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Werner Lütkebohmert

Rigid Geometry of Curves and Their Jacobians

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En hommage à Michel Raynaud

Preface

Projective algebraic curves or abelian varieties are defined as the vanishing locus of finite families of homogeneous polynomials in a projective space fulfilling certain conditions. Except for elliptic curves or hyperelliptic curves, it is difficult to pin down equations which give rise to curves or abelian varieties.

Over the complex numbers one has analytic tools to construct and to uniformize such objects. For example, every smooth curve of genus $g \geq 2$ has a representation $\Gamma \backslash \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane and $\Gamma \subset \mathrm{Aut}(\mathbb{H})$ is a group acting on $\mathbb{H}$. Similarly, every compact complex Lie group is of type $\mathbb{C}^n/\Lambda$, where Λ is a lattice in $\mathbb{C}^n$; the abelian varieties among the compact complex Lie groups can be characterized via polarizations. Moreover, one can construct curves and abelian varieties in this way via algebraization of the analytic quotients. Thus, the geometry and the construction of such objects are completely clarified.

Over a complete field K with respect to a non-Archimedean valuation, one can expect similar tools as in the complex case once a good theory of holomorphic functions has been established.

Historically, the theory started with the simplest case of an elliptic curve over K. One can define the elliptic curve by a minimal Weierstraß equation with integral coefficients. If this equation reduces to an elliptic curve over the residue field, we say that the given elliptic curve has good reduction. In this case there is no uniformization at all; such curves can be regarded as liftings of elliptic curves defined over the residue field. On the other hand, if the Weierstraß equation reduces to a cubic with an ordinary double point, then the situation looks better from the viewpoint of uniformization. As an abstract group its K-rational points are represented by a quotient $K^\times/q^\mathbb{Z}$ for some non-integral $q \in K^\times$ without any further structure. Originally Tate wanted to construct "analytic" quotients $\mathbb{G}_{m,K}/q^\mathbb{Z}$ of the multiplicative group of a non-Archimedean field K by the lattice $q^\mathbb{Z}$; a construction which cannot be carried out in the category of ordinary schemes directly.

Thus, there was the desire to create a theory of "analytic spaces" over a non-Archimedean field which allows such constructions. This was exactly the incentive of Tate to understand elliptic curves with multiplicative reduction by "analytic" means. In 1961 Tate gave a seminar at Harvard where he developed a theory of rigid analytic spaces; cf. [92].

Later on, using methods from formal algebraic geometry, Mumford generalized the construction of Tate's elliptic curve to curves of higher genus [75] – nowadays called Mumford curves – as well as to abelian varieties with split torus reduction [76]. Moreover, Mumford's constructions even work over complete Noetherian rings of higher dimension.

The relationship between formal algebraic geometry and rigid geometry was clarified by Raynaud in [80]. As a sort of reverse, Raynaud worked on the rigid analytic uniformization of abelian varieties and their duals over non-Archimedean fields [79].

The ideas of Mumford and Raynaud were picked up by Chai and Faltings and generalized to abelian varieties with semi-abelian reductions over fields of fractions of complete Noetherian normal rings of higher dimension. Whereas in the rigid analytic context, the periods of the uniformization enter the scene quite naturally even in the absence of a polarization, Chai and Faltings made the observation that the periods are encoded in the coefficients of the theta function associated to a principal polarization, in analogy to the complex case. So, for them it was not necessary to invoke rigid geometry.

Nevertheless, rigid geometry is a means to unfold the geometric ideas behind the formal constructions used by Mumford, Chai and Faltings. The results on uniformization and construction provide a method to parameterize polarized abelian varieties and their semi-abelian degeneration in a universal way. So, they became the essential ingredients for the construction of a toroidal compactification of the moduli space of polarized abelian varieties by Chai and Faltings; cf. [27].

This book thoroughly treats the main results on rigid geometry and their applications as they grew out of the notes of Tate. The focus of this book lies on the arithmetic geometry of curves and their Jacobians over non-Archimedean fields.

After an introduction to rigid geometry in Chap. 1, we directly concentrate on the main topic. Following ideas of Drinfeld and Manin [64], Mumford curves are treated in Chap. 2 via classical Schottky uniformization. Their Jacobians are rigid analytic tori which are constructed by automorphic functions. This is explained on an elementary level. Thus, we achieve the rigid analytic counterpart of the fascinating theory of Riemann surfaces and their Jacobians. The remainder of the book (Chaps. 3 to 7) deals with smooth rigid analytic curves and their semi-stable reductions or with proper smooth rigid analytic group varieties and their semi-abelian reductions. The intention here is to comprehensively present the rigid analytic uniformization and construction of curves and their Jacobians or of abelian varieties over non-Archimedean fields. Moreover, the structure of abeloid varieties, which are the counterparts of compact complex Lie groups, is presented in details.

The reader is assumed to be familiar with basic algebraic geometry in the style of Grothendieck and with standard facts about abelian varieties. The reader can consult [15, Chaps. 2 and 9], [60] and [74].

Since there are several books which deal with the foundations of rigid geometry, cf. [1, 9, 10], there is no need to develop it again. Therefore, the prerequisites on classical rigid geometry are only surveyed in Chap. 1 without giving proofs. In the same way the basic results on the relation between formal and rigid geometry are handled in Chap. 3, as they are presented in [14] and were revisited a few years ago in [1]. For the basic theory of formal and rigid geometry the reader may also consult [9] where it is carefully explained. There are other foundations of non-Archimedean analysis by Berkovich [6] and Huber [47], but these are not involved in this book. So, we concentrate on the main applications which are not touched or only partially studied in other books; cf. [30] and [35]. Compared to the existing literature, many proofs have been substantially improved and some new results have been added.

It is a pleasure for me to express my gratitude to my students Sophie Schmieg and Alex Morozov for proofreading and comments. Also I would like to thank colleagues, including Siegfried Bosch, Barry Green, Urs Hartl, Dino Lorenzini, Florian Pop, Stefan Wewers, for discussions and valuable suggestions. I am especially indebted to Ernst Kani, who helped me to edit the manuscript.

In particular I am glad to acknowledge here the extraordinary help from Michel Raynaud, who contributed many ideas to this book.

Münster, Germany Werner Lütkebohmert
September 2015

Contents

Introduction

Given a smooth projective curve over the field of fractions K of a discrete valuation ring R, one can find equations with coefficients in R which define the curve. Hereby it can happen that even the best possible choice of equations does not behave well under reduction to the residue field of R in the sense that the reduction of the equations no longer define a smooth curve over the residue field of R. After a suitable extension of R two types of an optimal reduction occur; either the reduction is a smooth curve or it is a configuration of smooth irreducible curves of arbitrary genus intersecting in ordinary double points.

In the first case, there is neither a uniformization nor a construction principle; but from the arithmetic point of view one has the best possible situation. In the second case, rigid geometry can analyze the structure and describe how such situations show up and how the combinatorial configuration is reflected by the structure of the associated Jacobian.

Whereas for the case of elliptic curves a simple theory of analytic functions in one variable would suffice, this is not the case for curves of higher genus. A good theory of rigid geometry should be general enough to cover analogs of Riemann surfaces and the analytification of schemes of finite type over K in the sense of Serre's Géométrie Algébrique et Géométrie Analytique.

The problem is that the topology associated to a non-Archimedean valuation of a field K is totally disconnected, and so the functions, which are only locally analytic, fail to satisfy global properties like the identity principle and the global expansion on polydiscs. Therefore, one has to require extra structures to define a (good) notion of holomorphic functions.

In 1961 Tate introduced such an extra structure and overcame the disconnectedness of the topology of a non-Archimedean field. Thus, he saved the analytic continuation and the identity principle over totally disconnected ground fields, hence *making the impossible possible*, as Remmert said. Tate himself writes at the beginning of [92, §10], where he introduces *rigid analytic spaces*, that his intent is to "*follow fully and faithfully a plan furnished by Grothendieck*". Later on, the theory was worked out by the school of Grauert and Remmert and by Kiehl; see the monograph [10].

Tate's uniformization of elliptic curves with non-integral j-invariant was presented in detail by Roquette [85]. A certain generalization of Tate's construction of elliptic curves to the case of higher dimensions was given by Morikawa [71].

Then in the early 1970's Tate's uniformization of elliptic curves was generalized by Mumford in two ways, on the one hand to curves of higher genus having split degenerate reduction [75] and on the other hand to abelian varieties of arbitrary dimensions with multiplicative reduction [76]. Furthermore, by adapting the classical construction of Schottky to rigid geometry, Drinfeld und Manin studied an analytic construction of projective algebraic curves in [64], which are quotients of domains in the projective line by a finitely generated free subgroup of $\mathrm{PGL}(2, K)$. Nowadays these curves are called *Mumford curves*. It is worth mentioning that Mumford worked over complete Noetherian rings and constructed families of curves and of abelian varieties of arbitrary dimensions [76], whereas Raynaud studied the rigid analytic uniformization of abelian varieties over non-Archimedean fields [79].

Over a non-Archimedean field with a non-discrete valuation, Mumford curves and totally degenerating abelian varieties were thoroughly investigated from the rigid analytic point of view by Gerritzen [32] and [33] with the main emphasis on the work of Drinfeld and Manin. The more general case of smooth projective curves and of abelian varieties was settled by Bosch and the author in [11–13].

It is worth noting that Tate's elliptic curve was an important tool for Deligne and Rapoport for studying the points at infinity of the arithmetic moduli scheme of elliptic curves [22]. Raynaud gave a geometric construction of the general Tate curve over the power series ring $\mathbb{Z}[[q]]$ in terms of formal schemes; cf. [22, Sect. VII]. While Mumford worked exclusively in the framework of formal algebraic geometry, Raynaud presented a program [80], which explained the connection between formal and rigid geometry. Hereby he explains the geometry behind the constructions of Mumford. Details of Raynaud's program were worked out by Mehlmann [67] as well as by Bosch and the author in [14]. It should be mentioned that the uniformization of semi-abelian varieties is a basis for the construction of the toroidal compactification for the moduli space of abelian varieties by Chai and Faltings [27]; cf. the discussion in Sect. 6.6.

The main purpose of this book is to rigorously analyze the uniformization of projective smooth curves and their Jacobians over a non-Archimedean field. Furthermore, we will relate the canonical polarization on the Jacobian to the geometric data given by the uniformization of the curve. The rigid analytic topology reflects the combinatorial configuration of the irreducible components of the stable reduction, and hence the multiplicative part of the reduction of its Jacobian is also related to it. One is automatically led to the analysis of abelian varieties and finally to abeloid varieties; cf. [62]. The rigid uniformization of abelian varieties is much easier to handle and does not require the technical work of Sects. 3.6 and 7.5. The results on abeloid varieties were conjectured by Raynaud who also gave some hints on the strategy for the proof. All these results have their roots in the foundation of rigid geometry which was proposed by Raynaud in 1974 at the "Table ronde d'analyse non-archimedienne"; cf. [80]. Of course, most of the applications are concerned only with the case of abelian varieties, but nevertheless one wants to see the complete picture of rigid uniformization for curves and for proper rigid analytic groups over a non-Archimedean field as it exists in complex analysis. An overview of the rigid analytic theory can also be found in [63].

Before discussing the contents of the book, it is worthwhile to remind the reader of the theory over the complex numbers. Over the complex numbers $\mathbb{C}$, a compact Riemann surface X of genus $g = 1$ has $\mathbb{C}$ as a universal covering and $\mathbb{Z} \oplus \mathbb{Z}\tau$ as Deck transformation group, so $X \simeq \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$. Moreover, X can also be represented as a quotient $X \cong \mathbb{C}^\times/M$ of the affine torus by a multiplicative lattice M of rank 1, by applying the map $z \mapsto \exp(2\pi\imath z)$.

If X has genus $g \geq 2$, a compact Riemann surface is hyperbolic. Its universal covering is the hyperbolic half plane $\mathbb{H}$ and X admits a representation $X \cong \mathbb{H}/\pi_1(X)$, where $\pi_1(X) \subset \mathrm{Aut}(\mathbb{H}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ is generated by $2g$ generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ with a single relation $\prod_{i=1}^{g}[\alpha_i, \beta_i] = 1$. Due to Abel's theorem the Jacobi variety of a compact Riemann surface has a canonical representation as a complex torus

$$\mathrm{Jac}\, X = \Gamma\big(X, \Omega^1_{X/\mathbb{C}}\big)'/H_1(X, \mathbb{Z}),$$

where $\Gamma(X, \Omega^1_{X/\mathbb{C}})'$ is the dual of the vector space of the global differential 1-forms, and where the cycle group $H_1(X, \mathbb{Z})$ is embedded into the dual by integrals along the closed cycles. The cup-product on cycles leads to Riemann's period relations which endow $\mathrm{Jac}\, X$ with a polarization. Thus, $\mathrm{Jac}\, X$ becomes a proper algebraic group variety; i.e. an abelian variety.

Every complex compact Lie group is isomorphic to $\mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice of rank $2g$. The algebraic ones among them are such Lie groups which admit a polarization. The latter means that there exists a positive definite hermitian form H on $\mathbb{C}^g$ such that its imaginary part $\mathfrak{Im}(H)$ takes values in $\mathbb{Z}$ on the lattice.

For the special class of Mumford curves one has the complete list of analogous results as in the complex case of Riemann surfaces. These results can be achieved by the methods of classical rigid geometry. The classical theory of rigid geometry was worked out in the late 1960's by researchers inspired from the analogy in complex analysis. This somehow restrictive view on rigid geometry is completely sufficient to understand Tate's elliptic curve from the geometric point of view, whereas Tate originally studied it in the context of function fields; cf. Sect. 2.1.

Here Tate's elliptic curves appear in the following way. One considers the field of fractions of the ring of Laurent series $\sum_{n\in\mathbb{Z}} c_n\zeta^n \in K[[\zeta, 1/\zeta]]$, which converge globally on $K^\times$, and its subring of functions, which satisfy the equation $f(q\zeta) = f(\zeta)$ for a fixed non-integral $q \in K^\times$. The latter is a function field $F(q)$ over K of genus 1. Moreover, two such function fields $F(q_1)$ and $F(q_2)$ are K-isomorphic if and only if $q_1 = q_2$.

