is a valuation ring of height 1. For $f \in B$ let $\mathfrak{c}(f) \subset A$ be the ideal of coefficients of Bf; the existence follows from 2.3. Then*
(a) *$\mathfrak{c}(af) = a\mathfrak{c}(f)$ for $a \in A$.*
(b) *If $f \in B$ is not zero on any fibre of $Z_0 \to T_0$, then $\mathfrak{c}(fg) = \mathfrak{c}(g)$ for all $g \in B$.*
(c) *$\mathfrak{c}(f_1 f_2) = \mathfrak{c}(f_1)\mathfrak{c}(f_2)$ for $f_1, f_2 \in B$ if $\mathfrak{c}(f_1)$ is monogenous.*

Proof. The assertion is local on T. So we may assume that A is connected and due to 1.8 that B has a topological basis $(e_i;\ i \in I)$ over A.
(a) We can write

$$f = \hat{\Sigma}_{i \in I}\, a_i e_i\,.$$

Then we have $\mathfrak{c}(f) = (a_i;\ i \in I)$ and $\mathfrak{c}(af) = (aa_i;\ i \in I) = a\mathfrak{c}(f)$. So (a) is clear.
(b) $\mathfrak{c}(fg) \subset \mathfrak{c}(g)$ because $fg \in Bg \subset B\mathfrak{c}(g)$ for trivial reasons. For the converse inclusion, we may assume $\mathfrak{c}(fg) = 0$. The latter implies $fg = 0$. Then we obtain $g = 0$ by 2.5 and, hence, $\mathfrak{c}(g) = 0$.
(c) Since $f_i \subset B\mathfrak{c}(f_i)$ for $i = 1, 2$ we obtain $f_1 f_2 \in B\mathfrak{c}(f_1)\mathfrak{c}(f_2)$ and, hence, by the definition of ideals of coefficients $\mathfrak{c}(f_1 f_2) \subset \mathfrak{c}(f_1)\mathfrak{c}(f_2)$. We have to show that for any $t \in T_0$

$$\mathfrak{c}(f_1 f_2)A_t = \mathfrak{c}(f_1)\mathfrak{c}(f_2)A_t\,.$$

So consider a point t of T_0; we will also write t for the prime ideal of A associated to t. We may assume that $\mathfrak{c}(f_1)\mathfrak{c}(f_2)A_t$ is not zero. We have $f_1 = ag_1$ for some $g_1 \in B$ and $a \in A$ with $Aa = \mathfrak{c}(f_1)$. If g_1 would vanish on the fibre of t, then $f_1 \in Bta$. This would imply $\mathfrak{c}(f_1) \subset ta = t\mathfrak{c}(f_1)$ by the definition of ideals of coefficients and, hence, $\mathfrak{c}(f_1)$ would vanish at t due to Nakayama's lemma. But this is impossible due to our assumption. Thus we see that g_1 is not zero on the fibre of t and, hence, also on any fibre over a neighborhood of t. Now the assertion follows from (a) and (b). □

Now we come to the *proof of 2.4*. Consider first the noetherian case. If suffices to prove that B is integral if A is integral. Consider $f_1, f_2 \in B$ with $f_1 f_2 = 0$ and $f_1 \neq 0$; in particular $\mathfrak{c}(f_1) \neq 0$. We have to show $f_2 = 0$; it suffices to show $\mathfrak{c}(f_2) = 0$. If $\mathfrak{c}(f_1)$ is monogenous, the assertion follows from 2.6(c) and the fact that A is integral. The general case follows by base change with a complete discrete valuation ring A' dominating a localization $A_{\mathfrak{m}}$ where $\mathfrak{m}$ is a maximal ideal of A; cf. [EGA, II, 7.1.7]. The inclusion of $A \subsetneq A'$ gives rise to a morphism of formal schemes $T' \to T$. Since the ideals of coefficients are compatible with base change, we have

$$A' \cdot \mathfrak{c}(f_i) = \mathfrak{c}(f'_i) \quad \text{for} \quad i = 1, 2$$

where f'_i is the pull-back of f_i to $Z' = Z \times_T T'$. Thus we obtain by 2.6(c)

$$0 = \mathfrak{c}(f_1 f_2)A' = \mathfrak{c}(f'_1 f'_2) = \mathfrak{c}(f'_1)\mathfrak{c}(f'_2)$$

since $\mathfrak{c}(f'_1)$ is monogenous. Since $\mathfrak{c}(f'_1) \neq 0$ we see $\mathfrak{c}(f'_2) = 0$ and, hence, $\mathfrak{c}(f_2) = 0$.

In the classical rigid case we do the same reasoning as before but now on rig-points. If $f_2 \neq 0$ there exists a rig-point t of T such that $t^*f_i \neq 0$ for $i = 1, 2$. Indeed, $(f_i)_{\mathrm{rig}}$ has an ideal of coefficients $\mathfrak{a}_i \subset A_{\mathrm{rig}}$ due to 2.3 which is not zero. Since A_{rig} is integral and jacobson, there exists a maximal ideal $\mathfrak{m} \subset A_{\mathrm{rig}}$ such that $\mathfrak{a}_i \not\subset \mathfrak{m}$ for $i = 1, 2$. Now $\mathfrak{m}$ gives rise to a rig-point we are looking for. The above reasoning, which works for valuation rings of height 1 also, shows that $t^*(f_1)t^*(f_2) \neq 0$ and, hence, $f_1 f_2 \neq 0$.
(b) It remains to determine the associated primes of B. Since B is flat over A, the associated primes of B are obtained as the associated primes of the fibres B/tB where t runs over the associated primes of A. In the noetherian case this follows directly by (a) from [AC, IV, par. 2, n°6, théorème 2]. Furthermore we supposed that the complement of the special fibre is schematically dense in Spec (A), so an associated prime lies in the complement of the special fibre. In the classical rigid case, any associated prime of B lies over the generic point of the valuation ring R as B is R-flat, so we can apply loc. cit. also, since $B \otimes K$ and $A \otimes K$ are noetherian. Then the assertion follows from (a).

(c) For any $f \in (B - \zeta)$ there exists an open neighborhood of the image t of z in T_0 such that f does not vanish on any fibre of $Z_0 \to T_0$ over that neighborhood. Then the assertion follows from 2.5(b). □

3 Dévissage

The purpose of this section is to recall some basic definitions and constructions for the flattening technique of Gruson and Raynaud [RG] and to adapt them to the formal case. The flattening technique will be worked out in the next section. As in Sect. 1 we fix a formal base scheme S. Let $f: X \to T$ be a morphism of quasi-compact formal S-schemes which are topologically of finite presentation over S.

Definition 3.1. A *T-dévissage* of a coherent $\mathcal{O}_X$-module $\mathcal{M}$ *in dimension n* at a point x of X_0 consists of the following data:

(a) a pointed closed immersion $(Y, y) \to (X, x)$ of finite presentation such that $\mathcal{M}$ is the direct image of a coherent $\mathcal{O}_Y$-module (i.e., the ideal of Y annihilates $\mathcal{M}$) which we will call $\mathcal{M}$ also,
(b) a pointed factorization $(Y, y) \to (Z, z) \to (T, t)$ of $Y \to T$ into a finite morphism $g: Y \to Z$ such that y is the only point above z and an affine morphism $Z \to T$ which is smooth with irreducible geometric fibres of dimension n,
(c) a homomorphism $\alpha: \mathcal{L} = \mathcal{O}_Z^r \to \mathcal{N} = g_* \mathcal{M}$ such that $\alpha \otimes k(\zeta)$ is bijective where ζ is the generic point of the fibre $Z \otimes k(t)$ of t. The cokernel of α is denoted by $\mathcal{P}$ which is a coherent $\mathcal{O}_Z$-module.

In short we will write

$$D = (g: (Y, y) \to (Z, z), \mathcal{M}, \alpha: \mathcal{L} \to \mathcal{N}, \mathcal{P})$$

for such a set of data.

We mention $\dim(\mathcal{P} \otimes k(t)) \leqq (n-1)$. It follows from Nakayama's lemma that there exists a closed subscheme Y' of Z of finite presentation with $\dim(Y' \otimes k(t)) \leqq (n-1)$ such that $\mathcal{P}$ is an $\mathcal{O}_{Y'}$-module.

Definition 3.2. A *T-dévissage* of a coherent $\mathcal{O}_X$-module $\mathcal{M}$ at a point x of X_0 *in dimensions $N = n^1 > n^2 > \ldots > n^r = n \geqq 0$ between N and n* consists of a sequence $(D^1, \ldots, D^r)$ of T-dévissages

$$D^i = (g^i: (Y^i, y^i) \to (Z^i, z^i), \mathcal{M}^i, \alpha^i: \mathcal{L}^i \to \mathcal{N}^i = g^i_* \mathcal{M}^i, \mathcal{P}^i)$$

where
D^1 is a T-dévissage of $\mathcal{M}$ in dimension n^1,
D^i is a T-dévissage of $\mathcal{P}^{i-1}$ in dimension n^i for $i = 2, \ldots, r$.
In particular, $\mathcal{M}^1 = \mathcal{M}|_{Y^1}$ and $\mathcal{M}^i = \mathcal{P}^{i-1}|_{Y^i}$.

A T-dévissage is called *complete* if $\mathcal{P}^r = \operatorname{coker}(\alpha^r) = 0$.

A (complete) dévissage is stable under base change. If a complete dévissage of an $\mathcal{O}_X$-module $\mathcal{M}$ is given as above, flatness conditions on $\mathcal{M}$ can be rephrased in conditions on the maps α^i and the quotients $\mathcal{P}^i$, since the free modules $\mathcal{L}^i$ over a smooth formal T-scheme behave quite well as $\mathcal{O}_T$-modules as we will see in 3.5. Note that the support of $\mathcal{M}^i$ is of relative dimension n^i and that the support of $\mathcal{P}^i$ is of relative dimension less than n^i. Therefore a dévissage provides a means to proceed for the flattening technique by induction on the relative dimension; cf. Sect. 4.

An important step is to show the existence of a complete dévissage for a given $\mathcal{M}$ at least after étale surjective base change.

Proposition 3.3. *Let $f:(X,x)\to(T,t)$ be a pointed formal S-morphism. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module with $\mathcal{M}_x \neq 0$. Let $N=\dim_x(\mathcal{M}\otimes k(t))$ and $r=$ $\operatorname{coprof}_{\mathcal{O}_{X,x}\otimes k(t)}(\mathcal{M}_x\otimes k(t))$. Then there exists a commutative diagram*

$$\begin{tikzcd} (X,x) \arrow[d, "f"] & (X',x') \arrow[l, "u"'] \arrow[d, "f'"] \\ (T,t) & (T',t') \arrow[l, "v"'] \end{tikzcd}$$

where u and v are elementary étale affine neighborhoods such that $\mathcal{M}'=u^*\mathcal{M}$ admits a complete T'-dévissage at x' in dimensions between N and $N-r$.

Proof. The proof in [RG, 1.2.3] can be copied by replacing the étale localization concerning Zariski's Main Theorem by 1.5. Therefore we will give only some indication of how to show the existence of such a dévissage.

We may assume that we start with an affine situation. Let Y be a closed subscheme of X which is of finite presentation such that $\mathcal{M}$ is an $\mathcal{O}_Y$-module and such that $\dim_x(Y\otimes k(t))=N$. So we can replace X by Y. After étale localization in the style of 1.5 we may assume that we have a factorization

$$(X,x)\xrightarrow{g}(Z,z)\xrightarrow{h}(T,t)$$

where h is smooth with geometrically irreducible fibres of dimension N, where g is finite and where x is the only point above z. Then set $\mathcal{N}=g_*\mathcal{M}$. Now consider $\mathcal{N}\otimes k(t)$.

If $r=\operatorname{coprof}_{\mathcal{O}_{Z,z}\otimes k(t)}(\mathcal{N}\otimes k(t))=0$ then $\mathcal{N}\otimes k(t)$ is free at z, say of rank ℓ; cf. [EGA, 0_{IV}, 17.3.4]. Choose a basis of $\mathcal{N}\otimes k(z)$ and take a lifting of this basis in $\mathcal{N}$. Such a system of elements gives rise to a morphism $\alpha:\mathcal{O}_Z^\ell\to\mathcal{N}$ inducing an isomorphism $\alpha\otimes k(t)$ at z.

Now assume $r\geqq 1$. Since the fibre $Z\otimes k(t)$ is integral, $(\mathcal{N}\otimes k(t))_\zeta$ is free at the generic point of the fibre, say of rank ℓ. Then there exists a morphism $\alpha:\mathcal{O}_Z^\ell\to\mathcal{N}$ such that $(\alpha\otimes k(t))_\zeta$ is an isomorphism. Set $\mathcal{P}=\operatorname{coker}(\alpha)$. Due to Nakayama's lemma α is surjective at ζ, so

$$\dim_z(\mathcal{P}\otimes k(t))\leqq(N-1)\,.$$

Moreover, since the fibres of Z_0/T_0 are integral, the morphism $\alpha\otimes k(t)$ is injective. The latter implies

$$\operatorname{prof}(\mathcal{P}\otimes k(t))=\operatorname{prof}(\mathcal{N}\otimes k(t))\,,$$

since $\mathcal{N}\otimes k(t)$ is not free; use [EGA, 0_{IV}, 16.4.4]. Since the relative dimension has dropped, we obtain

$$\operatorname{coprof}(\mathcal{P}\otimes k(t))\leqq(r-1)\,.$$

Now the claim follows by induction. □

If we are given a quasi-compact formal morphism $X\to T$, we can do this reasoning for any point of the fibre of a given point $t\in T_0$. By a compactness argument we see that only finitely many diagrams as in 3.3 are necessary to cover the whole fibre of t. Thus we get the following result.

Corollary 3.4. *Let (T, t) be a pointed formal S-scheme. Let $f\colon X \to T$ be a quasi-compact formal S-morphism. Let $\mathscr{M}$ be a coherent $\mathcal{O}_X$-module. Let $N = \dim(\mathscr{M} \otimes k(t))$ and $r = \operatorname{coprof}_{X \otimes k(t)}(\mathscr{M} \otimes k(t))$. Then there exists a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{u} & X' \\ \downarrow f & & \downarrow f' \\ T & \xleftarrow{v} & T' \end{array}$$

where X' and T' are affine and where v is induced by an elementary étale neighborhood $v\colon (T', t') \to (T, t)$, where u is an étale morphism whose image contains the support of $\mathscr{M} \otimes k(t)$ such that $\mathscr{M}' = u^*\mathscr{M}$ admits a complete T'-dévissage over t' in dimensions between N and $(N - r)$.

The next problem is to characterize T-flatness of an $\mathcal{O}_X$-module $\mathscr{M}$ in terms of data related to a dévissage. So let us consider a situation as in 3.3:

$$\begin{array}{ccc} (X, x) & \xleftarrow{u} & (X', x') \\ \downarrow f & & \downarrow f' \\ (T, t) & \xleftarrow{v} & (T', t') \end{array}$$

where u and v are étale affine neighborhoods such that $\mathscr{M}' = u^*\mathscr{M}$ admits a (complete) T'-dévissage $(D^1, \ldots, D^r)$ at x' in dimensions between $N = n^1$ and $n = n^r$ where, for $i = 1, \ldots, r$,

$$D^i = (g^i\colon (Y^i, y^i) \to (Z^i, z^i), \mathscr{M}^i, \alpha^i\colon \mathscr{L}^i \to \mathscr{N}^i = g^i_*\mathscr{M}^i, \mathscr{P}^i)\,.$$

In this situation we can characterize T-flatness of $\mathscr{M}$ at x by data of the dévissage.

Proposition 3.5. *Keep the situation of above. The following conditions are equivalent:*

(a) *$\mathscr{M}_x$ is $\mathcal{O}_{T,t}$-flat.*
(b) *$\mathscr{N}^i_{z^i}$ is $\mathcal{O}_{T',t'}$-flat for all i.*
(c) *α^i is bijective at the generic point ζ^i of $Z^i \otimes k(t')$ for all i and $\mathscr{P}^r_{z^r}$ is $\mathcal{O}_{T',t'}$-flat.*
(d) *α^i is injective at z^i for all i and $\mathscr{P}^r_{z^r}$ is $\mathcal{O}_{T',t'}$-flat.*
(e) *α^i is T'-universally injective over an open neighborhood of t' for all i and $\mathscr{P}^r_{z^r}$ is $\mathcal{O}_{T',t'}$-flat.*

For the proof we need a lemma.

Lemma 3.6. *Let $f\colon (Z, z) \to (T, t)$ be a smooth pointed formal S-morphism with geometric irreducible fibre $Z \otimes k(t)$. Let ζ be the generic point of $Z \otimes k(t)$. Let $\alpha\colon \mathscr{L} = \mathcal{O}_Z^\ell \to \mathscr{N}$ be a morphism to a coherent $\mathcal{O}_Z$-module $\mathscr{N}$ such that $\alpha \otimes k(\zeta)$ is bijective. The following conditions are equivalent:*

(a) *α_z is $\mathcal{O}_{T,t}$-universally injective.*
(b) *α_z is injective.*
(c) *α_ζ is bijective.*

Proof. Due to Nakayama's lemma α_ζ is surjective. Since $\mathcal{O}_{Z,z} \to \mathcal{O}_{Z,\zeta}$ is flat by [FI, 1.8(b)], the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear. Now assume (c); i.e., that α_ζ is bijective. Then consider the commutative diagram

$$\begin{tikzcd} \alpha_z\colon \mathscr{L}_z \arrow[r] \arrow[d] & \mathscr{N}_z \arrow[d] \\ \alpha_\zeta\colon \mathscr{L}_\zeta \arrow[r] & \mathscr{N}_\zeta \end{tikzcd} .$$

For showing (a) it suffices to verify that $\mathscr{L}_z \to \mathscr{L}_\zeta$ or that $\mathcal{O}_{Z,z} \to \mathcal{O}_{Z,\zeta}$ is $\mathcal{O}_{T,t}$-universally injective. So it is enough to show
(1) $\mathcal{O}_{Z,z} \to \mathcal{O}_{Z,\zeta}$ is injective and
(2) $\mathcal{O}_{Z,\zeta}/\mathcal{O}_{Z,z}$ is $\mathcal{O}_{T,t}$-flat.
For (2) we have to show that for any finitely generated ideal $\mathfrak{a}$ of $\mathcal{O}_{T,t}$ the canonical map

$$\mathfrak{a} \otimes (\mathcal{O}_{Z,\zeta}/\mathcal{O}_{Z,z}) \to \mathcal{O}_{Z,\zeta}/\mathcal{O}_{Z,z}$$

is injective. Since $\mathcal{O}_{Z,\zeta}$ is $\mathcal{O}_{T,t}$-flat, for the last assertion it suffices to show that

$$\mathcal{O}_{Z,z}/\mathfrak{a}\mathcal{O}_{Z,z} \to \mathcal{O}_{Z,\zeta}/\mathfrak{a}\mathcal{O}_{Z,\zeta}$$

is injective. Since a finitely generated ideal is induced by a coherent sheaf of ideals on a neighborhood of t, we can replace T by the subscheme associated to this ideal and we are reduced to $\mathfrak{a} = 0$. Thus it remains to show (1). Then we have to show that for any level $\lambda \in \mathbb{N}$
(1_λ) $\mathcal{O}_{Z_\lambda, z} \to \mathcal{O}_{Z_\lambda,\zeta}$
is injective. The latter follows from the proof of [RG, 2.2]. □

Now we come to the proof of 3.5: The proofs in [RG, 2.1 and 2.3] can be adapted to the formal case. So we will give only a rough outline.