More generally, one can also deal with discontinuous group actions Γ on subdomains Ω of the projective line in the style of Schottky [86]. The construction of the quotient $\Gamma\backslash\Omega$ and the construction of their Jacobians can be carried out in the framework of classical rigid geometry as well, because only simple subdomains of the projective line and of the affine torus $\mathbb{G}^g_{m,K}$ are involved. The advantage of rigid geometry is that the construction of the quotient can be carried out by geometric means, whereas Mumford explained a construction of such quotients in the framework of formal schemes, however in a much more general context.

Chapter 2 can thus be viewed as a counterpart of Riemann surfaces and their Jacobians. We provide the full picture of Mumford curves and their Jacobians which are rigid analytic tori. We also present the duality theory of rigid analytic tori. This enabled us to verify Riemann's period relations and even Riemann's vanishing theorem. The theory of vector bundles on Mumford curves due to Faltings [26] is not touched. Also, we do not present the result of Herrlich [46] on the size of the automorphism group of a Mumford curve, whose cardinality is surprisingly bounded by $12(g-1)$ if $g \geq 2$. We remind the reader that only a "small" part of the family of smooth projective curves consists of Mumford curves.

The remainder of the book deals with smooth rigid analytic curves and their semi-stable reduction and with proper smooth rigid analytic group varieties. The main new feature are rigid analytic spaces, which contain open subvarieties admitting smooth formal R-models over the valuation ring R with a non-rational reduction. Since such subvarieties behave like simply connected domains from the geometric point of view, formal analytic structures become unavoidable to understand their geometry. Raynaud's result on the relationship between rigid spaces and formal R-schemes of topological finite presentation paves the way to go beyond Mumford curves. Thus, a main point is to clarify how subdomains with good reduction are glued to build a rigid analytic space. The stable reduction theorem in Chap. 4 shows that, in the case of curves, the connection between such parts are defined via annuli. In the case of groups the situation is given by polyannuli of a certain type.

In Chap. 3 we explain the foundation of formal and rigid geometry. The central result is Theorem 3.3.3 of Raynaud. It states that any quasi-compact, quasi-separated rigid analytic space is the generic fiber of an admissible formal scheme. As a major result we show the Relative Reduced Fiber Theorem 3.4.8 that a flat morphism of affinoid spaces with geometrically reduced fibers admits a formal R-model with reduced fibers after a suitable base change. This is a deep result and has a long history. If the base field K is the field of fraction of discrete valuation ring, then this was settled by Epp [25]. If K is an algebraically closed non-Archimedean field, this was proved by Grauert and Remmert [36]. Bosch, Raynaud and the author treated the relative case [14, Part IV]. In particular, this is a first step to provide a semi-stable R-model of a curve in Theorem 4.4.3 and of a curve fibration in Theorem 7.5.2 as well.

The last Sect. 3.6 is designed to provide new methods on approximation which are only used in Chap. 7. This part is deeply related to the essence of properness of rigid analytic spaces and to Elkik's method on approximation of solutions of equations over restricted power series.

One of the main objectives of Chap. 4 is the description of the "boundary" of the formal fiber of a formal analytic curve in Proposition 4.1.11. It gives a precise description of how the interior of the formal fiber is connected to the remaining part of a curve. This is a cornerstone of the stable reduction theorem for projective curves in Theorem 4.4.3. It is remarkable to point out that we do not make use of the desingularization result of surfaces [59] as the usual proofs do in [5] or [21].

In Sect. 4.2 the result on the boundary is used to establish a genus formula in Proposition 4.2.6, which relates the genus of a projective rigid analytic curve to geometric data of the reduction. This formula allows us to define the genus of a formal fiber, which serves as a measure for the quality of the singularity in the reduction. From these results one deduces in Sect. 4.3 the stable reduction theorem for smooth projective curves by studying the behavior of meromorphic functions. Blowing-up and blowing-down of components in the reduction can easily be handled by changing formal analytic structures. Finally, the stable reduction theorem leads in Sect. 4.6 to a construction of a universal covering of a curve. In the case of split reduction the universal covering can be embedded into the projective line and its deck transformation group is a subgroup of $\mathrm{PGL}(2, K)$; this is, in fact, a Schottky group.

Chapter 5 begins with a survey on Jacobians of smooth projective curves with an emphasis on the autoduality of Jacobians. Moreover, for our purpose it is necessary to analyze the generalized Jacobian of a semi-stable curve $\widetilde{X}$, especially its representation as a torus extension of the Jacobian of the normalization $\widetilde{X}'$ of $\widetilde{X}$. The relationship between the torus part and the cycle group $H_1(\widetilde{X}, \mathbb{Z})$ is clarified in Proposition 5.2.3.

In the last part of Chap. 5 we consider a smooth projective curve X_K with a semi-stable reduction $\widetilde{X}$. In Sect. 5.3 it is shown that the generalized Jacobian $\widetilde{J} := \mathrm{Jac}\,\widetilde{X}$ has a lifting $\overline{J}_K$ as an open subgroup of $J_K := \mathrm{Jac}\,X_K$ and $\overline{J}_K$ has a smooth formal R-model $\overline{J}$ with semi-abelian reduction. The formal group scheme $\overline{J}$ is a formal torus extension of a formal abelian R-scheme B with reduction $\widetilde{B} = \mathrm{Jac}\,\widetilde{X}'$. The generic fiber $\overline{J}_K$ of $\overline{J}$ is the largest open subgroup of J_K, which admits a smooth formal model; this is discussed in Sect. 5.4 in a more general context.

The uniformization of J_K is obtained in the following way. The maximal torus $\widetilde{T}$ of $\widetilde{J}$ lifts to a formal torus $\overline{T}$ of $\overline{J}$. The inclusion $\overline{T}_K \hookrightarrow \overline{J}_K$ of the generic fibers extends to a morphism $T_K \to J_K$ from the associated affine torus T_K to the Jacobian J_K. The push-out $\widehat{J}_K := T_K \amalg_{\overline{T}} \overline{J}_K$ is a rigid analytic group, which contains $\overline{J}_K$ as an open rigid analytic subgroup, and the inclusion extends to a surjective group homomorphism $\widehat{J}_K \to J_K$. The kernel of the latter map is a lattice M in $\widehat{J}_K$ and makes $J_K = \widehat{J}_K/M$ into a quotient of the "universal covering" $\widehat{J}_K$. The representation $J_K = \widehat{J}_K/M$ is called *Raynaud representation*. Since any abelian variety is isogenous to a subvariety of a product of Jacobians, one can transfer the results to abelian varieties. For example, this implies Grothendieck's semi-abelian reduction theorem for abelian varieties; cf. [42].

The central objective in Chap. 6 is to show the algebraicity of the Raynaud extension $\widehat{J}_K$. A *Raynaud extension* is an affine torus extension of the generic fiber of a formal abelian R-scheme. In Sect. 6.1 we collect basic facts on Raynaud extensions and representations. For solving the main problem, one studies line bundles on a Raynaud extension $T \to E \to B$ with M-actions, where E is the torus extension of a formal abelian scheme B and where M is a lattice in E. For this, one introduces cubical structures on line bundles; necessary details are explained in Sect. A.3. This leads in Sect. 6.3 to the representability of the Picard functor Pic^{τ}_A of translation invariant line bundles on the Raynaud representation $A := E/M$; the representing space A' is called the dual of A.

In Theorem 6.3.3 it is shown that A' has the Raynaud representation $A' = E'/M'$. Here E' is the Raynaud extension given by the morphism $M \to B = B''$ from the lattice $M \subset E$ to B which is equal to the dual of B'. The map $M' \to E'$ is given via the canonical pairing between M and the character group M' of T. The algebraicity of the Raynaud extension $\widehat{J}_K$ is shown in Theorem 6.4.4, where the relationship of the polarization on J_K and the data on the Raynaud representation are considered. Of special interest is the canonical polarization of J_K. In Sect. 6.5 we discuss the theta polarization and its relation to the canonical pairing on the homology group $H_1(\widetilde{X}, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$.

In the final Chap. 7 we show that every abeloid variety A_K over an algebraically closed field admits a Raynaud representation in Theorem 7.6.4. There is also a representation theorem for bounded rigid analytic group varieties in Corollary 7.6.2. The latter is certainly the most difficult part of the book, since it achieves a much deeper result than the one of Grothendieck on the semi-abelian reduction of abelian varieties [42].

The proof requires advanced techniques, but it does not make use of Jacobian varieties. It mainly relies on the stable reduction theorem for smooth curve fibrations which are not necessarily proper. Therefore, one can cover the given group A_K by a finite family of such curve fibrations.

In a second step one deduces from such a covering the largest open subgroup $\overline{A}_K$, which admits a smooth formal R-model by well known techniques on group generation dating back to A. Weil; cf. Sect. 7.2. The formal group $\overline{A}$ is a formal torus extension of a formal abelian scheme B.

The prolongation of the embedding $\overline{T} \hookrightarrow \overline{A}$ of the formal torus to a group homomorphism of the associated affine torus $T_K \to A_K$ follows from the approximation theorem in Theorem 3.6.7 by a careful analysis of the convergence of the morphism $\overline{T} \to \overline{A}$, as we will see in Sect. 7.3. Thus, one obtains a group homomorphism from the push-out $\widehat{A}_K := T_K \amalg_{\overline{T}} \overline{A}$ to A_K.

The surjectivity of the map $\widehat{A}_K \to A_K$ is shown by studying the map from the curve fibration to A_K. More precisely, the torus part is related to the double points in the reduction of the stable curve fibration; cf. Sect. 7.4.

Until now, we were concerned only with the case where the base field is algebraically closed. But it is not difficult to see that the whole theory can be carried out after a suitable finite separable field extension if one starts with a non-Archimedean field which is not algebraically closed.

If the non-Archimedean field in question has a discrete valuation, there is a notion of a formal Néron model; cf. [16]. Then our result implies a semi-abelian reduction theorem for such Néron models. As a further application one can deduce that any abeloid variety has a dual; i.e., the Picard functor of translation invariant line bundles on A_K is representable by an abeloid variety.

Chapter 1
Classical Rigid Geometry

In this chapter we give a survey of rigid geometry over non-Archimedean fields. The foundation of the theory was laid by Tate in his private Harvard notes dating back to 1961, which were later published in Inventiones mathematicae [92]. Here we explain the main results from the classical point of view as studied in the late sixties; for proofs we refer to [9]. At that time rigid geometry was mainly inspired by complex analysis. Fundamental results were achieved by Kiehl, who introduced the Grothendieck topology and proved the basic facts concerning coherent sheaves. Moreover, Kiehl makes Serre's theory [87] of Géométrie Algébrique et Géométrie Analytique available for rigid analytic geometry, often referred to as GAGA; cf. [56].

We present the essential results on Tate algebras and affinoid spaces which are the building blocks of rigid geometry. By means of the Grothendieck topology we define rigid analytic spaces. Kiehl's results on coherent sheaves are stated without proofs. As a general reference we refer to [10] and the more recent account [9]. We always assume that K is a non-Archimedean field in the sense of Definition 1.1.1 unless otherwise stated.

1.1 Non-Archimedean Fields

In this section we will collect the basic definitions and results which are taken from various books; especially from [10]. Most of them are well documented, so we will not present their proofs. The proofs of more special results will be presented or delegated to the literature by exact references.

Definition 1.1.1. Let K be a field. A map $|\cdot| : K \to \mathbb{R}$ is called a *non-Archimedean absolute value* if for all $a, b \in K$ the following holds:

(i) $|a| \geq 0$,
(ii) $|a| = 0$ if and only if $a = 0$,

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(iii) $|ab| = |a| \cdot |b|$,
(iv) $|a+b| \leq \max\{|a|, |b|\}$.

We always assume that the absolute value is not the trivial one.

The subset $R := \{a \in K; |a| \leq 1\}$ is called *the valuation ring of* K with respect to the given absolute value $|\cdot|$. Then R is a ring with $\mathbb{Z} \cdot 1_K \subset R$. Its *residue field* is $k := R/\mathfrak{m}_R$, where $\mathfrak{m}_R \subset R$ is the maximal ideal of R.

A *non-Archimedean field* is a pair $(K, |\cdot|)$ consisting of a field K and a non-Archimedean absolute value such that K is complete with respect to the absolute value $|\cdot|$.

If R is a discrete valuation ring with field of fractions K, its valuation $v : R \to \mathbb{Z}$ induces a non-Archimedean absolute value on K by defining

$$|x|_e := e^{-v(a)} \quad \text{for } x \in K,$$

where e is a real number with $e > 1$. Thus, the set R is the set of the elements of K with absolute value less or equal to 1.

Example 1.1.2. Fix a prime number p. Then

$$|a|_p := \begin{cases} 0 & \text{if } a = 0, \\ p^{-r} & \text{if } a = p^r \frac{m}{n} \text{ with } r, m, n \in \mathbb{Z} \text{ and } p \not| mn, \end{cases}$$

is a non-Archimedean absolute value on $\mathbb{Q}$. The corresponding completion $\mathbb{Q}_p$ is called the *field of p-adic numbers*. Let $K/\mathbb{Q}_p$ be a finite extension and $N_{K/\mathbb{Q}_p}$ the norm of K over $\mathbb{Q}_p$. Then

$$|a| := \sqrt[[K:\mathbb{Q}_p]]{\left|N_{K/\mathbb{Q}_p}(a)\right|_p}$$

is the unique extension of $|\cdot|_p$ to K. It is a non-Archimedean absolute value on K and K is complete with respect to this value.

Since a non-Archimedean field is Henselian, the absolute value of K extends to an absolute value on every finite field extension L/K in a unique way, and hence it extends to an absolute value on the algebraic closure and on the completion of the algebraic closure of K. The completion of the algebraic closure is algebraically closed due to the continuity of roots.

For $K = \mathbb{Q}_p$ the completed algebraic closure of $\mathbb{Q}_p$ is denoted by $\mathbb{C}_p$; it is complete and algebraically closed. $\mathbb{C}_p$ is the p-adic analogue of $\mathbb{C}$.