Proceeding by induction it is only necessary to explain the first step where the dévissage is given only in dimension N. In the following we drop the index i.
(a)$\Leftrightarrow$(b) is easy and is left to the reader.
(b)$\Rightarrow$(c) Let $\mathscr{R} = \mathrm{Ker}(\alpha) \subset \mathscr{L}$ be the kernel of α. Since $\mathscr{P}_\zeta = 0$, we have an exact sequence

$$0 \to \mathscr{R}_\zeta \to \mathscr{L}_\zeta \to \mathscr{N}_\zeta \to 0 .$$

Because $\mathscr{N}_z$ is T-flat, $\mathscr{N}_\zeta = \mathscr{N}_z \otimes_{\mathcal{O}_{Z,z}} \mathcal{O}_{Z,\zeta}$ is also T-flat. So we obtain the exact sequence

$$0 \to \mathscr{R}_\zeta \otimes k(t) \to \mathscr{L}_\zeta \otimes k(t) \to \mathscr{N}_\zeta \otimes k(t) \to 0 .$$

Since $\alpha \otimes k(t)_\zeta$ is an isomorphism, we see $\mathscr{R}_\zeta \otimes k(t) = 0$ and by Nakayama's lemma $\mathscr{R}_\zeta = 0$. Thus α_ζ is bijective. Then α_z is universally injective by 3.6 and, hence, $\mathscr{P}_z$ is T-flat as $\mathscr{N}_z$ is T-flat.
(c)$\Leftrightarrow$(d) follows from 3.6.
(c)$\Rightarrow$(e) Again set $\mathscr{R} = \mathrm{Ker}(\alpha) \subset \mathscr{L}$. Since α_ζ is an isomorphism, there exists an open neighborhood U of ζ such that $\mathscr{R}|_U = 0$ and $\mathscr{P}|_U = 0$. The image of U in T is an open neighborhood V of t. For all points $v \in V_0$ the morphism α_v is an isomorphism at the generic point $v \in Z \otimes k(v)$ of the fibre of v. So the assertion follows from 3.6.
(e)$\Rightarrow$(d) is trivial.
(d)$\Rightarrow$(b) follows from the flatness of $Z \to T$ and the exact sequence

$$0 \to \mathscr{L}_z \to \mathscr{N}_z \to \mathscr{P}_z \to 0 .$$

□

Proposition 3.5 provides a means to characterize the relative dimension of the non-flat locus of a coherent $\mathcal{O}_X$-module by the data of a dévissage. By the flat locus of a coherent $\mathcal{O}_X$-module $\mathcal{M}$ over T we mean the set

$$F(\mathcal{M}/T) = \{x \in X_0;\ \mathcal{M}_x \text{ flat over } \mathcal{O}_{T,f(x)}\} \subset X_0 .$$

Due to the local criterion of flatness [FI, 1.6] one has

$$F(\mathcal{M}/T) = \bigcap\nolimits_{\lambda \in \mathbb{N}} F(\mathcal{M}_\lambda/T_\lambda)$$

where $F(\mathcal{M}_\lambda/T_\lambda)$ is the flat locus of $\mathcal{M}_\lambda$ over T_λ which is open due to [EGA, IV_3, 11.3.1]. By 3.4 and 3.5 we can show that even $F(\mathcal{M}/T)$ is open.

Corollary 3.7. *Let $f\colon X \to T$ be a formal S-morphism as above. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Then the flat locus $F(\mathcal{M}/T)$ is open.*

Proof. Consider a point $x \in F(\mathcal{M}/T)$ and set $t = f(x)$. Due to 3.3 there exists a commutative diagram

$$\begin{tikzcd}(X, x) \arrow[d, "f"] & (X', x') \arrow[l, "u"'] \arrow[d, "f'"] \\ (T, t) & (T', t') \arrow[l, "v"']\end{tikzcd}$$

where u and v are elementary étale affine neighborhoods such that $\mathcal{M}' = u^*\mathcal{M}$ admits a complete T'-dévissage at x'. It is clear that we have $F(\mathcal{M}/T) \times_T T' = F(\mathcal{M}'/T')$. Since $X'_0 \to X_0$ is open, it suffices to show that there is a neighborhood U' of x' such that $\mathcal{M}'$ is T'-flat over U'. This follows from 3.5(e), since the dévissage is complete. □

Moreover a dévissage of a coherent $\mathcal{O}_X$-module $\mathcal{M}$ at a point x in dimensions between N and n can be used to determine the flat locus $F(\mathcal{M}/T)$ at x up to a closed subscheme of the relative dimension n. There is the following result.

Corollary 3.8. *Let $f\colon (X, x) \to (T, t)$ be a pointed formal S-morphism. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module with $N = \dim_x(\mathcal{M} \otimes k(t))$. Let n be an integer with $0 \leq n < N$. Consider a commutative diagram*

$$\begin{tikzcd}(X, x) \arrow[d, "f"] & (X', x') \arrow[l, "u"'] \arrow[d, "f'"] \\ (T, t) & (T', t') \arrow[l, "v"']\end{tikzcd}$$

as in 3.3 such that $\mathcal{M}' = u^*\mathcal{M}$ admits a T'-dévissage $(D^1, \ldots, D^r)$ at x' in dimensions between $N = n^1$ and $n = n^r$ where

$$D^i = (g^i\colon (Y^i, y^i) \to (Z^i, z^i), \mathcal{M}^i, \alpha^i\colon \mathcal{L}^i \to \mathcal{N}^i = g^i_* \mathcal{M}^i, \mathcal{P}^i) .$$

Let $F(\mathcal{M}/T) \subset X$ be the open subscheme where $\mathcal{M}$ is T-flat. The following conditions are equivalent:

(a) $\dim_x((X_0 - F(\mathcal{M}/T)) \otimes k(t)) < n$,
(b) *α^i is bijective at the generic point ζ^i of $Z^i \otimes k(t')$ for all i,*
(c) *α^i is injective at z^i for all i,*
(d) *α^i is T'-universally injective over a formal neighborhood of t' for all i.*

Proof. We know $F(\mathcal{M}/Y)\times_T T' = F(\mathcal{M}'/T')$. Furthermore due to 3.5 we have

$$g^1(F(\mathcal{M}'/T')) = F(\mathcal{M}^1/T') .$$

Since $g^1\colon Y^1 \to Z^1$ is finite and $\{x'\} = (g^1)^{-1}(z^1)$ we have

$$\dim_x(F(\mathcal{M}/T)\otimes k(t)) = \dim_{z^1}(F(\mathcal{M}^1/T')\otimes k(t')) .$$

Now we proceed by induction on N. The case $N < 0$ is trivial. Due to 3.5 any of the condition (a)–(d) implies $\zeta^1 \notin F(\mathcal{M}^1/T')$ and, hence, after shrinking T' we may assume that α^1 is injective. Then we obtain $F(\mathcal{M}^1/T') = F(\mathcal{P}^1/T')$. Thus the assertion follows by induction. □

An $\mathcal{O}_X$-module is said to be *T-flat in dimension* $\geqq n$ at x if it satisfies the conditions of 3.8. For example if $\dim(X/T) \leqq n$ then X is T-flat in dimension $\geqq (n+1)$.

Next we want to introduce the notion of rig-flatness in dimension $\geqq n$ and to relate it to flatness in dimension $\geqq n$. Therefore we have to define the relative rig-dimension for a formal morphism. In the following by a rig-point of a formal scheme we always mean a locally closed rig-point in the sense of [FI, 3.1].

Definition 3.9. (a) Let X_K be a classical rigid space over a field K. For a point x of X_K the dimension of X_K at x is defined as the dimension of the local ring $\mathcal{O}_{X_K,x}$ at x. It is denoted by $\dim_x(X_K)$.
(b) For a formal morphism $X \to T$ and a (locally closed) rig-point x of X with image t in T, the relative dimension $\dim_x(X/T)$ is the dimension of the generic fibre of $(X\times_T t)$ at x. Note that the generic fibre of $(X\times_T t)$ is a rigid space in the classical sense and that x gives rise to a closed point of this space in a canonical way.

For a rigid space X_K of dimension n at a closed point x, there exists an open affinoid neighborhood $U_K = \mathrm{Sp}(A_K)$ of x such that U_K admits a finite surjective morphism to a ball of dimension n. Moreover the dimension of the localization of A_K at the maximal ideal of A_K which corresponds to x is n; [BGR, 7.3.2/8].

We mention the semi-continuity theorem for the fiber-dimension; a proof can be obtained by adapting the methods of Kiehl [K3] to our situation. The latter is possible if our schemes under consideration are universally catenarian. Therefore we assume that locally our schemes are closed subschemes of regular schemes; cf. [EGA, IV_2, 5.6.4].

Proposition 3.10. *Asssume that S locally is a closed subscheme of a regular scheme or that S is the formal spectrum of a valuation ring of height one.*

Let $f\colon X \to T$ be a formal S-morphism as above and let r be an integer. Then there is a closed subscheme Y of X with the following property:

A (locally closed) rig-*point x of X factors through Y if and only if* $\dim_x(X/T) \geqq r$.

Again by the method of Kiehl [K3] one shows the following fact.

Proposition 3.11. *Let $f\colon X \to T$ be a formal S-morphism as above. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Then there is a closed subscheme Y of X with the following property:*

A rig-*point x of X factors through Y if and only if $\mathcal{M}$ is not T-flat at x.*

The rigid space Y_{rig} associated to Y is uniquely determined and is called the locus where $\mathcal{M}$ is not rig-*flat over T.*

$\mathcal{M}$ is called rig-*flat over T in dimension* $\geqq n$ *if* $\dim_x(Y/T) \leqq (n-1)$ *for all* rig-*points of X.*

We want to mention that Propositions 3.10 as well as Proposition 3.11 are not necessary for showing the flattening theorem, but it may be helpful to know this fact.