If $\overline{K}$ is the complete algebraic closure of a non-Archimedean field K, then $|\overline{K}^{\times}| = \sqrt{|K^{\times}|}$ is the divisible hull of the value group $|K^{\times}|$ of K.

One easily shows that the strong inequality in Definition 1.1.1(iv) implies the following.

Lemma 1.1.3. *Let K be a non-Archimedean field. Then the following holds*:

(a) *A series $\sum_{\nu=0}^{\infty} a_\nu$ with elements $a_\nu \in K$ converges in K if and only if* $\lim_{\nu\to\infty} a_\nu = 0$.
(b) *If $\alpha, \beta \in K$ and $|\beta| \neq |\alpha|$, then* $|\alpha+\beta| = \max\{|\alpha|, |\beta|\}$.

Furthermore, the strong inequality induces a very specific topology.

Remark 1.1.4. Let K be a non-Archimedean field. Let $\overline{K}$ be an algebraic closure of K. A *closed disc* of K is a subset of type

$$D(a,r)^+ := \left\{z \in \overline{K}; |z-a| \leq r\right\} \quad \text{with } a \in K,\ r \in \mathbb{R}^\times.$$

An *open disc* is a subset of type

$$D(a,r)^- := \left\{z \in \overline{K}; |z-a| < r\right\} \quad \text{with } a \in K,\ r \in \mathbb{R}^\times.$$

Then the following holds:

(a) If D_1, D_2 are discs with $D_1 \cap D_2 \neq \emptyset$, then $D_1 \subset D_2$ or $D_2 \subset D_1$.
(b) Any open disc $D(a,r)^-$ is open and closed.
(c) The topological space K is totally disconnected.

One has to be careful with the notions open and closed discs, because they do not have the same behavior as in the usual topology. Quite often we will restrict the radius r to belong to $\sqrt{|K^\times|}$. Then a suitable power r^n belongs to the value group $|K^\times|$. In this case the disc is called *affinoid*. Only in these cases $D(a,r)^+$ is an affinoid space; cf. Example 1.3.2(e). Therefore, we will usually consider only radii r belonging to $\sqrt{|K^\times|}$. Moreover, a disc $D(a,r)$ is isomorphic to the unit disc in the category of rigid analytic spaces over K if and only if r belongs to the value group $|K^\times|$.

As a consequence of the total disconnectedness, we remind that the definition of a holomorphic function via local representability as a convergent power series does not lead to a well behaved theory of analytic functions. In fact, the identity principle would fail, and hence such a definition would not ensure meaningful global properties.

1.2 Restricted Power Series

As was pointed out at the end of the last section, it is necessary to introduce an extra topological structure in order to obtain a well behaved theory of holomorphic functions. More precisely, such a theory is based on two fundamental principles. Firstly, two holomorphic functions on a polydisc which coincide on some non-empty open subset should be equal. Secondly, every holomorphic function on a polydisc should be representable by a power series which converges globally on the whole polydisc.

The building blocks of rigid geometry over a non-Archimedean field K are the *Tate algebras*

$$T_n := K\langle\xi_1,\dots,\xi_n\rangle := \left\{\sum_{i\in\mathbb{N}^n} a_i\xi^i \in K[[\xi_1,\dots,\xi_n]];\ \lim_{|i|\to\infty} a_i = 0\right\}$$

and their residue algebras, the so-called *affinoid algebras* $A := T_n/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of T_n. The K-algebra T_n is the subset of the formal power series which converge on the unit polydisc.

Proposition 1.2.1. *With the above notations we have*:

(a) *T_n is Noetherian, factorial, regular and Jacobson.*
(b) *The units of T_n are of the following type*

$$T_n^\times = \left\{\sum_{i\in\mathbb{N}^n} c_i\xi^i \in T_n;\ |c_0| > |c_i| \text{ for all } i\neq 0\right\}.$$

(c) *T_n is a K-Banach algebra with respect to the Gauss norm*

$$\left|\sum_{i=0}^{\infty} a_i\xi^i\right| := \max\left\{|a_i|;\ i\in\mathbb{N}^n\right\}.$$

(d) *Each ideal $\mathfrak{a}\subset T_n$ is closed with respect to the Gauss norm.*
(e) *The residue algebra $A := T_n/\mathfrak{a}$ is Noetherian and Jacobson.*
(f) *The residue morphism $\alpha : T_n \to A$ induces a norm $|\cdot|_\alpha$ on A, which takes values in the value group $|K|$ of K. The Banach structure does not depend on the representation of A as a residue K-algebra of some T_n.*
(g) *Any morphism of affinoid K-algebras is continuous.*

These properties are mainly a consequence of the Weierstraß division theorem for restricted power series; cf. [9, §2.1–3].

Definition 1.2.2. An element $f\in T_n$ is called *ξ_n-distinguished of order $s\in\mathbb{N}$* if f has an expansion $f=\sum_{\nu=0}^{\infty} g_n\xi^\nu \in T_{n-1}\langle\xi_n\rangle$ with coefficients $g_\nu\in T_{n-1}$, where g_s is a unit in T_{n-1} and $|g_s| = |f| > |g_\nu|$ for $\nu\geq s+1$.

A monic polynomial $\omega\in T_{n-1}[\xi_n]$ which is ξ_n-distinguished of order $\deg_{\xi_n}\omega$ as an element of T_n is called a *Weierstraß polynomial.*

Remark 1.2.3. In the situation of Definition 1.2.2 we have:

(a) f is a unit if and only if $s=0$.
(b) For every $f\in T_n$ with $f\neq 0$ there exists a K-automorphism σ of T_n of type $\sigma(\xi_n)=\xi_n$ and $\sigma(\xi_\nu)=\xi_\nu+\xi_n^{\rho_\nu}$ with suitable $\rho_\nu\in\mathbb{N}$ for $\nu=1,\dots,n-1$ such that $\sigma(f)$ is ξ_n-distinguished.

Theorem 1.2.4 (Weierstraß division). *Let $g \in T_n$ be ξ_n-distinguished of order s. Then for every $f \in T_n$ there exists an element $q \in T_n$ and a polynomial $r \in T_{n-1}[\xi_n]$ of degree $r < s$ such that $f = qg + r$.*

The elements q and r are uniquely determined by these conditions. Furthermore, we have

$$|f| = \max\{|q| \cdot |g|, |r|\}.$$

Theorem 1.2.5 (Weierstraß preparation). *If $g \in T_n$ is ξ_n-distinguished of order s, then there exists a unique Weierstraß polynomial ω in $T_{n-1}[\xi_n]$ of degree s and a unit $e \in T_n^\times$ such that $g = e \cdot \omega$.*

Corollary 1.2.6 (Noether normalization). *If $\mathfrak{a} \subset T_n$ is a proper ideal, then there exists a K-algebra homomorphism $T_d \to T_n/\mathfrak{a}$ which is a finite monomorphism. The integer d equals the dimension of $T_n/\mathfrak{a}$.*

Corollary 1.2.7 (Hilbert's Nullstellensatz). *If $\mathfrak{n} \subset T_n$ is a maximal ideal, then $T_n/\mathfrak{n}$ is a finite field extension of K.*

Moreover, $\mathfrak{n}$ is generated by the polynomials in $\mathfrak{m} := \mathfrak{n} \cap K[\xi_1, \dots, \xi_n]$. The $\mathfrak{m}$-adic completion of $K[\xi_1, \dots, \xi_n]$ coincides with the $\mathfrak{n}$-adic completion of T_n. The extension $K[\xi_1, \dots, \xi_n] \to K\langle\xi_1, \dots, \xi_n\rangle$ is flat.

More generally, we will also consider relative Tate algebras.

Definition 1.2.8. Let $A := K\langle\zeta_1, \dots, \zeta_m\rangle/\mathfrak{a}$ be an affinoid algebra and let $\eta_1, \dots, \eta_n$ be a further set of variables. A *relative Tate algebra over A* is

$$A\langle\eta_1, \dots, \eta_n\rangle := \left\{ \sum_{i \in \mathbb{N}^n} a_i \eta^i \in A[[\eta_1, \dots, \eta_n]];\ \lim_{i \to \infty} a_i = 0 \right\}$$

the set of formal power series over A with coefficients $a_i \in A$ tending to zero. This is an affinoid algebra as well; in fact, $A\langle\eta\rangle = K\langle\zeta, \eta\rangle/\mathfrak{a} \cdot K\langle\zeta, \eta\rangle$.

1.3 Affinoid Spaces

The associated geometric object of an affinoid algebra A is defined by

$$\operatorname{Sp} A := \operatorname{MaxSpec} A := \{\mathfrak{m};\ \mathfrak{m} \subset A \text{ maximal ideal}\},$$

the set of maximal ideals of A_K. It is called an *affinoid space*. The field extension $K \to A/\mathfrak{m}$ is finite due to Corollary 1.2.7. The absolute value on K extends uniquely to every finite field extension due to Example 1.1.2. So denote by $|\cdot|$ the extension of the absolute value of K to the residue field $A/\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A. As in classical algebraic geometry, one views the residue class

$$f(\mathfrak{m}) := f \bmod \mathfrak{m}$$

as the evaluation of f at the maximal ideal $\mathfrak{m}$ of A. Thus, Corollary 1.2.7 implies

$$\mathbb{B}_K^n := \mathrm{Sp}(T_n) = \{x \in \mathbb{A}_K^n;\ |\xi_i(x)| \le 1 \text{ for } i = 1, \ldots, n\},$$

$\mathbb{B}_K^n$ is the set of points in the affine n-space $\mathbb{A}_K^n := \mathrm{MaxSpec}\, K[\xi_1, \ldots, \xi_n]$ over K, where the coordinates ξ_i take absolute values ≤ 1. One regards T_n as the set of *holomorphic functions* on the n-dimensional polydisc $\mathbb{B}_K^n$.

The affinoid space associated to an affinoid algebra $A = T_n/\mathfrak{a}$ is the locus

$$\mathrm{Sp}\, A = V(\mathfrak{a}) := \{x \in \mathbb{B}_K^n;\ f(x) = 0 \text{ for all } f \in \mathfrak{a}\}$$

of the ideal $\mathfrak{a} \subset T_n$. One views A as the set of *holomorphic functions* on $\mathrm{Sp}\, A$. Furthermore, by Corollary 1.2.7 every morphism $\varphi : B \to A$ of affinoid algebras induces a map

$$\mathrm{Sp}\,\varphi : \mathrm{Sp}\, A \longrightarrow \mathrm{Sp}\, B,\ x \longmapsto \varphi^{-1}(x).$$

The affinoid spaces are the building blocks of rigid geometry. Next, one carefully introduces the theory of holomorphic functions on an affinoid space. In order to create such a theory over a non-Archimedean field K, one has to require at least two principles: the analytic continuation and the global expansion of analytic functions on polydiscs. This poses a serious problem, because the natural topology given by the absolute value of K is totally disconnected. Therefore, one has to provide analytic spaces with an extra topology which forces a non-trivial notion of connectedness.

In the original paper of Tate [92] the definition of the extra structure was very clumsy. Substantial simplifications are due to Gerritzen and Grauert who introduced the notion of rational domains; cf. [34]. As a result, admissible open sets and coverings of rigid spaces became easier to handle.

Definition 1.3.1. Let $X = \mathrm{Sp}(A)$ be an affinoid space and $f_0, \ldots, f_r \in A$ without common zeros. Then

$$X_j := X(f_0/f_j, \ldots, f_r/f_j) := \{x \in X;\ |f_i(x)| \le |f_j(x)| \text{ for } i = 0, \ldots, r\}$$

is called a *rational affinoid subdomain* of X with structure ring

$$\begin{aligned}\mathcal{O}_X(X_j) &:= A\langle f_0/f_j \ldots, f_r/f_j\rangle \\ &:= A\langle \zeta_0/\zeta_j, \ldots, \zeta_r/\zeta_j\rangle/(\zeta_i f_j - \zeta_j f_i;\ i = 0, \ldots, r).\end{aligned}$$

The family $(X_0, \ldots, X_r)$ is called a *rational covering of* X. The geometric meaning of such a covering will become clear in Proposition 3.3.2.

In the case where $f_0 = 1$, the domain $X(f_1, \ldots, f_r) := X(f_1/1, \ldots, f_r/1)$ is called a *Weierstraß domain* in X. The image of A in $\mathcal{O}_X(X_0)$ is dense.

Rational subdomains are open subsets of X with respect to the topology of X, which is induced by the absolute value. The canonical morphism

$\varphi_j : A \to A\langle f_0/f_j \ldots, f_r/f_j\rangle$ induces a bijective map

$$\operatorname{Sp} A\langle f_0/f_j \ldots, f_r/f_j\rangle \xrightarrow{\sim} X(f_0/f_j, \ldots, f_r/f_j).$$

Indeed, f_j has no zeros on X_j and $|f_i(x)/f_j(x)| \le 1$ for all $x \in X_j$.

To illustrate the notion of rational domains, let us look at some subdomains of the disc, which will be important in Chap. 2.

Example 1.3.2. Let $A := T_1 = K\langle\xi\rangle$ be the 1-dimensional Tate algebra. Then set $X := \mathbb{B}^1_K = \operatorname{Sp} K\langle\xi\rangle$ and let $f = (\xi - a_1)\ldots(\xi - a_s) \in K[\xi]$ be a polynomial.

(a) f is a Weierstraß polynomial of order s if and only if $|a_i| \le 1$ for $i = 1, \ldots, s$.
(b) If f is as in (a), then

$$X(1/f) = \mathbb{B}^1_K - \big(D(a_1)^- \cup \cdots \cup D(a_s)^-\big)$$

with $D(a_i)^- := \{x \in \mathbb{B}^1_K;\ |\xi(x) - a_i| < 1\}$ for $i = 1, \ldots, s$.
(c) If the residue classes $\tilde{a}_1, \ldots, \tilde{a}_s$ of $a_1, \ldots, a_s$ in the residue field of R are distinct, then the partial fraction decomposition yields

$$\begin{aligned} A\langle 1/f\rangle &= K\langle\xi, \zeta_1, \ldots, \zeta_s\rangle/\big(1 - \zeta_1(\xi - a_1), \ldots, 1 - \zeta_s(\xi - a_s)\big) \\ &= \left\{\sum_{\nu=0}^{\infty} c_{0,\nu}\xi^\nu + \sum_{i=1}^{s}\sum_{\nu=1}^{\infty} c_{i,\nu}(\xi - a_i)^{-\nu};\ c_{i,\nu} \in K,\ \lim_{\nu\to\infty} c_{i,\nu} = 0\right\}. \end{aligned}$$

(d) In the special case that $f_0 = \pi \in R - \{0\}$ and $f_1 = \xi$ we have

$$\begin{aligned} X(f_0/f_1) &= \big\{x \in \mathbb{B}^1_K;\ |\pi| \le \big|\xi(x)\big| \le 1\big\}, \\ X(f_1/f_0) &= \big\{x \in \mathbb{B}^1_K;\ \big|\xi(x)\big| \le |\pi|\big\}, \end{aligned}$$

and

$$\begin{aligned} A\langle f_0/f_1\rangle &= \left\{\sum_{\nu\in\mathbb{Z}} c_\nu\xi^\nu;\ \lim_{\nu\to\infty}|c_\nu| = 0,\ \lim_{\nu\to-\infty}\big|c_\nu\pi^\nu\big| = 0\right\}, \\ A\langle f_1/f_0\rangle &= \left\{\sum_{\nu\in\mathbb{N}} c_\nu\xi^\nu;\ \lim_{\nu\to\infty}\big|c_\nu\pi^\nu\big| = 0\right\} = K\langle\xi/\pi\rangle. \end{aligned}$$

(e) Let $r \in \mathbb{R}$ with $0 < r < 1$ and assume that there is an $m \in \mathbb{N}$ with $r^m = |c|$ for some $c \in K^\times$. Then the disc $D(0, r)$ is the affinoid space associated to the affinoid algebra $K\langle\xi, \eta\rangle/(\xi^m - c\eta)$.

Definition 1.3.3. Let ξ be a coordinate function of the projective line $\mathbb{P}^1_K$. A domain of type

$$A(r_1, r_2) := \big\{x \in \mathbb{P}^1_K;\ r_1 \le \big|\xi(x)\big| \le r_2\big\}$$

with $r_1, r_2 \in \mathbb{R}^\times$ and $0 < r_1 \le r_2$ is called a *closed annulus* and

$$A(r_1, r_2)^- := \big\{x \in \mathbb{P}^1_K;\ r_1 < \big|\xi(x)\big| < r_2\big\}$$

is called an *open annulus* if $r_1 < r_2$.

The ratio r_1/r_2 is a biholomorphic invariant of $A(r_1, r_2)$ in the sense that for every isomorphism $A(r_1, r_2) \xrightarrow{\sim} A(\rho_1, \rho_2)$ with $\rho_1 \le \rho_2$ we have that $\rho_1/\rho_2 = r_1/r_2$. The ratio r_1/r_2 is called the *height of the annulus*. The subset of $A(r_1, r_2)$ consisting of the points z with $|z| = r_1$ or $|z| = r_2$ is called the *boundary* of $A(r_1, r_2)^+$.

As was already mentioned at the end of Sect. 1.1, one has to be careful with the notions open, closed and boundary in the context of annuli. We will mainly use the notion $A(r_1, r_2)$ in the case where the radii r_1, r_2 belong to $\sqrt{|K^\times|}$; in this case $A(r_1, r_2)$ is an affinoid domain. Then $A(r_1, r_2)$ is called an *affinoid annulus*.

Proposition 1.3.4. *Let $r_1 \le r_2$ be elements of $\sqrt{|K^\times|}$ and let*

$$A := A(r_1, r_2) := \left\{ z \in \mathbb{P}^1_K ; r_1 \le \left|\zeta(z)\right| \le r_2 \right\}$$

be a closed annulus. If $X = \operatorname{Sp} B$ is a connected and reduced affinoid space, then the algebra of holomorphic functions on $X \times A$ is given by

$$\mathcal{O}_{X \times \mathbb{P}^1_K}(X \times A) = \left\{ f = \sum_{i \in \mathbb{Z}} b_i \zeta^i \in B[[\zeta, 1/\zeta]]; \begin{array}{c} \lim\limits_{i \to \infty} |b_i| r_2^i = 0 \\ \lim\limits_{i \to -\infty} |b_i| r_1^i = 0 \end{array} \right\},$$

where $|.|$ is the sup-norm of B. Moreover, we have

$$|f| = \max_{x \in X} \max_{i \in \mathbb{N}} \left\{ \left|b_{-i}(x)\right| \cdot r_1^{-i}, \left|b_i(x)\right| \cdot r_2^i \right\}.$$

Its group of invertible functions is

$$\mathcal{O}_{X \times \mathbb{P}^1_K}(X \times A)^\times = \left\{ b \cdot \zeta^n \cdot (1 + h); \begin{array}{l} b \in B^\times, \ n \in \mathbb{Z}, \ |h| < 1 \\ h \in \mathcal{O}_{X \times \mathbb{P}^1_K}(X \times A) \end{array} \right\}.$$

In particular, $\mathcal{O}_{X \times \mathbb{P}^1_K}(X \times \mathbb{G}_{m,K})^\times = B^\times \cdot \zeta^{\mathbb{Z}}$.

Proof. The meaning of connectedness is here related to rational coverings; this is the right notion of connectedness in rigid geometry due to Theorem 1.3.8.

First consider the case where $B = K$ is a non-Archimedean field. Then it is easy to see that a unit has a dominating term $b_n \zeta^n$, and hence the assertion is true in this case. In the relative case, there is fiber by fiber a dominating term $b_{n(x)} \zeta^{n(x)}$ for each $x \in X$. The map $n : X \to \mathbb{Z}, x \mapsto n(x)$, is locally constant with respect to a rational covering, and hence constant as X being connected. Multiplying f with $b_n^{-1} \zeta^{-n}$ shows that is equal to $1 + h$ as asserted. □

Definition 1.3.5. Let $X = \operatorname{Sp} A$ be an affinoid space. A subset $U \subset X$ is called an *affinoid subdomain* if there exists a morphism $\iota : X' = \operatorname{Sp} A' \to X$ with $\iota(X') \subset U$ such that the following universal property is satisfied:

If $\psi : Y \to X$ is a morphism of affinoid spaces with $\psi(Y) \subset U$, then ψ factorizes through ι uniquely.

Obviously, if X' is an affinoid subdomain of X and if X'' is an affinoid subdomain X', then X'' is an affinoid subdomain X. One easily shows that rational subdomains are affinoid subdomains and that affinoid subdomains are open subsets of X; see [9, §3.3].

Proposition 1.3.6. *In the situation of Definition 1.3.5 the following holds*:

(i) *The map ι is injective and satisfies $\iota(X') = U$. Thus, it induces a bijection of sets $X' \xrightarrow{\sim} U$.*
(ii) *For $x \in X'$ one has $\mathfrak{m}_x = \mathfrak{m}_{\iota(x)} A'$ where $\mathfrak{m}_x \subset A'$ is the maximal ideal of x and $\mathfrak{m}_{\iota(x)} \subset A$ the one of $\iota(x)$.*
(iii) *For every $x \in X'$ and $n \in \mathbb{N}$, the map ι^* induces an isomorphism of affinoid K-algebras $A/\mathfrak{m}^n_{\iota(x)} \to A'/\mathfrak{m}^n_x$.*
(iv) *The induced map of $\mathfrak{m}$-adic completions $\widehat{A}_{\mathfrak{m}_{\iota(x)}} \to \widehat{A'}_{\mathfrak{m}_x}$ is bijective.*
(v) *The map $A \to A'$ is flat.*
(vi) *If A is regular, normal or reduced, so is A'.*

Proof. For the assertion (i), (ii) and (iii) see [9, §3.3/20]. These implies the remaining assertion by standard facts from commutative algebra. □

There is the following result on the structure of subdomains; cf. [9, §3.3/20].

Theorem 1.3.7 (Gerritzen-Grauert). *Let X be an affinoid space and let X' be an affinoid subdomain of X. Then there exists a finite covering $\{X_1, \ldots, X_n\}$ of X by rational subdomains such that $X' \cap X_i$ is a rational subdomain of X.*

On an affinoid space $X = \operatorname{Sp} A$ one considers the functor $\mathcal{O}_X$, which associates to affinoid subdomains the associated affinoid algebras with the canonical restriction morphisms. Moreover, one allows only *finite* coverings by affinoid subdomains. Then $\mathcal{O}_X$ is a sheaf on the category of finite affinoid coverings; cf. [9, §4.3/10].

Theorem 1.3.8 (Tate's acyclicity theorem). *Let $X = \operatorname{Sp} A$ be an affinoid space. The structure sheaf $\mathcal{O}_X$ is acyclic for finite coverings by affinoid subdomains. More precisely, if $\{X_0, \ldots, X_n\}$ is a finite covering of X by affinoid subdomains, then the augmented Čech complex*

$$0 \to \mathcal{O}_X(X) \to \prod_{i=0}^{n} \mathcal{O}_X(U_i) \to \prod_{i_0,i_1=0}^{n} \mathcal{O}_X(X_{i_0} \cap X_{i_1}) \to \cdots$$

is exact.

The proof for rational coverings is done by concrete computation and the general case is reduced to the special case by Theorem 1.3.7. Indeed, a finite covering by affinoid subdomains admits a refinement by a rational covering.

The result of Tate can be generalized to finitely generated A-modules; cf. [9, §4.3/11].

Corollary 1.3.9. *Let $X = \operatorname{Sp} A$ be an affinoid space and let M be a finitely generated A-module. Then the presheaf $\mathcal{F}^M$, which associates to an affinoid subdomain $X' = \operatorname{Sp} A'$ of X the A'-module*

$$\mathcal{F}^M(X') := M \otimes_A A',$$

and the induced restriction $\mathcal{F}^M(U) \to \mathcal{F}^M(V)$ for subdomains $V \subset U$ of X, is a sheaf of $\mathcal{O}_X$-modules. Furthermore, $\mathcal{F}^M$ is acyclic for finite coverings of X by affinoid subdomains.

1.4 The Maximum Principle

The elements f of an affinoid algebra A are viewed as holomorphic functions on the associated affinoid space $X := \operatorname{Sp} A$, as was explained above. The *supremum norm* $|\cdot|_X$ of f is defined by

$$|f|_X := \sup\{|f(x)|; x \in X\}.$$

We refer to it as the sup-norm. In general, $|\cdot|_X$ is only a semi-norm. However, the sup-norm is a Banach norm on A if and only if A is reduced; cf. [10, 6.2.1/4(iii)]. For $A = T_n$ the sup-norm coincides with the Gauss norm which was defined in Proposition 1.2.1(c). For an affinoid algebra there is a monomorphism $\varphi : T_d \hookrightarrow A$ making A into a finitely generated T_d-module of rank r which is defined by $\dim_{Q(T_d)} A \otimes_{T_d} Q(T_d)$, where $Q(T_d)$ is the field of fractions of T_d; cf. Corollary 1.2.6. Then one has the following lemma [10, 6.2.2/2].

Lemma 1.4.1. *In the above situation assume that A is a domain. Then there is a minimal polynomial*

$$\Phi(\eta) = \eta^r + a_1\eta^{r-1} + \cdots + a_r \in T_d[\eta]$$

with $\Phi(f) = 0$, and we have

$$|f|_{F(y)} = \max\{\sqrt[i]{|a_i(y)|}; i = 1, \ldots, r\},$$

where $F(y)$ is the fiber of $y \in \mathbb{B}^d_K$ with respect to the map $\operatorname{Sp} \varphi$.

In particular, one obtains the maximum principle.

Theorem 1.4.2 (Maximum principle). *In the situation of Lemma 1.4.1 we have*

$$|f|_X = \max\{\sqrt[i]{|a_i|}; i = 1, \ldots, r\},$$

and the maximum is attained on a non-empty open subset of X.

More generally, for every affinoid algebra A and every $f \in A$ there exists a point $x \in \operatorname{Sp} A$ such that $|f(x)| = |f|_X$.

Remark 1.4.3. Let A be an affinoid algebra and $f_1, \dots, f_n \in A$. Then there exists a morphism $\varphi : T_n \to A$ sending the variables $\xi_i \mapsto f_i$ if and only if $|f_i|_X \le 1$ for $i = 1, \dots, n$. If $f \in A$, then we have $|f|_X \le |f|_\alpha$ for every residue norm $|\_|_\alpha$ defined by a surjective morphism $\alpha : T_n \to A$.

Definition 1.4.4. Let A be an affinoid algebra, $X := \operatorname{Sp} A$, and put

$$\mathring{A} := \{f \in A; |f|_X \le 1\}.$$

Then $\mathring{A}$ is an R-algebra, called the *algebra of power bounded functions* of A. In general $\mathring{A}$ is not of topologically finite type over R; for more details see Sect. 3.1. Since $|f|_X^m = |f^m|_X$ for all $f \in A$, an element $f \in A$ is contained in $\mathring{A}$ if and only if f is power bounded; i.e., $|f|_X \le c$ for all $n \in \mathbb{N}$ and some $c \in \mathbb{R}$. The subset

$$\check{A} := \{f \in A; |f|_X < 1\}$$

is a reduced ideal of $\mathring{A}$, the residue ring

$$\widetilde{A} := \mathring{A}/\check{A}$$

is a reduced affine k-algebra over the residue field k of R. Denote by $\widetilde{f}$ the image of $f \in \mathring{A}$ under the reduction map $\mathring{A} \to \widetilde{A}$.

The following examples will illustrate the definition.