Now we want to relate rig-flatness to flatness on the formal level at least in the special case where we consider a smooth morphism with geometrically reduced fibres. So let us fix the situation for the following: Let

$$\begin{array}{ccc} X & \hookleftarrow & Y \\ \Big\downarrow & & \downarrow \\ & & Z = \mathrm{Spf}(B) \\ & & \downarrow \\ T & \longleftarrow & T = \mathrm{Spf}(A) \end{array}$$

be a commutative diagram where $Y \subsetneq X$ is a closed immersion where

$$m = \dim(Y/T) = \dim(Z/T)$$

such that $\mathscr{M}$ is induced by a coherent $\mathcal{O}_Y$-module which is called $\mathscr{M}$ again. Let M be the module of global sections $\Gamma(Y, \mathscr{M})$. Assume that $\mathscr{M}$ admits a dévissage in dimension m over T, say

$$(g: Y \to Z, \mathscr{M}, \alpha: \mathscr{L} \to \mathscr{N}, \mathscr{P}) .$$

In particular $Z \to T$ is smooth with geometrically connected fibres of dimension m and $Y \to Z$ is finite. Set $N = \Gamma(Z, \mathscr{N})$.

Let t be a rig-point of A which is viewed as a prime ideal of A. Due to 2.4(a) the ideal $\zeta = Bt$ is a prime ideal which is the generic point of the fibre over t. Then we have the following result.

Proposition 3.12. *Keep the situation of above. The following conditions are equivalent*:

(a) $\mathscr{M}$ *is* rig-*flat over T in dimension m above t.*
(b) $\mathscr{N}$ *is* rig-*flat over T in dimension m above t.*
(c) N_ζ *is flat over* A_t.
(d) N_ζ *is free over* B_ζ.

Proof. (a)$\Leftrightarrow$(b) is clear, since M and N are equal as A-modules.
(b)$\Rightarrow$(c) Since $\mathscr{N}$ is flat above t in dimension m, there exists a point z above t such that $\mathscr{N}_z$ is flat over $\mathcal{O}_{T,t}$. So N_z is flat over A_t due to [FI, 5.3]. Henceforth, N_ζ is flat over N_z, because it is a localization of the latter.
(c)$\Rightarrow$(d) In our special situation $(B/tB)_\zeta$ is a field, so $(N/tN)_\zeta$ is free over $(B/tB)_\zeta$. The morphism $A_t \to B_\zeta$ is flat and local. Since t is a rig-point, the localizations A_t resp. B_ζ are localizations of A_{rig} and B_{rig}; so they are noetherian rings. Then it follows from [AC, III, par. 5, $n°4$, proposition 3] that N_ζ is B_ζ-flat and, hence, free over B_ζ.
(d)$\Rightarrow$(a) is clear, since $A_t \to B_\zeta$ is flat. □

Now we will relate rig-flatness of M over T in dimension $\geqq m$ to flatness of $\mathscr{M}$ over T in dimension $\geqq m$. Therefore we need the notion of Fitting ideals $F_r(M)$. More generally we define this notion for coherent modules.

Definition 3.13. Let X be a formal scheme and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. For an integer $r \in \mathbb{N}$ the *r-th Fitting ideal* $F_r(\mathcal{M})$ is a coherent ideal of $\mathcal{O}_X$ which is defined as follows:
On affine open subschemes U of X choose an exact sequence

$$\mathcal{O}_X^m|_U \xrightarrow{\varphi} \mathcal{O}_X^n|_U \longrightarrow \mathcal{M}|_U \longrightarrow 0$$

and set $F_r(\mathcal{M})(U)$ the ideal generated by the $(n-r)\times(n-r)$ minors of φ. One knows that this definition is independent of the chosen exact sequence for $\mathcal{M}|_U$ and that it is compatible with base change. Thus it is clear that the collection of the $F_r(\mathcal{M})(U)$ gives rise to a coherent sheaf $F_r(\mathcal{M})$ of ideals of $\mathcal{O}_X$ and that $F_r(\mathcal{M})$ is compatible with base change.

An important application of Fitting ideals in our context is the following lemma; for the definition of associated prime ideals in the non-noetherian case see [RG, 3.2.1].

Lemma 3.14 [RG, 5.4.3]. *Let M be a finitely presented A-module. Denote by $F_r(M)$ the r-th Fitting ideal of M. Assume that M is free of rank r at the points of* $\mathrm{Ass}(A)$. *If $F_r(M)$ is invertible, the quotient $M/\mathrm{Ann}_M(F_r(M))$ is locally free of rank r.*

Now it is easy to explain the relationship between rig-flatness in dimension $\geq m$ and flatness in dimension $\geq m$ for modules on a smooth T-scheme Z with geometrically connected fibres over T of dimension m.

Proposition 3.15. *Keep the situation of above. Assume that $\mathcal{M}$ is* rig-*flat in dimension m and that T has no $\mathfrak{I}$-torsion. Then consider the following conditions:*

(a) *$\mathcal{M}$ is flat over T in dimension $\geq m$.*
(b) *$\mathcal{N}$ is flat over T in dimension $\geq m$.*
(c) *The $\mathfrak{I}$-torsion of N is supported on a closed subscheme of relative dimension $\leq (m-1)$ over T_0.*

The following implications hold:

$$(a) \Leftrightarrow (b) \Rightarrow (c).$$

Since $\mathcal{N}$ is rig-*flat, the module N_ζ is a free B_ζ-module for the generic point ζ above any* rig-*point t of T. Assume that this rank is r (independent of t).*

If, in addition, the r-th Fitting ideal of N is invertible on a T-dense open subscheme of Z, all the conditions (a), (b), (c) *are equivalent.*

Proof. (a) $\Leftrightarrow$ (b) is contained in 3.5.
(b) $\Rightarrow$ (c) Since T has no $\mathfrak{I}$-torsion, a flat $\mathcal{O}_T$-module has no $\mathfrak{I}$-torsion either. Due to 3.7, the non-flat locus is closed, so the assertion is clear.
(c) $\Rightarrow$ (b) At the generic points of the rigid fibres, $F_r(N)$ coincides with B, since N is free over B at such points due to 3.12. Due to 2.3 the coherent ideal $F_r(N)$ has an ideal of coefficients $\mathfrak{a}$ which is open. Since $F_r(N)$ and $\mathfrak{a}$ coincide at the generic points of fibres of Z/T, we see that $F_r(N)$ is open on a T-dense open subscheme of Z. So, after replacing Z by a T-dense open subscheme, we may assume that $F_r(N)$ is invertible and open.

Since N has no $\mathfrak{I}$-torsion and since $F_r(N)$ is open, we have $\mathrm{Ann}_N(F_r(N)) = 0$. Since T has no $\mathfrak{I}$-torsion, the associated primes of B are the generic points of fibres of points $t \in \mathrm{Spec}(A)$ which do not belong to the special fibre T_0, due 2.4(b). Thus we see by 3.12 that N is free at all associated primes of B. Since $F_r(N)$ is invertible, N is a flat Z-module due to 3.14 and, hence, flat over T, as Z is flat over T. □

The last proposition describes precisely what has to be done for transforming a module $\mathcal{M}$, which is rig-flat over T in dimension $\geqq m$, into a module which is flat over T in dimension $\geqq m$. Namely, one has to blow-up the ideal of coefficients of $F_r(\mathcal{N})$ to make $F_r(\mathcal{N})$ invertible over a T-dense open subscheme of Z and to divide out the $\mathfrak{I}$-torsion of $\mathcal{N}$ and, hence, of $\mathcal{M}$.

4 The flattening technique

The main result of this section is the following theorem.

Theorem 4.1. *Let S be an admissible formal base scheme. Let T be a quasi-compact admissible formal S-scheme. Let $f: X \to T$ be a morphism of formal S-schemes of topologically finite presentation. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Let $n \in \mathbb{N}$ be an integer such that $\mathcal{M}$ is* rig-*flat over T in dimension $\geqq n$.*

Then there exists a formal blowing-up $T' \to T$ such that the strict transform $\bar{\mathcal{M}}'$ of $\mathcal{M}$ on $X' = X \times_T T'$ is T'-flat in dimension $\geqq n$. In particular, there exists an open subscheme U' of X'_0 with

$$\dim((X'_0 - U'_0)/T'_0)) \leqq (n-1)$$

such that

$$\bar{\mathcal{M}}'|_{U'} = \mathcal{M}'/(\mathcal{I}\text{-torsion})|_{U'}$$

where $\mathcal{M}'$ is the pull-back of $\mathcal{M}$ to X'.

A coherent $\mathcal{O}_X$-module $\mathcal{M}$ is called rig-flat over T in dimension $\geqq n$ if there exists a closed subscheme Y of X which is of finite presentation over X with $\dim(Y/T)_{\mathrm{rig}} \leqq (n-1)$ such that $\mathcal{M}|_{(X-Y)}$ is rig-flat over T; see 3.11. For example, if $\dim(X/T)_{\mathrm{rig}} \leqq (n-1)$ then $X \to T$ is rig-flat over T in dimension $\geqq n$.

Proceeding by induction on the relative dimension of the non-flat locus of $\mathcal{M}$ over T, it suffices to show the following proposition, since a composition of admissible formal blowing-ups is an admissible formal blowing-up again.

Proposition 4.2. *Assume a situation as in* 4.1. *Let m be an integer with $m \geqq n$ and assume that $\mathcal{M}$ is T-flat in dimension $\geqq (m+1)$. Then there exists a formal blowing-up $T' \to T$ such that the strict transform $\bar{\mathcal{M}}'$ of $\mathcal{M}$ on $X' = X \times_T T'$ is T'-flat in dimension $\geqq m$.*

Moreover, the center of the blowing-up is disjoint from the flat locus in dimension $\geqq m$ of $\mathcal{M}$.