Example 1.4.5. (a) If $A = T_n$, then its ring of power bounded functions is

$$\mathring{T}_n = R\langle \xi_1, \dots, \xi_n \rangle := \left\{ f = \sum_i a_i \xi^i \in T_n; a_i \in R \right\}$$

and its reduction is the polynomial ring over the residue field

$$\widetilde{T}_n = k[\tilde{\xi}_1, \dots, \tilde{\xi}_n].$$

(b) If $L_{m,n} = K\langle \xi_1, \dots, \xi_m, \eta_1, 1/\eta_1, \dots, \eta_n, 1/\eta_n \rangle$ is the ring of the Laurent series, then

$$\mathring{L}_{m,n} = R\langle \xi_1, \dots, \xi_m, \eta_1, 1/\eta_1, \dots, \eta_n, 1/\eta_n \rangle,$$
$$\widetilde{L}_{m,n} = k[\tilde{\xi}_1, \dots, \tilde{\xi}_m, \tilde{\eta}_1, 1/\tilde{\eta}_1, \dots, \tilde{\eta}_n, 1/\tilde{\eta}_n].$$

The residue map $\mathring{A} \to \widetilde{A}$ induces a map

$$\widetilde{\ }: \operatorname{Sp} A \longrightarrow \widetilde{X} := \operatorname{MaxSpec} \widetilde{A},\ x \longmapsto x \cap \mathring{A} \bmod \check{A},$$

which is surjective. $\widetilde{X}$ is called the *canonical reduction* of $X = \operatorname{Sp} A$. Moreover, the reduction is a functor which also applies to morphism of affinoid algebras; cf. Sect. 3.1.

Remark 1.4.6. Let A be a reduced affinoid algebra. Then $\widetilde{A}$ is a domain if and only if the sup-norm on A is multiplicative; i.e., $|f \cdot g|_X = |f|_X \cdot |g|_X$ for all $f, g \in A$, cf. [10, 6.2.3/5]. In particular, if $\widetilde{A}$ is a domain, then for every invertible $f \in A$ the absolute value function $|f|$ on $\mathrm{Sp}(A)$ is constant.

1.5 Rigid Analytic Spaces

In [50] and [51] Kiehl simplified Tate's theory by introducing Grothendieck topologies. This is a means to restrict the notion of open coverings in a topology; cf. [9, §5.1].

Definition 1.5.1. A *weak Grothendieck topology* $\mathfrak{T}$ consists of a category $\mathrm{Cat}\,\mathfrak{T}$ and a set $\mathrm{Cov}\,\mathfrak{T}$ of families $(U_i \to U)_{i\in I}$ of morphisms in $\mathrm{Cat}\,\mathfrak{T}$ such that the following properties are satisfied:

(i) If $(\varphi : U \to V)$ is an isomorphism, then $\varphi \in \mathrm{Cov}\,\mathfrak{T}$.
(ii) If $(U_i \to U)_{i\in I}$ belongs to $\mathrm{Cov}\,\mathfrak{T}$ and each $(V_{i,j} \to U_i)_{j\in J_i}$ belongs to $\mathrm{Cov}\,\mathfrak{T}$, then the composition $(V_{i,j} \to U)_{j\in J_i, i\in I}$ belongs to $\mathrm{Cov}\,\mathfrak{T}$.
(iii) If $(U_i \to U)_{i\in I}$ belongs to $\mathrm{Cov}\,\mathfrak{T}$ and if $(V \to U)$ belongs to $\mathrm{Cat}\,\mathfrak{T}$, then the fibered products $U_i \times_U V$ exist in $\mathrm{Cat}\,\mathfrak{T}$ and the restriction $(U_i \times_U V \to V)_{i\in I}$ belongs to $\mathrm{Cov}\,\mathfrak{T}$.

Definition 1.5.2. A *presheaf $\mathcal{F}$ on a Grothendieck topology* $\mathfrak{T}$ with values in an abelian category $\mathfrak{A}$ is a contravariant functor on $\mathcal{F} : \mathrm{Cat}\,\mathfrak{T} \to \mathfrak{A}$. A presheaf $\mathcal{F}$ is a *sheaf* if the following sequence

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

is exact for all $(U_i \to U; i \in I) \in \mathrm{Cov}\,\mathfrak{T}$.

Example 1.5.3. If $X = \mathrm{Sp}\,A$ is an affinoid space, then let $\mathrm{Cat}\,\mathfrak{T}$ be the category of affinoid subdomains of X and let $\mathrm{Cov}\,\mathfrak{T}$ be the set of all finite coverings $(U_i \to U)_{i\in I}$ by affinoid subdomains of affinoid subdomains U of X. Then $\mathfrak{T}$ is a Grothendieck topology. The presheaf $\mathcal{O}_X$ introduced in Theorem 1.3.8 is a sheaf on $\mathfrak{T}$. For every finitely generated A-module M the presheaf $\mathcal{F}^M$ introduced in Corollary 1.3.9 is a sheaf on $\mathfrak{T}$ as well.

In order to define global rigid spaces and especially to allow gluing techniques on rigid and/or affinoid spaces, one has to enlarge the Grothendieck topology which was just defined.

Definition 1.5.4. Let $X = \mathrm{Sp}\,A$ be an affinoid space. The *strong Grothendieck topology* on X is given as follows:

(0) $\emptyset, X \in \operatorname{Cat}\mathfrak{T}$.

(i) A subset U of X is called *admissible open* if there exists a (not necessarily finite) covering $(U_i)_{i\in I}$ of U by affinoid subdomains U_i of X such that for all morphisms of affinoid spaces $\varphi : Y \to X$ with $\varphi(Y) \subset U$ the covering $(\varphi^{-1}(U_i))_{i\in I}$ admits a refinement by a finite covering with affinoid subdomains of Y.

(ii) A covering $(V_j)_{j\in J}$ of some admissible open subset V of X by admissible open subsets V_j is *admissible* if for every morphism $\varphi : Y \to X$ of an affinoid space Y with $\varphi(Y) \subset V$ the covering $(\varphi^{-1}(V_j))_{j\in J}$ admits a refinement by a finite covering of Y by affinoid subdomains.

Obviously, the strong Grothendieck topology is a weak Grothendieck topology. The strong Grothendieck topology on X restricts to the strong Grothendieck topology on every affinoid subdomain of X and every morphism $\varphi : Y \to X$ of affinoid spaces is continuous with respect to their strong Grothendieck topologies in the sense that pull-backs of admissible open subsets and/or admissible open coverings are admissible. One easily shows that Zariski open subsets of X are admissible open and coverings by Zariski open subsets are admissible; see [9, 5.1.9].

Remark 1.5.5. We now define the strong Grothendieck topologies in general. This can be done locally:

Let X be a set and $(X_i \to X)_{i\in I}$ a covering of X. Consider strong Grothendieck topologies $\mathfrak{T}_i$ on X_i for $i \in I$. Assume that the restrictions $\mathfrak{T}_i|_{X_i\cap X_j}$ yield the same Grothendieck topology on $X_i \cap X_j$ for all $i, j \in I$. Then there is a unique strong Grothendieck topology on X which restricts to $\mathfrak{T}_i$ for all $i \in I$.

Lemma 1.5.6. *Let $X = \operatorname{Sp} A$ be an affinoid space. Then every sheaf $\mathcal{F}$ with respect to the weak Grothendieck topology defined in Example 1.5.3 extends to a sheaf with respect to the strong topology in a unique way. In particular, the structure sheaf $\mathcal{O}_X$ defined in Example 1.5.3 has a unique extension to a structure sheaf with respect to the strong Grothendieck topology defined in Definition 1.5.4.*

In the following text of this book, by an *affinoid space* $X = \operatorname{Sp} A$ we mean the space $(X, \mathcal{O}_X)$ equipped with a strong Grothendieck topology and the structure sheaf as defined in Lemma 1.5.6.

Definition 1.5.7. A *rigid analytic space* is a pair $(X, \mathcal{O}_X)$ equipped with a strong Grothendieck topology such that there exists an admissible covering $(X_i; i \in I)$ such that the restriction $(X_i, \mathcal{O}_X|_{X_i})$ is isomorphic to an affinoid space. We will also use the notion *holomorphic topology* for the strong Grothendieck topology.

A *morphism of rigid analytic spaces* $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism in the sense of locally ringed spaces with Grothendieck topologies.

Any morphism $\varphi : A \to B$ of affinoid spaces induces a morphism

$$\operatorname{Sp}\varphi : Y = \operatorname{Sp} B \longrightarrow X = \operatorname{Sp} A$$

of the associated rigid analytic spaces and one can show that the canonical morphism

$$\mathrm{Hom}_K(A,B) \xrightarrow{\sim} \mathrm{Mor}_{\mathrm{RigSp}}(Y,X)$$

is bijective; cf. [9, 5.3/2]. More generally, we have the following result (cf. [9, 5.3/7]):

Proposition 1.5.8. *Let X be a rigid analytic space and Y an affinoid space. Then the canonical map*

$$\mathrm{Mor}_{\mathrm{RigSp}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_K\big(\mathcal{O}_Y(Y), \mathcal{O}_X(X)\big), \quad \varphi \longmapsto \varphi^*,$$

is bijective.

For a Grothendieck topology $\mathfrak{T}$ on X and a presheaf $\mathcal{F}$ on $\mathfrak{T}$ one defines the *stalk at a point $x \in X$* by

$$\mathcal{F}_x := \varinjlim_{x \in U \in \mathrm{Cat}\,\mathfrak{T}} \mathcal{F}_x .$$

The stalk $\mathcal{O}_{X,x}$ of the structure sheaf $\mathcal{O}_X$ on a rigid analytic space is a local ring. Since it is a residue ring of a convergent power series ring $K\{\xi_1, \dots, \xi_n\}$ which is Noetherian, the ring $\mathcal{O}_{X,x}$ is Noetherian and Henselian.

Remark 1.5.9. Without further explanation we will use the notion of *smooth, étale and unramified morphisms*. All these notions are defined in the usual way be using the differential forms as in the context of algebraic geometry. However, we here consider the following module of differential forms

$$\Omega^1_{T_n/K} := T_n d\xi_1 \oplus \cdots \oplus T_n d\xi_n$$

in the case where $T_n = K\langle \xi_1, \dots, \xi_n \rangle$ and the differential

$$d : T_n \longrightarrow \Omega^1_{T_n/K}, \ f \longmapsto df := \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} d\xi_i .$$

If $A := T_n/\mathfrak{a}$ is a residue algebras, then we put

$$\Omega^1_{A/K} := \Omega^1_{T_n/K}/(T_n d\mathfrak{a} + \mathfrak{a}\Omega_{T_n/K})$$

and the differential is the induced map $d : A \to \Omega^1_{A/K}$.

The pair $(\Omega^1_{A/K}, d)$ has the usual universal property with respect to finitely generated A-modules. Details can be found in [14, Part III, §1–2]. Note that $\Omega^1_{A/K} \otimes_A B = \Omega^1_{B/K}$ if $\mathrm{Sp}\, B \to \mathrm{Sp}\, A$ is an affinoid subdomain; i.e., this implies that $\Omega^1_{A/K}$ is a coherent sheaf; cf. Definition 1.6.1.

1.6 Coherent Sheaves

Kiehl obtained some fundamental results on coherent modules; specifically, the rigid analytic analogs of Grauert's Direct Image Theorem and of Theorems A and B of Cartan and Serre; cf. [50] and [51]. Also Serre's GAGA-Theorems could be carried over to rigid geometry verbatim. The acronym GAGA are the initials of Serre's celebrated article [87].

Definition 1.6.1. Let X be a rigid analytic space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if there is an admissible covering $(X_i, i \in I)$ by affinoid subspaces $X_i = \operatorname{Sp} A_i$ and if there exist finitely generated A_i-modules M_i such that the restriction $\mathcal{F}|_{X_i}$ is induced by M_i for all $i \in I$; i.e., $\mathcal{F}|_{X_i}$ is of type $\mathcal{F}^{M_i}$ as was defined in Corollary 1.3.9.

The following result is a theorem of Kiehl [51] which corresponds to Cartan-Serre's Theorem A and B. Proofs can be found in [9, §6.1/4 and 6.2/7].

Theorem 1.6.2 (Kiehl). *If $X = \operatorname{Sp} A$ is an affinoid space, then every coherent $\mathcal{O}_X$-module $\mathcal{F}$ is associated to a finitely generated A-module M.*

More precisely, $M = \Gamma(X, \mathcal{F})$ is finitely generated and the canonical morphism

$$\mathcal{F}^M \longrightarrow \mathcal{F}$$

is an isomorphism. Furthermore, the cohomology groups $H^q(X, \mathcal{F})$ vanish for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $q \geq 1$.

Definition 1.6.3. A morphism $\varphi : X \to Y$ is *separated* if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion.

A morphism of rigid analytic spaces $\varphi : X \to Y$ is called *proper* if φ is separated and if there exists an admissible affinoid covering $(Y_i, i \in I)$ such that for each $i \in I$ there exist two finite affinoid coverings $(X_{i,1}, \ldots, X_{i,n_i})$ and $(X'_{i,1}, \ldots, X'_{i,n_i})$ of $\varphi^{-1}(Y_i)$ with $X_{i,j} \Subset_{Y_i} X'_{i,j}$ for $j = 1, \ldots, n_i$.

Here the notion $X \Subset_Y X'$ for affinoid spaces means that there exists a closed immersion $X' \hookrightarrow \mathbb{B}^n_Y(1)$ into the relative unit ball over Y such that X is mapped into $\mathbb{B}^n_Y(r)$ for a strictly smaller radius $r < 1$. The latter means that the coordinate functions $\xi_1, \ldots, \xi_n$ of $\mathbb{B}^n_Y(1)$ take values $|\xi_i(x)| \leq r < 1$ for $x \in X$ and $i = 1, \ldots, n$.

The notion of properness was introduced by Kiehl in [50]. For a long time it was unknown whether a composition of proper maps is proper as well until it was proved in [61] by relating Kiehl's notion to properness of associated formal schemes in Theorem 3.3.12. It is easy to see that a factorization $\varphi = \alpha \circ \beta$ of a proper morphism φ, where α is separated, implies that β is proper. Kiehl's notion of properness seems to have been designed such that one can apply methods from functional analysis for the proof of the finiteness theorem. Proofs can be found in [9, §6.3/9].