Before we start with the proof we want to explain how the method works in a simple case.

Example 4.3. Let A be a ring and let $\varphi: A^m \to A^n$ be a linear map. Denote the cokernel of φ by M, so $M = \mathrm{coker}(\varphi)$. We may view φ as an $(n \times m)$-matrix with coefficients in A. For $\nu = 1, \ldots, N$ with $N = \min\{m, n\}$ consider the ideal $\mathfrak{d}_\nu$ of A generated by all $(\nu \times \nu)$-minors of φ.

If A is a principal domain, then due to the theory of elementary divisors we may assume that φ is of "diagonal" form with entries

$$\varphi_{ij} = a_i \delta_{ij} \quad \text{with} \quad (a_N) \subset (a_{N-1}) \subset \ldots \subset (a_1) .$$

Let t be the greatest integer with the property that $a_t \neq 0$ and set $r = n - t$. In such a situation, the cokernel is given by

$$M \cong A/(a_1) \times A/(a_2) \times \ldots \times A/(a_t) \times A^r = T \oplus A^r$$

where A^r is the free A-module of rank r and where T is a module of torsion. In this case we have

$$\mathfrak{d}_\nu = (a_1 a_2 \ldots a_\nu)$$

$$\mathfrak{d}_\nu = 0 \Leftrightarrow \nu \geqq t + 1 .$$

In particular, $M/\mathrm{Ann}_M(\mathfrak{d}_t)$ is free of rank $(n - t)$.

Now consider an arbitrary ring A, but assume that all non-zero ideals $\mathfrak{d}_\nu$ are invertible. Then, locally on $\mathrm{Spec}(A)$, one can transform the bases of A^m resp. of A^n as in the case of a principal domain such that the matrix associated to φ with respect to the new bases is of "diagonal" form. Thus we obtain in this case that $M/\mathrm{Ann}_{\mathfrak{d}_t}(M)$ is locally free of rank $(n - t)$ also.

In the general case, consider the blowing-up $X' \to X = \mathrm{Spec}(A)$ of all ideals $\mathfrak{d}_\nu$ which are not zero. So we obtain for the blowing-up X' a situation as discussed above. We see that the cokernel of a morphism $\varphi: \mathcal{O}_X^m \to \mathcal{O}_X^n$ of $\mathcal{O}_X$-modules becomes locally free after blowing up the non-trivial ideals of minors and by killing the torsion with respect to the ideal generated by the minors of biggest size.

The ideal $\mathfrak{d}_{n-r}$ is an invariant of the cokernel $M = \mathrm{coker}(\varphi)$ and it is the *Fitting ideal* $F_r(M)$; cf. 3.13. It is not necessary to consider the Fitting ideals $F_r(M) = \mathfrak{d}_{n-r}$ for r greater than the generic rank of M as seen by Lemma 3.14. □

Now we want to treat an important special case of the main theorem which will play a key role in the proof.

Special case 4.4. *Keep the assumption of* 4.2. *In addition, suppose that* $X = Z = \mathrm{Spf}(B) \to T = \mathrm{Spf}(A)$ *is smooth with irreducible geometric fibres of constant dimension* m. *Then the assertion of* 4.2 *is true.*

Proof. The $\mathcal{O}_Z$-module $\mathcal{M}$ is associated to a coherent B-module M. Let us consider the map of the ordinary spectra

$$\bar{Z} = \mathrm{Spec}(B) \to \bar{T} = \mathrm{Spec}(A) .$$

For $t \in \bar{T} - T_0$ the ideal $Bt \subset B$ is prime due to 2.4(a) and, hence, $\zeta = Bt$ is the generic point of the fibre over t. In particular $(B/tB)_\zeta$ is a field. Then $(M/tM)_\zeta$ is a free $(B/tB)_\zeta$-module; let $r(t)$ be its rank. Thus we obtain a function

$$r: (\bar{T} - T_0) \to \mathbb{N} , \; t \mapsto r(t) = \mathrm{rk}((M/tM)_\zeta) .$$

We want to show that r is locally constant. Since $\mathcal{M}$ is rig-flat over T in dimension $\geqq m$ where $m = \dim(Z/T)$, the module M_ζ is free over B_ζ of rank $r(t)$ due to 3.12. So there exists an element $b \in B - \zeta$ such that M_b is B_b-free of rank $r(t)$ if $\zeta = Bt$. Let $\mathfrak{a} \subset A$ be the ideal of coefficients of Bb; in the classical rigid case choose a finitely generated ideal $\mathfrak{a} \subset A$ such that $\mathfrak{a}_{\mathrm{rig}}$ is an ideal of coefficients of $(Bb)_{\mathrm{rig}}$; cf. 2.3. Clearly $t \in \bar{T} - V(\mathfrak{a})$ and for any $t' \in \bar{T} - T_0$ with $t' \notin V(\mathfrak{a})$ we have $r(t') = r(t)$. Indeed, let ζ' be the generic point above t'. Since $\mathfrak{a} \not\subset t'$ we have $b \notin Bt'$ as $\mathfrak{a}$ resp. $\mathfrak{a}_{\mathrm{rig}}$ is the ideal of coefficients of Bb resp. of $(Bb)_{\mathrm{rig}}$. Since $\zeta' = Bt'$ we see $b \notin \zeta'$. Thus $M_{\zeta'} = (M_b)_{\zeta'}$ is a free $B_{\zeta'}$-module of rank equal to $\operatorname{rank}(M_b) = r(t)$. This yields our claim.

Now due to [RG, 5.1.5], we can perform a formal blowing-up $T' \to T$ to reduce to the case where the function $r(t)$ is constant, say $r(t) = r$.

Then consider the r-th Fitting ideal $F_r(M)$. This is a finitely generated ideal of B. For the generic point $\zeta \in Z$ above $t \in \bar{T} - T_0$ we have $B_\zeta = F_r(M_\zeta)$ because M_ζ is free of rank r. In the noetherian case there exists an ideal of coefficients $\mathfrak{a} \subset A$ of $F_r(M)$ by 2.3(a). The ideal $\mathfrak{a}$ is open, since on any fibre of a point $t \in \bar{T} - T_0$ there exists a point ζ with $F_r(M)_\zeta = B_\zeta$ because $F_r(M_\zeta) = B_\zeta F_r(M)$. In the classical rigid case, the latter forces that there exists an (open) ideal of coefficients $\mathfrak{a} \subset A$ of $F_r(M)$ by 2.3(b) also.

Let $T' \to T$ be the admissible formal blowing-up of $\mathfrak{a}$. The center is disjoint from the flat locus in dimension $\geqq m$. Indeed, let M be T-flat at the generic point ξ over a point $t \in T_0$. Below we show that M_ξ is free over B_ξ and, hence, $F_r(M)_\xi = B_\xi$. Since $\mathfrak{a}$ is the ideal of coefficients of $F_r(M)$, we have $\mathfrak{a}A_t = A_t$ and, thus the center of $T' \to T$ is disjoint from the flat locus in dimension $\geqq m$.

Now let us show that M_ξ is free over B_ξ. We know that $(B/tB)_\xi$ is a field. So there exists an exact sequence

$$0 \longrightarrow K_\xi \longrightarrow B_\xi^r \xrightarrow{\alpha} M_\xi \longrightarrow 0$$

such that $\alpha \otimes k(t)$ is bijective. K_ξ is finitely generated over B_ξ. The latter is clear in the noetherian case. In the classical rigid case, we know that M_ξ is also flat over A and, hence, over the valuation ring R, as A is flat over R. So K_ξ is finitely generated due to [FI, 1.2(c)]. Since M is A-flat at ξ, the kernel of $\alpha \otimes k(t)$ vanishes at ξ and so $K_\xi/tK_\xi = 0$. By Nakayama's lemma we get $K_\xi = 0$. Thus, see that M_ξ is free over B_ξ.

After replacing T by the formal blowing-up $T' \to T$ of $\mathfrak{a}$, we may assume that $\mathfrak{a}$ is invertible; note that $F_r(M)$ and its ideal of coefficients are compatible with base change. Then $F_r(M)$ is invertible at the generic point ξ of any fibre of $\bar{Z} \to \bar{T}$; even for points $t \in T_0$. Namely we have $\mathfrak{a}B_\xi = F_r(M)B_\xi$, since $\mathfrak{a}$ is the ideal of coefficients of $F_r(M)$. Since $\mathfrak{a}$ is invertible and $A \to B$ is flat, $F_r(M)$ is invertible at ξ.

Due to 2.4 and by what we have proved so far M is free of rank r at all associated prime ideals of B; cf. 2.4(b). Since

$$\operatorname{Ann}_M(F_r(M))_\xi = \operatorname{Ann}_M(\mathfrak{a})_\xi$$

we see that $M/\operatorname{Ann}_M(\mathfrak{a})$ is free at the generic points of $Z_0 \to T_0$ due to 3.14. The latter means that the strict transform $\tilde{M}$ is flat in dimension $\geqq m$. □

The general case of 4.2 will be handled by dévissage and by reducing stepwise to the special case we just treated. At each step we need the following lemma.

Lemma 4.5. *Keep the situation of* 4.4. *Consider an exact sequence*

$$0 \to \mathscr{L} \to \mathscr{N} \to \mathscr{P} \to 0$$

of coherent $\mathcal{O}_Z$-modules where $\mathcal{L}$ is free and where $\dim(\mathcal{P}/T) < m$. *Denote by $\mathcal{T}(\mathcal{N})$ resp. $\mathcal{T}(\mathcal{P})$ the submodules of the $\mathcal{I}$-torsion in $\mathcal{N}$ resp. in $\mathcal{P}$ where $\mathcal{I} \subset \mathcal{O}_Z$ is the ideal of definition. Then*
(a) $\mathcal{L} \cap \mathcal{T}(\mathcal{N}) = 0$ *and, hence,* $\mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{P})$ *is injective,*
(b) $\mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{P})$ *is bijective,*
(c) $0 \to \mathcal{L} \to \mathcal{N}/\mathcal{T}(\mathcal{N}) \to \mathcal{P}/\mathcal{T}(\mathcal{P}) \to 0$ *is exact.*

Proof. We may assume $m \geqq 1$. Denote by $\bar{Z} = \operatorname{Spec}(B) \to \bar{T} = \operatorname{Spec}(A)$ the map of the ordinary spectra and denote by L, resp. N, resp. P the coherent B-modules associated to $\mathcal{L}$, resp. $\mathcal{N}$, resp. $\mathcal{P}$. Let $T(N)$ resp. $T(P)$ the submodules consisting of $\mathfrak{I}$-torsion.