Theorem 1.6.4 (Kiehl). *Let $\varphi : X \to Y$ be a proper morphism of rigid analytic spaces and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then the direct images $R^q\varphi_*\mathcal{F}$ are coherent $\mathcal{O}_Y$-modules for all $q \in \mathbb{N}$.*

Similarly as in complex analysis one has the usual consequences of the finiteness theorem like the Proper Mapping Theorem of Remmert and the Stein factorization of proper morphisms. A *closed analytic subset* of a rigid analytic space X is, locally with respect to the Grothendieck topology of X, the vanishing locus of finitely many holomorphic functions.

Corollary 1.6.5. *If $\varphi : X \to Y$ is a proper morphism of rigid analytic spaces, then the image $\varphi(A)$ is a closed analytic subset of Y for every closed analytic subset A of X.*

Corollary 1.6.6. *Let $\varphi : X \to Y$ be a proper morphism of rigid analytic spaces. Then the coherent $\mathcal{O}_Y$-module $\varphi_*\mathcal{O}_X$ gives rise to the rigid analytic space $Y' := \operatorname{Sp}\varphi_*\mathcal{O}_X$ such that $\varphi : X \to Y$ factorizes into $\varphi = \psi \circ \varphi'$ where $\psi : Y' \to Y$ is finite and $\varphi' : X \to Y'$ is proper with connected fibers.*

Corollary 1.6.7. *If $\varphi : X \to Y$ is a proper and quasi-finite morphism of rigid analytic spaces, then φ is finite.*

Corollary 1.6.8. *Let $p : X \to \operatorname{Sp} K$ be a proper rigid analytic space over K. Assume that $X \otimes_K \overline{K}$ is reduced and connected where $\overline{K}$ is a complete algebraic closure of K. Then the canonical morphism $K \to \Gamma(X, \mathcal{O}_X)$ is bijective. Furthermore, if S is a rigid analytic space, then $\mathcal{O}_S \to (p_S)_*\mathcal{O}_{X\times S}$ is bijective as well.*

Proof. Since $\Gamma(X, \mathcal{O}_X)$ commutes with field extensions, we may assume that $K = \overline{K}$ is algebraically closed. Due to Theorem 1.6.4 the K-algebra $\Gamma(X, \mathcal{O}_X)$ is a K-vector space of finite dimension. Since $\Gamma(X, \mathcal{O}_X)$ is reduced, it is a finite product of copies of K. Since X is connected, it consists of one copy. Therefore, $K \to \Gamma(X, \mathcal{O}_X)$ is bijective. The additional assertion follows from the fact that $p_*\mathcal{O}_X$ commutes with flat base change. □

The finiteness result is sufficient to carry over Serre's theorems about Géométrie Algébrique et Géométrie Analytique [87] to the category of rigid analytic spaces. The latter was worked out by Köpf [56].

First let us recall the definition of Serre's GAGA-functor in the setting of rigid geometry. Let $X = \operatorname{Sp} A$ be an affinoid space, $\xi_1, \ldots, \xi_n$ variables, and let $c \in K^\times$ be a constant with $|c| > 1$. Now consider

$$\mathbb{B}^n_X(c) := X \times \mathbb{B}^n(c) := \operatorname{Sp} A\langle \xi_1/c \ldots, \xi_n/c\rangle.$$

Then there is an increasing sequence

$$\mathbb{B}^n_X(c) \subset \mathbb{B}^n_X\left(c^2\right) \subset \cdots \subset \mathbb{B}^n_X\left(c^i\right) \subset \cdots$$

of affinoid space, where each one is an affinoid subdomain of the bigger ones. So, one obtains a rigid analytic structure on the affine n-space over X by

$$\mathbb{A}_X^n := \bigcup_{i\in\mathbb{N}} \mathbb{B}_X^n(c^i).$$

A similar approach works for affine algebraic A-schemes of finite type. The idea behind this construction is that one regards a polynomial with coefficients in A as a holomorphic function. Since Zariski open subsets are admissible, the algebraic construction of gluing carries over to the category of rigid geometry. Thus, to every scheme Z of finite type over A there is associated a rigid analytic X-space Z^{an}. The closed points of Z corresponds one-to-one to the points of Z^{an} in a canonical way. One can characterize the analytification of an A-scheme of finite type by a universal property.

Definition 1.6.9. Let A be an affinoid K-algebra, $X := \operatorname{Sp} A$, and Z an A-scheme of locally finite type. An *analytification* of Z is a rigid analytic X-space Z^{an} together with a morphism of locally Grothendieck-ringed spaces $\iota : Z^{\text{an}} \to Z$ satisfying the universal property:

For every rigid analytic X-space Y, each morphism of locally Grothendieck-ringed spaces $Y \to Z$ factorizes through ι via a unique morphism $Y \to Z^{\text{an}}$ of rigid analytic spaces.

If it is clear from the context, then we will drop the sup-index "an", when we view a scheme of finite type over an affinoid space as a rigid analytic space. For example this applies to algebraic curves, the projective space $\mathbb{P}_K^n$ or the affine n-space $\mathbb{A}_K^n$ over a non-Archimedean field K.

The ring extension $\mathcal{O}_{Z,z} \to \mathcal{O}_{Z^{\text{an}},z}$ is faithfully flat, because for every maximal ideal $\mathfrak{m} \subset A[\xi_1,\ldots,\xi_n]$ and $n \in \mathbb{N}$ the canonical morphism

$$A[\xi_1,\ldots,\xi_n]/\mathfrak{m}^n \xrightarrow{\sim} A\langle \xi_1/c,\ldots,\xi_n/c\rangle/\mathfrak{m}^n A\langle \xi_1/c,\ldots,\xi_n/c\rangle$$

is bijective if the associated point to $\mathfrak{m}$ belongs to $\mathbb{B}_X^n(c)$. Therefore, every algebraic coherent sheaf $\mathcal{F}$ gives rise to a rigid analytic coherent sheaf $\mathcal{F}^{\text{an}}$. In the case of a proper A-scheme Z the converse is also true.

Remark 1.6.10. The analytification of a proper morphism between K-schemes of finite type is a proper morphism of rigid analytic spaces.

Proof. It suffices to look at a proper algebraic A-scheme Y where A is an affinoid algebra. If Y is projective over A, then we can assume that $Y = \mathbb{P}_A^n$ is already the projective space. For $c \in |K^\times|$ consider the subsets

$$Y_i(c) := \left\{ x \in \mathbb{P}_A^n,\ \left|\xi_j(x)\right| \le c \cdot \left|\xi_i(x)\right| \text{ for } j = 0,\ldots,n \right\}$$

for $i = 1,\ldots,n$. Then $Y(c) := \{Y_0(c),\ldots,Y_n(c)\}$ is an admissible affinoid covering of $\mathbb{P}_A^n$ for $c \ge 1$ with $Y_i(1) \Subset_A Y_i(c)$ for $i = 0,\ldots,n$ and $c > 1$. So, $Y(1)$ and $Y(c)$

with $c > 1$ satisfy the condition for properness in the sense of Definition 1.6.3. In the general case, due to the theorem of Chow [39, II, 5.6.2] there exists a surjective A-morphism $f : Z \to Y$ from a projective A-scheme Z to Y. By what we have shown above it follows that $Z^{\mathrm{an}} \to \operatorname{Sp} A$ is proper. Since Y is of finite type over A, one can cover the analytification Y^{an} of Y by finitely many affine A-schemes $Y_1, \ldots, Y_n$ of finite type. Each Y_i admits an exhausting affinoid covering $\{Y_{i,j};\ j \in \mathbb{N}\}$ with $Y_{i,j} \Subset_A Y_{i,j+1}$. Then $\{Y_{i,j};\ j \in \mathbb{N}, i = 1, \ldots, n\}$ is an admissible covering of Y^{an}. So $\{f^{-1}(Y_{i,j});\ j \in \mathbb{N}, i = 1, \ldots, n\}$ is an admissible covering of Z^{an}. Since the morphism $Z^{\mathrm{an}} \to \operatorname{Sp} A$ is proper, a finite subfamily will cover Z^{an} also. Therefore, a finite subfamily of $\{Y_{i,j};\ j \in \mathbb{N}, i = 1, \ldots, n\}$ will cover Y^{an} as well. Thus, we see that $Y^{\mathrm{an}} \to \operatorname{Sp} A$ is proper in the sense of Definition 1.6.3. □

Theorem 1.6.11 (GAGA). *Let A be an affinoid algebra and let X be a proper A-scheme. Consider coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ and denote by $\mathcal{F}^{\mathrm{an}}$, $\mathcal{G}^{\mathrm{an}}$ the associated rigid analytic $\mathcal{O}_{X^{\mathrm{an}}}$-modules. Then we have the following results.*

(a) *The canonical morphism*

$$H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q\left(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}\right)$$

is an isomorphism for every $q \in \mathbb{N}$.

(b) *The canonical homomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{an}}}}\left(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\right)$$

is an isomorphism.

(c) *If $\mathcal{F}'$ is a coherent $\mathcal{O}_{X^{\mathrm{an}}}$-module, then there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and an isomorphism $\mathcal{F}^{\mathrm{an}} \xrightarrow{\sim} \mathcal{F}'$. The $\mathcal{O}_X$-module $\mathcal{F}$ is uniquely determined up to canonical isomorphism.*

Proof. See [56]. □

As a corollary one has the analogue of Chow's theorem.

Corollary 1.6.12. *Let $Y \subset \mathbb{P}_A^n$ be a closed analytic subset of the relative projective space $\mathbb{P}_A^n$. Then Y is the locus of finitely many homogeneous polynomials $f_1, \ldots, f_N$ of the coordinate ring $A[\xi_0, \ldots, \xi_n]$.*

One can use Theorem 1.6.4 to show that the ring of holomorphic functions on a product of polydiscs by a polyannulus is factorial. For $\rho \in |K^\times|$ with $\rho \le 1$ we denote by $A\langle \eta^{\pm} \rangle_{\rho,1}$ the ring of Laurent series $\sum_{\nu \in \mathbb{Z}} c_\nu \eta^\nu$ with coefficients in A which converge for $\rho \le |\eta| \le 1$.

Proposition 1.6.13. *Let $\rho_\nu \in |K^\times|$ with $\rho_\nu \le 1$ for $\nu = 1, \ldots, n$. Then the Laurent ring $K\langle \xi_1, \ldots, \xi_m \rangle \langle \eta_1^{\pm} \rangle_{\rho_1,1} \ldots \langle \eta_n^{\pm} \rangle_{\rho_n,1}$ is factorial.*

Proof. For technical reasons we consider the affinoid algebra

$$K\langle\xi_1,\dots,\xi_m\rangle\langle\zeta_1^{\pm},\dots,\zeta_\ell^{\pm}\rangle\langle\eta_1^{\pm}\rangle_{\rho_1,1}\cdots\langle\eta_n^{\pm}\rangle_{\rho_n,1},$$

where the series in the variables $\zeta_1,\dots,\zeta_\ell$ in the middle should converge for $|\zeta_\lambda|=1$ for $\lambda=1,\dots,\ell$. These K-algebras are regular as follows from Propositions 1.2.1(a) and 1.3.6, and hence they are locally factorial. So it remains to see that their divisorial ideals are principal. For the proof we proceed by induction of n.

Let us start with $n=0$. Let $\mathfrak{a}_K \subsetneqq L_{m,\ell}:=K\langle\xi,\zeta^{\pm}\rangle$ be a divisorial ideal. The reduction of $L_{m,\ell}$ is $\widetilde{L}_{m,\ell}=k[\tilde{\xi},\tilde{\zeta}^{\pm}]$ which is a factorial ring. Now look at the ideal

$$\mathfrak{a}:=\mathfrak{a}_K\cap R\langle\xi,\zeta^{\pm}\rangle\subset R\langle\xi,\zeta^{\pm}\rangle.$$

Then $R\langle\xi,\zeta^{\pm}\rangle/\mathfrak{a}$ has no R-torsion, and hence it is flat over R. Thus one obtains an exact sequence

$$0\to\mathfrak{a}\otimes_R k\to k[\tilde{\xi},\tilde{\zeta}^{\pm}]\to k[\tilde{\xi},\tilde{\zeta}^{\pm}]/\mathfrak{a}\otimes_R k\to 0.$$

Due to the flatness, $\mathfrak{a}\otimes_R k$ has codimension 1. Next we show that $\mathfrak{a}\otimes_R k$ has no embedded components. This follows from the fact that $\mathfrak{a}$ is a reflexive $R\langle\xi,\zeta^{\pm}\rangle$-module. Indeed, consider the canonical inclusion $\iota:\mathfrak{a}\to\mathfrak{a}^{**}$ of $\mathfrak{a}$ into its bi-dual $\mathfrak{a}^{**}\subset R\langle\xi,\zeta^{\pm}\rangle$. Since $\mathfrak{a}\otimes_R K$ is locally principal, $\iota\otimes_R K$ is an isomorphism. So, there exists a power π^N such that $\pi^N\mathfrak{a}^{**}\subset\mathfrak{a}$. Since $R\langle\xi,\zeta^{\pm}\rangle/\mathfrak{a}$ is R-flat, the multiplication by π is injective on the $R\langle\xi,\zeta^{\pm}\rangle/\mathfrak{a}$. Thus, we get $\mathfrak{a}=\mathfrak{a}^{**}$. Since $k[\tilde{\xi},\tilde{\zeta}^{\pm}]$ is factorial, $\mathfrak{a}\otimes_R k$ is principal. So there exists an element $f\in\mathfrak{a}$ which generates $\mathfrak{a}\otimes_R k$. Then f generates $\mathfrak{a}$. Indeed, $\mathfrak{a}=(g_1,\dots,g_N)$ is finitely generated due to Theorem 3.2.1. Since f generates $\mathfrak{a}\otimes_R k$, there are relations $g_i=r_i f+ta_i$ for $i=1,\dots,N$ with $r_i\in R\langle\xi,\zeta^{\pm}\rangle$, $a_i\in\mathfrak{a}$ for a suitable $t\in R$ with $|t|<1$. Then an iteration process shows that $\mathfrak{a}$ is generated by f.