(a) $\mathcal{L}$ has no $\mathcal{I}$-torsion, since Z has no $\mathfrak{I}$-torsion due to 2.4(b).
(b) Let W be the complement of the support of $T(P)$ and let $\iota: W \to \bar{Z}$ be the inclusion. Set

$$R = \operatorname{Ker}(N \to P/T(P)) .$$

Indicate by the superfix " $\sim$ " associated (algebraic) coherent modules on $\bar{Z}$. Let $\beta: \tilde{L} \to \tilde{R}$ be the inclusion. We obtain an isomorphism

$$\iota^*\beta: \iota^*\tilde{L} \simeq \iota^*\tilde{R} .$$

Since $\dim(\mathcal{P}/T) < m$ the open subscheme W contains all generic points of the fibres of $\bar{Z} \to \bar{T}$. Furthermore W contains $\bar{Z} - Z_0$. Due to 2.5(c) the canonical map $\tilde{L} \to \iota_*\iota^*\tilde{L}$ is bijective. Then look at the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{K} & \to & \tilde{R} & \xrightarrow{\tilde{\alpha}} & \iota_*\iota^*\tilde{R} \\
 & & & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle \iota_*\iota^*\beta} \\
 & & & & \tilde{L} & \xrightarrow{\sim} & \iota_*\iota^*\tilde{L} .
\end{array}$$

Thus we see that β has a section and, hence $T(P)$ has a representative in N; namely $K = \operatorname{Ker}(\alpha)$. The associated formal sheaf $\mathcal{K}$ of K is $\mathcal{T}(\mathcal{N})$ and, hence, $\mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{P})$ is bijective.
(c) follows from (a) and (b). □

Lemma 4.6. *Keep the assumptions of* 4.2. *Let x be a point of X. Let $(D^1, \ldots, D^r)$ be a T-dévissage of $\mathcal{M}$ at x in dimension $n^i \geqq m = n^r$ as in* 3.2, *say*

$$D^i = (g^i: (Y^i, y^i) \to (Z^i, z^i), \mathcal{M}^i, \alpha^i: \mathcal{L}^i \to \mathcal{N}^i = g^i_*\mathcal{M}^i, \mathcal{P}^i) .$$

After shrinking the pointed spaces
(a) α^i *is universally injective for $i \leqq (r-1)$; i.e., $n^i \geqq (m+1)$.*
(b) $\mathcal{N}^i$ *is rig-flat over T in dimension $\geqq n$ for all $i \leqq r$.*

Proof. (a) follows from 3.8, since $\mathcal{M}$ is T-flat in dimension $\geqq (m+1)$.
(b) Obviously $\mathcal{M}^1$ is rig-flat over T in dimension $\geqq n$ if $\mathcal{M}$ is. Since $\mathcal{M}^1$ and $\mathcal{N}^1$ are equal as $\mathcal{O}_T$-modules and since g^1 is finite, it it easy to see that $\mathcal{N}^1$ is rig-flat over T in dimension $\geqq n$. If $r \geqq 2$ we proceed by induction on r. Then it suffices to treat the case $r = 2$. Due to (a) we have the exact sequence

$$0 \to \mathcal{L}^1 \to \mathcal{N}^1 \to \mathcal{P}^1 \to 0 .$$

Since $\mathscr{L}^1$ is T-flat and α^1 is universally injective, we see that $\mathscr{N}^1$ is rig-flat over T in dimension $\geqq n$ if and only if $\mathscr{P}^1$ is. As above it follows that $\mathscr{M}^2$, which is equal to $\mathscr{P}^1$ as $\mathcal{O}_T$-module, is rig-flat in dimension $\geqq n$ and so $\mathscr{N}^2$ is also. □

Now we start with the proof of 4.2: Since T_0 is quasi-compact, it suffices to show the following assertion:

Lemma 4.7. *Keep the assumptions of* 4.2 *and consider a cartesian diagram*

$$\begin{array}{ccc} X & \longleftarrow & X(1) \\ \downarrow & & \downarrow \\ T & \longleftarrow & T(1) \end{array}$$

where $T(1) \to T$ is an admissible formal blowing-up. Let $\mathscr{M}(1)$ be the pull-back of $\mathscr{M}$ to $X(1)$ and let $\bar{\mathscr{M}}(1)$ be the strict transform of $\mathscr{M}(1)$. Let $U_0 \subset T_0$ be an open subset with $U_0 \neq T_0$ such that $\bar{\mathscr{M}}(1)$ is flat over $U_0(1)$ in dimension $\geqq m$ where $U_0(1) = U_0 \times_T T(1)$.

Then there exists an admissible formal blowing-up $T(2) \to T(1)$ with center disjoint from U_0 and an open subscheme W_0 of T_0 with $U_0 \subset W_0$ and $U_0 \neq W_0$ such that $\bar{\mathscr{M}}(2)$ is flat over $W_0(2)$ in dimension $\geqq m$. As above $\bar{\mathscr{M}}(2)$ is the strict transform of the pull-back $\mathscr{M}(2)$ of $\mathscr{M}$ to $X(2) = X \times_T T(2)$ and $W_0(2) = W_0 \times_T T(2)$.

Proof. Let t be a generic point of $T_0 - U_0$. We consider the locus of the fiber above t where $\mathscr{M}$ is not flat over T. We choose a generic point $x \in X_0$ of this locus of dimension m over t. It suffices to show that there exists an admissible formal blowing-up $T(2) \to T(1)$ such that $\bar{\mathscr{M}}(2)$ is $T(2)$-flat at all points above x. Due to the extension property of blowing-ups [FI, 2.5] it suffices to work locally around t. Due to 3.3 there exist elementary étale affine neighborhoods

$$\begin{array}{ccc} (X, x) & \xleftarrow{u} & (X', x') \\ \downarrow & & \downarrow \\ (T, t) & \longleftarrow & (T', t') \end{array}$$

such that there exists a dévissage of $\mathscr{M}' = u^*\mathscr{M}$ in dimension $\geqq m$ at x'.

We can reduce to the case where the dévissage exists already for $\mathscr{M}$ over T. Namely, if there exists a blowing-up $T(2)' \to T(1)' = T(1) \times_T T'$ of a coherent ideal $\mathscr{J}' \subset \mathcal{O}_{T(1)'}$ with center outside $U_0 \times_T T' = U(1)'_0$ satisfying the required properties, then the blowing-up is induced by a blowing-up $T(2) \to T(1)$ of a coherent ideal $\mathscr{J} \subset \mathcal{O}_{T(1)}$ after shrinking (T', t'). Indeed, the locus of $\mathscr{J}'$ is contained in $T(1)'_0 - U(1)'_0$. Since $(T', t') \to (T, t)$ is elementary étale, we may assume that the morphisms between the closures $\overline{t'} \to \overline{t}$ of the points in question is an open immersion. So $V(\mathscr{J}')$ is isomorphic to a subscheme of $T(1)$. Thus we see that $\mathscr{J}'$ is induced by some open ideal $\mathscr{J}$ of $\mathcal{O}_T$. It is clear that $\mathscr{M}'$ is rig-flat over T' in dimension $\geqq n$ if $\mathscr{M}$ is rig-flat over T in dimension $\geqq n$. Conversely, the T'-flatness of the strict transform $\bar{\mathscr{M}}(2)'$ descends to the T-flatness of the strict transform $\bar{\mathscr{M}}(2)$, since $T' \to T$ is étale. So we can drop all the primes.

Now consider a T-dévissage $(D^1, \ldots, D^r)$ of $\mathscr{M}$ at x in dimension $n^i \geqq m = n^r$ as in 3.2, say

$$D^i = (g^i\colon (Y^i, y^i) \to (Z^i, z^i), \mathscr{M}^i, \alpha^i\colon \mathscr{L}^i \to \mathscr{N}^i = g^i_* \mathscr{M}^i, \mathscr{P}^i)\,.$$

Working locally around x, we may assume by 4.6 that the morphisms α^i are T-universally injective for $i \leqq (r-1)$. All α^i are surjective at the generic point ζ^i of $Z^i \otimes k(t)$ for $i \leqq r$ due to Nakayama's lemma. So α^i are isomorphisms at ζ^i for $i \leqq (r-1)$.