For the induction step "$n-1\to n$" consider a divisorial ideal

$$\mathfrak{a}\subset K\langle\xi_1,\dots,\xi_m\rangle\langle\zeta_1^{\pm},\dots,\zeta_\ell^{\pm}\rangle\langle\eta_1^{\pm}\rangle_{\rho_1,1}\cdots\langle\eta_n^{\pm}\rangle_{\rho_n,1}.$$

Now restrict the associated sheaf of $\mathfrak{a}$ to the subset $\{|\eta_1(x)|=\rho_1\}$. Then the induction hypothesis implies that the ideal

$$\mathfrak{a}K\langle\xi_1,\dots,\xi_m\rangle\langle\zeta_1^{\pm},\dots,\zeta_\ell^{\pm}\rangle\langle\eta_1^{\pm}\rangle_{\rho_1,\rho_1}\langle\eta_2^{\pm}\rangle_{\rho_2,1}\cdots\langle\eta_n^{\pm}\rangle_{\rho_n,1}$$

is principal. Then it can be pasted with the trivial invertible sheaf to build an invertible sheaf on $\mathrm{Sp}(K\langle\xi_1,\dots,\xi_m,\eta_1\rangle\langle\zeta_1^{\pm},\dots,\zeta_\ell^{\pm}\rangle\langle\eta_2^{\pm}\rangle_{\rho_2,1}\dots\langle\eta_n^{\pm}\rangle_{\rho_n,1})$. Due to Theorem 1.6.2, this sheaf is associated to an invertible ideal. Then the induction hypothesis shows that $\mathfrak{a}$ is principal. □

1.7 Line Bundles

In this section we review some basic facts about line bundles and invertible sheaves.

Definition 1.7.1. Let X be a rigid analytic space.

(a) An *invertible sheaf* $\mathcal{L}$ on X is a locally free $\mathcal{O}_X$-module of rank 1.
(b) A *line bundle* on X is a morphism $\pi : L \to X$ of rigid analytic space such that there exists an admissible covering $\mathfrak{U} = \{U_i; i \in I\}$ of X and isomorphisms $\pi_i : L|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{A}^1_K, \ell \longmapsto (\pi(\ell), \pi_i(\ell))$, which respect the fibers over U_i such that the transition functions $\tau_{i,j}$

$$\begin{array}{ccc} U_i \times \mathbb{A}^1_K \cong L|_{U_i} & \dashrightarrow & L|_{U_j} \cong U_j \times \mathbb{A}^1_K \\ \uparrow & & \uparrow \\ (U_i \cap U_j) \times \mathbb{A}^1_K & \xrightarrow{\tau_{j,i}} & (U_j \cap U_i) \times \mathbb{A}^1_K \end{array}$$

$$\begin{array}{ccc} \ell & \longmapsto & \ell \\ \downarrow \pi_i & & \downarrow \pi_j \\ (x, \pi_i(\ell)) & \longrightarrow & (x, \pi_i(\ell) \cdot t_{i,j}(x)) \end{array}$$

are morphisms of fiber bundles and linear isomorphisms on the stalks.
(c) A $\mathbb{G}_m$*-torsor* is a faithfully flat morphism $L \to X$ which admits local sections with respect to the Grothendieck topology of X, equipped with a $\mathbb{G}_{m,X}$-action such that $\mathbb{G}_{m,X}|_U \longrightarrow L|_U, t \longmapsto t \cdot \ell$, is an isomorphism if $\ell : U \to L$ is a section.
(d) Assume that X is a rigid analytic group variety. A line bundle L on X is called *translation invariant* if for every point $a \in X(\overline{K})$ the line bundle $\tau_a^* L$ is isomorphic to L, where $\tau_a : X \otimes_K \overline{K} \to X \otimes_K \overline{K}$ is the translation by a. Here $\overline{K}$ is an algebraic closure of K.

Remark 1.7.2. There is a canonical correspondence between line bundles, invertible sheaves, and $\mathbb{G}_m$-torsors. We will make no difference between these notions. For the convenience of the reader we explain the correspondence below; cf. [39, II, §1.7] or [69, Chap. III, §4].

The sections of a line bundle L constitute an invertible sheaf $\mathcal{S}(L)$. The invertible sheaf $\mathcal{L}$ associated to L is the dual of $\mathcal{S}(L)$. If $\mathcal{L}$ is an invertible sheaf, then the associated line bundle L is the spectrum of the symmetric algebra $\bigoplus_{m \in \mathbb{N}} \mathcal{L}^{\otimes m}$ also denoted by $\mathcal{A}(\mathcal{L})$. Thus, one has

$$L := V(\mathcal{L}) := \mathcal{S}pec\Bigg(\bigoplus_{m \in \mathbb{N}} \mathcal{L}^{\otimes m}\Bigg);$$

more precisely, the analytification of $V(\mathcal{L})$. The association $\mathcal{L} \mapsto V(\mathcal{L})$ is a contravariant functor from the category of invertible sheaves on X to the category of line bundles on X.

Since $\mathrm{Mor}_X(U, V(\mathcal{L})) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}(\mathcal{L}), \mathcal{O}_X)(U) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)(U)$ for an admissible open subset U of X, there is a canonical identification

$$\mathcal{S}(L) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) = \mathcal{L}^{-1}.$$

The functor $\mathcal{S}$ from the category of line bundle on X to the category of invertible sheaves on X is covariant. In particular, there are canonical isomorphisms

$$\mathcal{L}(U) \xrightarrow{\sim} \mathrm{Hom}_X\big(L\big|_U, \mathbb{A}^1_U\big),$$
$$\mathcal{S}(L)(U) \xrightarrow{\sim} \mathrm{Hom}_X\big(\mathbb{A}^1_U, L\big|_U\big)$$

for every admissible open subset U of X.

The $\mathbb{G}_m$-torsor $L^\times := L - \{0\}$ associated to a line bundle L is obtained by removing the zero section from L; i.e., for $\mathcal{L} = \mathcal{S}(L)^{-1}$ it is

$$L^\times = \mathcal{S}pec\Big(\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}\Big).$$

Every line bundle induces a cocycle $(t_{i,j}) \in H^1(X, \mathcal{O}_X^\times)$ via the transition functions $(t_{i,j})$. A collection $(f_i \in \mathcal{S}(L)(U_i))$ gives rise to a global section if and only if

$$f_i \cdot t_{i,j} = f_j \quad \text{for all } i, j.$$

So the invertible sheaf $\mathcal{S}(L)$ is associated to the cocycle $(t_{i,j})$. The invertible sheaf $\mathcal{L}$ associated to L is given by the cocycle $(t_{i,j}^{-1})$.

If D is a Cartier divisor on X (cf. [10, p. 212]), then the line bundle $L(D)$ associated to the invertible sheaf $\mathcal{O}_X(D)$ has the sheaf of sections given by

$$\mathcal{S}\big(L(D)\big)(U) : \big\{f : U \dashrightarrow \mathbb{A}^1_K; \text{ meromorphic with } \operatorname{div} f - D \geq 0\big\}$$

for admissible open subvarieties U of X.

Definition 1.7.3. An invertible sheaf $\mathcal{L}$ on a proper rigid analytic space X is called *ample* if there exists an integer $n \geq 1$ such that $\mathcal{L}^{\otimes n}$ is generated by global sections and yields an embedding into a projective space $\mathbb{P}^N_K$.

The latter means the following. If $(f_0, \ldots, f_N)$ is a basis of the vector space $\Gamma(X, \mathcal{L}^{\otimes n})$ of the global sections of $\mathcal{L}^{\otimes n}$, then the mapping

$$\underline{f} := (f_0, \ldots, f_N) : X \longrightarrow \mathbb{P}^N_K,\ x \longmapsto \underline{f}(x) := \big(f_0(x), \ldots, f_N(x)\big),$$

is a closed embedding.

A line bundle L is called *ample* if the invertible sheaf $\mathcal{L} := \mathcal{S}(L)^{-1}$ is ample.

Definition 1.7.4. Let X be proper smooth rigid analytic curve over K assumed to be geometrically connected. A *divisor* D on X is an formal linear combination

$D = \sum_{i=1}^{r} n_i x_i$ over points $x_i \in X$ with $n_i \in \mathbb{Z}$. Its *degree* is defined by

$$\deg D := \sum_{i=1}^{r} n_i \cdot \left[K(x_i) : K\right].$$

The support supp(D) of D is the set consisting of all the x_i with $n_i \neq 0$. The divisor D is called *effective* if $n_i \geq 0$ for $i = 1, \ldots, n$. One defines $D \leq D'$ for a divisor $D' = \sum_{i=1}^{r} n'_i x_i$ if $n_i \leq n'_i$ for $i = 1, \ldots, n$.

Let $\mathcal{L}$ be an invertible sheaf on X and D a divisor on X. Put

$$\mathcal{L}(D) := \left\{ f \in (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}_X)(X); \operatorname{div}(f) + D \geq 0 \right\},$$

where $\mathcal{M}_X$ is the sheaf of meromorphic functions on X. Then $\mathcal{L}(D)$ is again an invertible sheaf on X. A *meromorphic section* of $\mathcal{L}$ is a global section of $\mathcal{L}(D)$ for some divisor D.

If f is a meromorphic section of an invertible sheaf $\mathcal{L}$ on X with $f \neq 0$, then its divisor is defined by

$$\operatorname{div}(f) := \sum_{x \in X} \operatorname{ord}_x(f)$$

where $\operatorname{ord}_x(f)$ is the vanishing order of f at the point x.

Lemma 1.7.5. *Let X be a proper smooth rigid analytic curve over K. Then every invertible sheaf $\mathcal{L}$ admits a non-trivial meromorphic global section. In particular, $\mathcal{L}$ is isomorphic to $\mathcal{O}_X(D)$ for a suitable divisor D on X.*

Proof. Theorem 1.6.4 yields $d := \dim_K H^1(X, \mathcal{L}) < \infty$, since X is proper. For every effective divisor D there is a long exact sequence

$$\begin{aligned} 0 \to H^0(X, \mathcal{L}) \to H^0\big(X, \mathcal{L}(D)\big) \to H^0\big(X, \mathcal{L}(D)/\mathcal{L}\big) \to \\ \to H^1(X, \mathcal{L}) \to H^1\big(X, \mathcal{L}(D)\big) \to 0, \end{aligned}$$

because $H^1(X, \mathcal{L}(D)/\mathcal{L}) = 0$ vanishes as $\mathcal{L}(D)/\mathcal{L}$ is concentrated on the support of D which consists of finitely many points. For effective divisors D with $\deg D > d$ the map $H^0(X, \mathcal{L}(D)/\mathcal{L}) \to H^1(X, \mathcal{L})$ cannot be injective. Thus, we see that $H^0(X, \mathcal{L}(D))$ is not zero, and hence that $\mathcal{L}$ has a non-trivial meromorphic section. □

By the usual techniques [29, 16.9] one deduces from Lemma 1.7.5 the formula of Riemann-Roch.

Theorem 1.7.6 (Riemann-Roch). *Let X be a smooth proper rigid analytic variety of dimension* 1 *which is geometrically connected.*

If D is a divisor on X, then

$$\dim_K H^0\big(X, \mathcal{O}_X(D)\big) = \deg D + 1 - g + \dim_K H^1\big(X, \mathcal{O}_X(D)\big),$$

where $g := \dim_K H^1(X, \mathcal{O}_X)$ is the genus of X.

Proof. For the convenience of the reader we provide a proof here.

(1) The formula is true for the empty divisor because of $H^0(X, \mathcal{O}_X) = K$ by Corollary 1.6.8.

(2) Consider two divisors D and D' and assume $D \leq D'$. Then we have a long exact sequence

$$0 \to H^0\big(X, \mathcal{O}_X(D)\big) \to H^0\big(X, \mathcal{O}_X\big(D'\big)\big) \to H^0\big(X, \mathcal{O}_X\big(D'\big)/\mathcal{O}_X(D)\big) \to$$
$$\to H^1\big(X, \mathcal{O}_X(D)\big) \to H^1\big(X, \mathcal{O}_X\big(D'\big)\big) \to 0,$$

because $H^1(X, \mathcal{O}_X(D')/\mathcal{O}_X(D)) = 0$ vanishes as $\mathcal{O}_X(D')/\mathcal{O}_X(D)$ is concentrated on finitely many points. Moreover,

$$\dim_K H^0\big(X, \mathcal{O}_X\big(D'\big)/\mathcal{O}_X(D)\big) = \deg D' - \deg D.$$

Therefore, if the formula is true for one of the divisors, then it is true for the other one. Thus, by (1) the formula holds all effective divisors.

(3) For every divisor D there exists a divisor D' such that $D \leq D'$ and $D' \geq 0$. So, the formula follows from (2). □

Corollary 1.7.7. *Let $\mathcal{L}$ be an invertible sheaf and D a divisor with $\mathcal{L} \cong \mathcal{O}_X(D)$.*

(a) *If $\mathcal{L} \cong \mathcal{O}_X$, then* $\deg D = 0$.
(b) $\deg \operatorname{div}(f) = 0$ *for every meromorphic function $f \neq 0$ on X.*

In particular, $\deg \mathcal{L} := \deg D$ does not depend on the choice of the divisor D with $\mathcal{L} \cong \mathcal{O}_X(D)$ and it is called the *degree of the invertible sheaf.* Moreover, if L is a line bundle on X, then the degree of L is defined by $\deg L = \deg \mathcal{S}(L)$.

Definition 1.7.8. Let X and S be a rigid analytic spaces over K and let $x_0 \in X(K)$ be a K-rational point. Denote by $x_S := (x_0, \mathrm{id}_S) \in X$ the S-valued point of $X \times S$. A *rigidified line bundle* along x_0 on $X \times S$ is a couple (L, ℓ) consisting of a line bundle on $X \times S$ and a section ℓ of $x_S^* L$ over S outside the zero section of L. The section ℓ is called a *rigidificator*. A *morphism of rigidified line bundles* is a bundle morphism which maps the indicated rigidificators to each other.

Remark 1.7.9. If X is geometrically reduced, connected and proper, the only automorphism of a rigidified line bundle on $X \times S$ is the identity. The automorphisms of a line bundle are given by the invertible functions on S. In fact, since $\mathcal{O}_S^\times \to (p_S)_* \mathcal{O}_{X \times S}^\times$ is bijective by Corollary 1.6.8, we see that all the automorphism are given by a scalar multiplication with elements in $\mathcal{O}_S^\times$. If the rigidificator is fixed, then the automorphism is the identity.