After the base change $T(1) \to T$ we obtain a $T(1)$-dévissage of $\mathscr{M}(1)$ in dimension $\geqq m$ at every point above x in $X(1)$. Note that the morphisms α^i remain injective for $i \leqq (r-1)$ under this base change. From now on we work over $X(1) \to T(1)$, but we do not make this explicit in our notion for the dévissage. Now set

$$\bar{\mathscr{N}}^i = \text{strict transform of } \mathscr{N}^i \text{ with respect to } T(1) \to T$$

$$\bar{\mathscr{P}}^i = \text{strict transform of } \mathscr{P}^i \text{ with respect to } T(1) \to T$$

$$\bar{\mathscr{M}}^i = \text{strict transform of } \mathscr{M}^i \text{ with respect to } T(1) \to T\,.$$

It follows from 4.5 that $(\bar{D}^1, \ldots, \bar{D}^{r-1})$ is a T-dévissage in dimension $n^i \geqq (m+1)$ for $i \leqq (r-1)$ where we have set

$$\bar{D}^i = (g^i : (Y^i, y^i) \to (Z^i, z^i), \bar{\mathscr{M}}^i, \bar{\alpha}^i : \mathscr{L}^i \to \bar{\mathscr{N}}^i = g^i_* \bar{\mathscr{M}}^i, \bar{\mathscr{P}}^i)\,.$$

For $i = r$ it can happen that $\bar{\alpha}^r \otimes k(t)$ is not an isomorphism at the generic point. But we can still consider $\bar{\mathscr{N}}^r = g^r_* \bar{\mathscr{P}}^{r-1}$ as a coherent module on Z^r. Now $\bar{\mathscr{N}}^r$ is T-flat over U_0 in dimension $\geqq m$ due to 3.5 and rig-flat in dimension $\geqq n$ due to 4.6. Then we are in the situation of our special case 4.4. So there exists a formal blowing-up $T(2) \to T(1)$ with center in $T(1)_0 - U(1)_0$ such that

$$\begin{aligned} \bar{\mathscr{N}}^r(2) &= \text{strict transform of } \bar{\mathscr{N}}^r \text{ with respect to } T(2) \to T(1) \\ &= \text{strict transform of } \mathscr{N}^r \text{ with respect to } T(2) \to T \end{aligned}$$

is $T(2)$-flat in dimension $\geqq m$. Now we pull back the dévissage $(D^1, \ldots, D^{r-1})$ to $T(2)$ and obtain a $T(2)$-dévissage

$$(D^1, \ldots, D^{r-1}) \times_T T(2)$$

of the pull-back $\mathscr{M}(2)$ of $\mathscr{M}$ with respect to $T(2) \to T$. The homomorphisms $\bar{\alpha}^i$ remain injective for $i < r$ after the base change $T(2) \to T$. Taking strict transforms with respect to $T(2) \to T$ we obtain a $T(2)$-dévissage

$$(\tilde{\bar{D}}^1, \ldots, \tilde{\bar{D}}^{r-1})$$

of $\bar{\mathscr{M}}(2)$ as above due to 4.5. Now the strict transform $\bar{\mathscr{N}}^r(2)$ of $\mathscr{N}^r$ is $T(2)$-flat in dimension $\geqq m$. Due to 4.5 the $\mathscr{I}$-torsion of $\mathscr{N}^r(2)$ is induced by the $\mathscr{I}$-torsion of $\mathscr{N}^{r-1}(2)$ and, hence, by the $\mathscr{I}$-torsion of $\mathscr{M}(2)$. Thus we see that taking the strict transform of $\mathscr{M}$ effects to take the strict transform of $\mathscr{N}^r$. By 3.5 we see that $\bar{\mathscr{M}}(2)$ is T-flat at x in dimension $\geqq m$. This finishes the proof of 4.2. □

Thereby we have also nearly finished the proof of Theorem 4.1. It remains to show the additional claim. But it follows easily from the preceding one. Namely let $T' \to T$ be the blowing-up of an open coherent ideal $\mathscr{J}$ of $\mathcal{O}_T$. There exists an integer $n \in \mathbb{N}$ such that $\mathscr{I}^n \subset \mathscr{J}$, so

$$\mathrm{Ann}_{\mathscr{M}'}(\mathscr{J}) \subset (\mathscr{I}\text{-torsion of } \mathscr{M}')\,.$$

Since $\bar{\mathcal{M}}'$ has no $\mathcal{I}$-torsion over the T'-flat locus, we see that $\mathrm{Ann}_{\mathcal{M}'}(\mathcal{I})$ coincides with the $\mathcal{I}$-torsion over the T-flat locus of $\bar{\mathcal{M}}'$. The latter yields the claim. □

5 Existence of formal models

As before let us fix an admissible quasi-compact formal scheme S. We will consider morphisms $f: X \to Y$ of quasi-compact admissible formal S-schemes. We are looking for transformations of f without changing the associated rigid map f_{rig} in order to save certain properties of f_{rig} for a suitable model of f_{rig}. For example, in the case of flatness, such a procedure is provided by the main result 4.1 of the last section. As an easy example how the procedure of flattening works in the case of morphisms we want to mention the following example.

Example 5.1. Let A be a ring and consider a polynomial

$$P(\xi) = a_n\xi^n + a_{n-1}\xi^{n-1} + \ldots + a_0 \in A[\xi]\,.$$

Let S be $\mathrm{Spec}(A)$, let $\mathfrak{a} = (a_0, \ldots, a_n)$ be the ideal generated by the coefficients of $P(\xi)$, let X be the vanishing locus $V(P) \subset \mathbb{A}^1_S$ of P, and set $U = S - V(\mathfrak{a})$. Now conider the projection

$$f: X = V(P) \to S$$

to the base S. Over U the map f has finite fibres and over $V(\mathfrak{a})$ the fibres are of dimension 1.

If $\mathfrak{a}$ is principal, say $\mathfrak{a} = Aa$, we can write $P = aQ$ for a polynomial $Q \in A[\xi]$. In this case we have $U = S - V(a)$ and we can replace P by Q over U for describing X over U. The projection

$$f': V(Q) \to S$$

is quasi-finite everywhere. We can obtain the S-scheme $V(Q)$ also by killing the $\mathfrak{a}$-torsion of X.

In the general case, we perform the blowing-up $S' \to S$ of the ideal $\mathfrak{a}$. Over U the map $S' \to S$ is an isomorphism. But for the projection

$$f': X = X \times_S S' \to S'$$

we obtain a similar situation as discussed before when we work locally on S'. So, by killing the $\mathfrak{a}$-torsion in X', we obtain a closed subscheme X^* of X' such that $X^* \to S'$ is quasi-finite.

If $P \neq 0$ the condition of f being quasi-finite is equivalent to f being flat. Thus we have transformed the map f into a flat map by blowing up the non-flat locus $V(\mathfrak{a})$ and by killing the $\mathfrak{a}$-torsion. The latter can also be described by taking the strict transform of X under blowing up the non-flat locus.

As an example consider the polynomial ring $A = K[\zeta_1, \zeta_2]$ in two variables and the polynomial $P(\xi) = \zeta_1\xi + \zeta_2$. In particular $A[\xi]/(P)$ is an integral domain. In this case the projection is quasi-finite outside the origin and one has to blow up the ideal (ζ_1, ζ_2) in order to simplify the equation. □

So far we have dealt only with polynomials, but the procedure works also with restricted formal power series when one assumes that the equation has finite rigid fibres. Namely this implies that the ideal of coefficients is open and, hence, that only finitely many coefficients are involved. Now let us turn to the formal case. Theorem 4.1 implies the following result.

Theorem 5.2. *Let $f: X \to Y$ be an S-morphism of quasi-compact admissible formal S-schemes. Let $n \in \mathbb{N}$ be an integer. If f_{rig} is flat (resp. flat in dimension $\geqq n$), there exists an admissible formal blowing-up $Y' \to Y$ such that the induced morphism $f': X' \to Y'$ is flat (resp. flat in dimension $\geq n$) where X' is the strict transform of X with respect to $Y' \to Y$.*

Corollary 5.3. *Let $f: X \to Y$ be an S-morphism of quasi-compact admissible formal S-schemes.*

(a) *If $\dim(X/Y)_{\mathrm{rig}} \leq n$ there exists an admissible formal blowing-up $Y' \to Y$ such that the induced morphism $f': X' \to Y'$ has relative dimension $\leqq n$.*
(b) *If f_{rig} is quasi-finite, there exists an admissible formal blowing-up $Y' \to Y$ such that the induced morphism $f': X' \to Y'$ is quasi-finite.*

Proof. (a) Due to 5.2 there exists $Y' \to Y$ such that $f': X' \to Y'$ is flat in dimension $\geqq (n+1)$. For any formal point $T \to Y'$ from the formal spectrum T of a valuation ring to Y', the morphism $f'_T = f' \times_{Y'} T$ induced by base change is flat in dimension $\geqq (n+1)$. Since the generic fibre has dimension $\leqq n$, the special fibre has dimension $\leqq n$; the latter easily follows as in [EGA, IV_3, 14.3.10].
(b) follows from (a) with $n = 0$. □

Corollary 5.4. *Let $f: X \to Y$ be an S-morphism of quasi-compact admissible formal S-schemes.*

(a) *If f_{rig} is a flat monomorphism, there exists an admissible formal blowing-up $Y' \to Y$ such that the induced morphism $f': X' \to Y'$ is an open immersion. In particular, if the generic fibre of $f: X \to Y$ is a morphism of classical rigid spaces, f_{rig} is an open immersion.*

In the following assume that the generic fibre of $f: X \to Y$ is a morphism of classical rigid spaces.
(b) *If f_{rig} is an immersion (resp. an isomorphism), there exists an admissible formal blowing-up $Y' \to Y$ such that the induced morphism $f': X' \to Y'$ is an open immersion (resp. an isomorphism).*
(c) *If f_{rig} is an immersion (resp. closed immersion), there exists an admissible formal blowing-up $Y' \to Y$ such that the induced morphism $f': X' \to Y'$ is an immersion (resp. closed immersion).*

X' is always the strict transform of X with respect to $Y' \to Y$.

Recall that a morphism $f_K: X_K \to Y_K$ of rigid spaces is an *open* (resp. *closed*) *immersion* if f_K is an isomorphism of X_K onto an open (resp. closed) subvariety of Y_K. Furthermore f_K is an *immersion* if it is a composition of an open immersion and a closed immersion.

Proof. (a) Due to 5.2 we may assume that f is flat. Then we have to show that f is an open immersion. That f_{rig} is a monomorphism means that the projection

$$(X \times_Y X)_{\mathrm{rig}} \to X_{\mathrm{rig}}$$

is an isomorphism. Thus $X \times_Y X \to X$ which has a section and which is flat is an isomorphism, and thus $X \to Y$ is an isomorphism.

(b) An open immersion is a flat monomorphism, so the assertion follows from (a) in this case. If f_{rig} is an isomorphism, f_0 is surjective, since any rigid point of Y lifts to a rigid point of X. The proof in (a) showed that f' is an isomorphism.
(c) Due to (b) we may assume that f_{rig} is a closed immersion. Then we can replace Y by the closed subscheme Z of Y which is defined by the kernel of the map $\mathcal{O}_Y \to f_* \mathcal{O}_X$. Obviously f factors through Z and the induced rigid map $X_{\mathrm{rig}} \to Z_{\mathrm{rig}}$ is an isomorphism. Now the assertion follows from (b), since an admissible formal blowing-up of a subscheme can be induced by an admissible formal blowing-up of the ambient scheme. □

Corollary 5.4 can be viewed as a globalized version of the theorem of Gerritzen and Grauert which was already mentioned in [FI, 4.2]. Recall that the theorem of Gerritzen and Grauert says that an affinoid map $\varphi: X \to Y$, which is injective as a map of sets and which induces bijections $\mathcal{O}_{Y,\varphi(x)} \xrightarrow{\sim} \mathcal{O}_{X,x}$ on the stalks, is an open immersion and the image of φ is a finite union of rational subdomains. Note that the assumptions of the theorem of Gerritzen and Grauert [GG] imply that φ is a flat monomorphism. So it is less than we prove in 5.4(a). The statement 5.4(a) was originally a question of Tate [T]. In the following we will use 5.4 to give a more advanced proof for the existence of models for classical rigid spaces as already done in [FI, Sect. 4].

Theorem 5.5. *Let X_K be a classical rigid space assumed to be quasi-compact and quasi-separated. Let $\mathcal{U}_K$ be a finite covering of X_K by open affinoid subvarieties of X_K. Then there exists a formal model X of X_K such that $\mathcal{U}_K$ is induced by an open covering $\mathcal{U}$ of X.*

For the proof we need the following lemmata where we are always concerned with admissible formal schemes whose generic fibre is a classical rigid space.

Lemma 5.6. *Let X and Y be admissible formal S-schemes and assume that X and Y are quasi-compact. Let $f_K: X_{\mathrm{rig}} \to Y_{\mathrm{rig}}$ be a classical rigid morphism. Then there exists a formal blowing-up $X' \to X$ such that f_K is induced by a morphism $f': X' \to Y$.*

Proof. First assume that $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(B)$ are affine. In this case f_K corresponds to a morphism $B \otimes_R K \to A \otimes_R K$. Let Γ_K be the graph of f_K in $\mathrm{Spec}(A \hat{\otimes}_R B) \otimes_R K)$. Let $\Gamma \subset \mathrm{Spec}(A \hat{\otimes}_R B)$ be the schematic closure of Γ_K. Then Γ can be viewed as a model of Γ_K; use [FI, 1.1(c)] The projection $p_1: \Gamma \to X$ is a rig-isomorphism. After replacing X by an admissible formal blowing-up and taking the strict transform of Γ, we may assume due to 5.4(c) that the projection $p_1: \Gamma_{\mathrm{rig}} \to X_{\mathrm{rig}}$ extends to an isomorphism from Γ to X. Then $p_2 \circ (p_1|_\Gamma)^{-1}$ yields a model for f_K.

Due to [FI, 2.6], any admissible formal blowing-up $U' \to U$ of an open subscheme U of X can be extended to an admissible blowing-up of X and finitely many admissible formal blowing-ups can be dominated by an admissible blowing-up. So there exists a finite covering by open subschemes X^i for $i = 1, \ldots n$ and an admissible blowing-up $X' \to X$ such that $f_K|_{X^i_{\mathrm{rig}}}$ is induced by a morphism $f^i: X' \times_X X^i \to Y$ for $i = 1, \ldots, n$. Since f^i is uniquely determined by $f_K|_{X^i_{\mathrm{rig}}}$, these morphisms fit together to build a morphism $f: X' \to Y$. □

Lemma 5.7. *Let X be a quasi-compact admissible formal scheme and let U_K be a quasi-compact open rigid subspace of X_{rig}. Then there exists an admissible formal blowing-up $X' \to X$ and an open subscheme U' of X' such that U_K is induced by U'.*

Proof. If $U_K = \mathrm{Sp}(A_K)$ is affinoid, U_K has a model $U = \mathrm{Spf}(A)$ where A is a flat R-algebra A of topologically finite type with $A_K = A \otimes_R K$. Due to 5.6 and 5.4(c), there exists a blowing-up $X' \to X$ such that the open immersion $U_{\mathrm{rig}} \hookrightarrow X_{\mathrm{rig}}$ is induced by an open immersion $U' \hookrightarrow X'$. Since the given U_K can be covered by finitely many affinoid subspaces, the assertion is clear. □

Lemma 5.8. *Let U^i be quasi-compact admissible formal S-schemes for $i = 1, 2$. Let $V^i \subset U^i$ be open subschemes for $i = 1, 2$ such that there is a rigid isomorphism $\varphi_K \colon V^1_{\mathrm{rig}} \xrightarrow{\sim} V^2_{\mathrm{rig}}$. Let X_K be the rigid space obtained by gluing U^1_{rig} and U^2_{rig} along φ_K. Then there exist a formal blowing-up $\tilde{U}^i \to U^i$ for $i = 1, 2$ such that φ_{rig} extends to a formal isomorphism $\varphi \colon \tilde{V}^1 \to \tilde{V}^2$ where $\tilde{V}^i = \tilde{U}^i \times_{U^i} V^i$ for $i = 1, 2$. The gluing of $\tilde{U}^1$ and $\tilde{U}^2$ along φ yields a formal model of X_K.*

Proof. Due to 5.6 and [FI, 2.6] we may assume that φ_K extends to a morphism $\varphi \colon V^1 \to V^2$. Since φ_{rig} is an isomorphism, the assertion follows from 5.4(b) and [FI, 2.6]. □

Now we come to the *proof of 5.5*. Since X_K is quasi-compact, there exists a finite covering $\{U^1_K, \ldots, U^r_K\}$ by open affinoid subspaces U^i_K of X_K. Each U^i_K obviously admits a model U^i. So we may start with a finite open covering $\{U^1_K, \ldots, U^r_K\}$ of X_K and assume that each U^i_K admits a model U^i for $i \geqq 1$. Now proceeding by induction on r, we are reduced to the case $r = 2$.

The intersection $U^1_K \cap U^2_K$ is quasi-compact, since X_K is assumed to be quasi-separated. Due to 5.7 there exist admissible blowing-ups $\tilde{U}^i \to U^i$ such that $U^1_K \cap U^2_K$ is induced by an open subscheme of $\tilde{U}^1$ resp. by $\tilde{U}^2$. So we may assume that $U^1_K \cap U^2_K$ is induced by an open subscheme V^1 of U^1 and by an open subscheme V^2 of U^2. Now the assertion follows from 5.8. □

Corollary 5.9. *In the classical rigid case, the functor*

$$(\text{Formal } R\text{-schemes}) \to (\text{Rigid spaces}),\ X \mapsto X_{\mathrm{rig}}$$

gives rise to an equivalence of categories

$$(\text{Formal } R\text{-schemes})/(\text{blowing-up}) \xrightarrow{\sim} (\text{Rigid } K\text{-spaces})\,.$$

Proof. It follows from [FI, 2.1] that an admissible formal blowing-up induces a rigid isomorphism; see also Sect. 2. Due to 5.5 and 5.6 the functor is an epimorphism. Due to 5.4(b) it is an equivalence. □

Corollary 5.10. *Let $f_K \colon X_K \to Y_K$ be a morphism of quasi-compact, quasi-separated classical rigid spaces.*
(a) *There exists a model $f \colon X \to Y$ of f_K.*
(b) *If f_K has fibres of dimension $\leqq n$, one can choose f in such a way that f has fibres of dimension $\leqq n$.*
(c) *If f_K is flat, one can choose f to be flat.*
(d) *If f_K is an immersion (resp. closed, resp. open immersion), one can choose f to be an immersion (resp. closed, resp. open immersion).*

Proof. (a) follows from 5.9 and (b), (c), resp. (d) follow from 5.3, 5.2 resp. 5.4. □

Corollary 5.11. *Let $f_K \colon X_K \to Y_K$ be a morphism of quasi-compact classical rigid spaces. If f_K is flat, the image of f_K is a finite union of open affinoid subvarieties of Y_K. In particular, f_K is open.*

Proof. Due to 5.10 there exists a flat formal morphism $f: X \to Y$ inducing f_K. Since f_0 is flat and of finite presentation, the image of f_0 is an open subscheme V_0 of Y_0; cf. [EGA, IV, 2.4.6]. Let V be the open subscheme of Y associated to V_0. Then f factors through V and $f: X \to V$ is faithfully flat. Thus we see that the image of f_{rig} equals V_{rig}. Since V_0 is a finite union of affine open subschemes, V_{rig} is a finite union of affinoid subvarieties of Y_{rig}. □

Now assume that the valuation of R is discrete, and consider a formal morphism $f: X \to Y$. It is shown in [L, 3.1] that the morphism f_{rig} is proper if and only if f is proper. It is not difficult to see that f is proper if f_{rig} is proper; but the converse is difficult because one has to show a contraction theorem which requires the construction of certain maps. We mention that there is a much more general result in [L, 5.1] which reflects precisely the geometric meaning of properness in rigid geometry.

At least in the case where the valuation is discrete, the existence of formal models provides a means for showing the basics of rigid geometry as laid by Kiehl [K1] and [K2] by referring to [EGA] without any further efforts; cf. [L Sect. 2]. As we have seen in the present paper, using [EGA] and the methods of [RG] only a few further results of p-adic anlysis are needed to lay the foundations of rigid geometry. Moreover these techniques allow to achieve powerful results which are beyond the scope of the classical methods; for example, see 5.11 or [L, 5.1].

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