The latter is a necessary condition to expect a fine moduli space for the functor $\mathrm{Pic}_{X/K}$, which will be studied in Theorems 5.1.1 and 5.1.4. This functor associates to a rigid analytic space S the set of isomorphism classes of line bundles on $X \times S$ which are rigidified along the section of x_0.

In the following we will consider a rigid analytic variety Ω equipped with a group action $\Gamma \times \Omega \longrightarrow \Omega, (\gamma, z) \longmapsto \gamma z$, such that the rigid analytic quotient $X := \Gamma \backslash \Omega$ exists and such that $p : \Omega \to X$ is unramified in the topological sense. By this we mean the following.

Definition 1.7.10. A morphism $p : Y \to X$ of rigid analytic spaces is called *unramified in the topological sense* if there is an admissible covering $(U_i; i \in I)$ of X such that $p^{-1}(U_i) = \bigcup_{j \in I_i} V_{i,j}$ is a disjoint union and $p|_{V_{i,j}} : V_{i,j} \xrightarrow{\sim} U_i$ is an isomorphism for all $j \in I_i$ and $i \in I$.

We want to give a direct geometric description of the line bundles on a quotient $X = \Gamma \backslash \Omega$. This will be possible in the case of Mumford curves in Theorem 2.8.7 and in the case of analytic tori in Proposition 2.7.5. In both cases, line bundles on the universal covering Ω turn out to be trivial. For studying this, the following notions will be useful; cf. [74, I, §2].

Definition 1.7.11. A *Γ-linearization on a line bundle* L on Ω consists of a family of isomorphisms $(c_\alpha; \alpha \in \Gamma)$ of the line bundle L

$$\begin{array}{ccccccc} L & \xrightarrow{c_\alpha} & L & \qquad & \ell & \longmapsto & c_\alpha(z)(\ell) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow \\ \Omega & \xrightarrow{\alpha} & \Omega & & z & \longmapsto & \alpha z \end{array}$$

which is associative; i.e., $c_{\alpha\beta}(z)(\ell) = c_\alpha(\beta z) \circ c_\beta(z)(\ell)$, where $\ell \in L_z$ is a point above the point $z \in \Omega$, and satisfies $c_{\mathrm{id}} = \mathrm{id}$.

There is an evident notion of isomorphism of Γ-linearized line bundles.

If $L = \Omega \times \mathbb{A}^1$, then $c_\alpha : \Omega \to K^\times$ corresponds to a holomorphic function without zeros which acts as multiplication on a section $\ell \in L$. Therefore, the associativity is equivalent to

$$c_{\alpha\beta}(z) = c_\alpha(\beta z) \cdot c_\beta(z) \quad \text{for all } z \in \Omega,\ \alpha, \beta \in \Gamma.$$

A linearized line bundle (L, c) is *trivial* if L is trivial and if there exists an invertible holomorphic function u on Ω such that $c_\alpha(z) = u(\alpha(z)) \cdot u(z)^{-1}$. On the isomorphism classes of linearized line bundles there is a group law by putting

$$(L_1, c_1) \otimes (L_2, c_2) := (L_1 \otimes L_2, c_1 \otimes c_2).$$

Two Γ-linearized line bundles (L_1, c_1) and (L_2, c_2) are *isomorphic* if the tensor product $(L_1 \otimes L_2^{-1}, c_1 \otimes c_2^{-1})$ is trivial. If L_1 and L_2 are trivial, then $c_1 \otimes c_2$ is simply the multiplication by the product $c_1(z) \cdot c_2(z)$.

Remark 1.7.12. Let c be a Γ-linearization on $\Omega \times \mathbb{A}^1$. If c_α is constant for all elements $\alpha \in \Gamma$, then the linearization is equivalent to a group homomorphism $c : \Gamma \to K^\times, \alpha \mapsto c_\alpha$.

Example 1.7.13. Let Ω be a rigid analytic group variety and $\Gamma \subset \Omega$ be a discrete subgroup with respect to the Grothendieck topology. Let Γ act on Ω by left translations. Consider a group homomorphism

$$\lambda : \Gamma \longrightarrow \mathrm{Hom}(\Omega, \mathbb{G}_{m,K}),\ \alpha \longmapsto \big[z \longmapsto \big\langle \lambda(\alpha), z\big\rangle\big],$$

where $\langle \lambda(\alpha), z\rangle := \lambda(\alpha)(z)$. Then the rule $c_\alpha(z) = r_\alpha \cdot \langle \lambda(\alpha), z\rangle$, where the constants $r_\alpha \in K^\times$ satisfy

$$\big\langle \lambda(\alpha), \beta\big\rangle = \frac{r_{\alpha\beta}}{r_\alpha \cdot r_\beta},$$

defines a Γ-linearization on the trivial line bundle $\Omega \times \mathbb{A}^1$. If λ is trivial, then the Γ-linearization is translation invariant and $r : \Gamma \to K^\times$ is a group homomorphism.

Proposition 1.7.14. *In the situation of Definition 1.7.11, assume that the quotient $X = \Gamma \backslash \Omega$ exists and that $\Omega \to X$ is a unramified covering in the topological sense.*

(a) *Let $c := (c_\alpha; \alpha \in \Gamma)$ be a Γ-linearization on a line bundle L. Then the quotient $L(c) := L/c$ exists as a rigid analytic space, the residue map $p_L : L \to L(c)$ is an unramified covering in the topological sense and $L(c) \to X$ is a line bundle on X.*
(b) *For every line bundle $\overline{L}$ on X there is a canonical Γ-linearization c on $p^*\overline{L}$ such that $p^*\overline{L}/c$ is isomorphic to $\overline{L}$.*
(c) *The constructions in (a) and (b) constitute a one-to-one correspondence*

$$\left\{\begin{array}{l}\textit{Isoclasses of } \Gamma\textit{-linearized}\\ \textit{line bundles } (L,c) \textit{ on } \Omega\end{array}\right\} \stackrel{\sim}{\longleftrightarrow} \left\{\begin{array}{l}\textit{Isoclasses of line}\\ \textit{bundles } \overline{L} \textit{ on } X\end{array}\right\}.$$

Proof. (a) Since the covering $p : \Omega \to X$ is unramified in the topological sense, it is easy to see that the quotient $L(c) = L/c$ exists and that $L(c) \to X$ is a line bundle.

(b) There are obvious maps from $p^*\overline{L} \longrightarrow \alpha^* p^*\overline{L}$ and they give rise to a linearization with respect to Γ, because p is a covering in the topological sense. In fact, the Γ-linearization sends a point $(w, \ell) \in p^{-1}\overline{L}$ over $w \in \Omega$ to the point $(\gamma(w), \ell)$ over $\gamma(w)$. More precisely, choose a trivialization of $\overline{L}$ with respect to an admissible covering $\mathfrak{V} = \{V_i; i \in I\}$ of X, where each V_i is given by a chart $p : U_i \xrightarrow{\sim} V_i$ with $U_i \subset \Omega$. Then $W_i := p^{-1}(V_i)$ decomposes $\bigcup_{\alpha\in\Gamma} \alpha(U_i)$ as a disjoint union. Let $\tau_{i,j}$ be the transition functions on $V_i \cap V_j$. The mapping $(w, \ell) \mapsto (\gamma(w), \ell)$ maps the point $(w, \ell) \in \alpha(U_i) \times \mathbb{A}^1_K$ to $(\gamma(w), \ell) \in \gamma\alpha(U_i) \times \mathbb{A}^1_K$. This map is obviously compatible with the transition functions, because they are Γ-invariant.

(c) Due to (b), it suffices to show that a linearized line bundle (L, c) is isomorphic to $p^*L(c)$ equipped with the linearization which was defined in (b). It is easy to see that the linearization c can be used to define an isomorphism between these linearized line bundle. □

If every line bundle on Ω is trivial, then the left-hand side of Proposition 1.7.14(c) can be identified with the cohomology group $H^1(\Gamma, \mathcal{O}_\Omega(\Omega)^\times)$ and the right hand side with $H^1(X, \mathcal{O}_X^\times)$.

Lemma 1.7.15. *Let L be a line bundle on X and let $\mathcal{L}$ be the associated invertible sheaf. Let $(c_\gamma; \gamma \in \Gamma)$ be a Γ-linearization of L. Then the induced Γ-linearization on $\mathcal{L}$ is given by*

$$\varphi_\gamma : \mathcal{L}_z \longrightarrow \mathcal{L}_{\gamma z}, \ g \longmapsto g \circ c_\gamma^{-1},$$

and the Γ-linearization on $\mathcal{L}$ is given by $\varphi_\gamma(z) = c_\gamma(z)^{-1}$.

Proof. By Remark 1.7.2 there is a canonical identification of $\mathcal{L}(U)$ with group $\mathrm{Hom}_X(L|_U, , \mathbb{A}^1_X)$. So, if g belongs to $\mathrm{Hom}_X(L|_U, , \mathbb{A}^1_X)$, then the composition $g \circ c_\gamma^{-1}$ lies in $\gamma^*\mathcal{L}(U) = \mathrm{Hom}_X(L|_{\gamma U}, , \mathbb{A}^1_X)$. Thus, we see that the Γ-linearization on $\mathcal{L}$ is given by $(c_\gamma^{-1}; \gamma \in \Gamma)$. □

1.8 Algebraization of Proper Rigid Analytic Curves

Over the complex numbers, every compact Riemann surface is the analytification of a smooth projective algebraic curve. In this section we show that the analogous statement is valid in rigid geometry as well.

Theorem 1.8.1. *Let X be a smooth proper rigid analytic variety of dimension* 1, *and assume that X is geometrically connected.*

(a) *Then there exists an irreducible smooth projective algebraic curve X^{alg} and an isomorphism from X to the analytification of X^{alg}.*
(b) *Every meromorphic function on X is induced by a rational function on X^{alg}.*
(c) *If $g := \dim_K H^1(X, \mathcal{O}_X)$ denotes the genus of X, then $2g - 2$ is the degree of* $\mathrm{div}\,\omega$ *for every non-vanishing differential form ω on X. In particular, the genus of X equals the genus of X^{alg}.*

Proof. (a) First of all, due to the Riemann-Roch Theorem 1.7.6, there exists a meromorphic function f on X which is not constant. Moreover, we may assume that the differential of f does not vanish identically. Indeed, since $\dim_K H^0(X, \mathcal{O}_X(D))$ grows with $\deg D$, the differential $d : H^0(X, \mathcal{O}_X(D)) \to H^0(X, \Omega^1_{X/K}(D'))$ cannot be identically zero, where $D' = \sum_i (n_i + 1)x_i$ if $D = \sum_i n_i x_i$ is an effective divisor. Then f gives rise to a non-constant rigid analytic morphism

$$\varphi : X \longrightarrow \mathbb{P}^1_K,$$

which is obviously quasi-finite and proper, and hence finite by Corollary 1.6.7. Let n be the degree of φ. In particular, it is generically étale and surjective.

It is easy to see that the field of meromorphic functions on $\mathbb{P}^1_K$ coincides with the field $K(\zeta)$ of rational functions over K in a coordinate function ζ of $\mathbb{P}^1_K$. Now every meromorphic function h on X satisfies an equation

$$h^n + a_{n-1} \cdot h^{n-1} + \cdots + a_0 = 0$$

with coefficients in $K(\zeta)$. Indeed, $\varphi_* \mathcal{O}_X$ is a locally free $\mathcal{O}_{\mathbb{P}^1_K}$-module, and hence h satisfies such an equation with coefficients $a_i \in \Gamma(\mathbb{P}^1_K - S, \mathcal{O}_{\mathbb{P}^1_K})$, where S is the image by φ of the set of poles of h; for example, one can take the characteristic polynomial of the multiplication by h. Then it is easy to see that the coefficients are meromorphic on $\mathbb{P}^1_K$. Due to the existence of a primitive element there exists a meromorphic function h such that $\mathcal{M}(X) = \mathcal{M}(\mathbb{P}^1_K)(h)$. Thus, we see that h satisfies an equation as above and $\mathcal{M}(X) = K(\zeta, h)$. All coefficients $a_i = r_i/s_i$ have a representation with relatively prime polynomials $r_i, s_i \in K[\zeta]$. Then let s be the least common multiple $s := \operatorname{lcm}(s_0, \ldots, s_{n-1})$. Now consider the map

$$\psi := (1, f, h) : X \longrightarrow \mathbb{P}^2_K.$$

This map is defined everywhere, since X is smooth of dimension 1, and hence all local rings of X are valuation rings. Now look at the polynomial

$$F(\zeta, \eta) := s(\zeta) \cdot \left(\eta^n + a_{n-1}\eta^{n-1} + \cdots + a_0\right) \in K[\zeta, \eta].$$

If we write $\zeta := T_1/T_0$ and $\eta := T_2/T_0$, we obtain

$$G(T_0, T_1, T_2) := T_0^{\deg F} \cdot F(T_1/T_0, T_2/T_0) \in K[T_0, T_1, T_2],$$

which is a homogeneous polynomial. Its vanishing locus

$$Y := V(G) \subset \mathbb{P}^2_K$$

is an irreducible closed algebraic subset and ψ maps X onto Y. To prove this, we may assume that K is algebraically closed. From Theorem 1.7.6 follows that there exists a meromorphic function g which takes n distinct values on a fiber of φ over an unramified point, and hence its minimal polynomial over $K(\zeta)$ has degree n. Since g belongs to $\mathcal{M}(X) = K(\zeta, h)$, we see that the degree $[\mathcal{M}(X) : K(\zeta)] = n$, and hence that the polynomial F is irreducible.

Moreover, the map $\psi : X \to Y$ is surjective. Since ψ is proper, it suffices to show that there is a non-empty open algebraic subset $Y' \subset Y$ contained in the image of ψ. Therefore, we can restrict our attention to the affine part $\mathbb{A}^2_K := \mathbb{P}^2_K - V(T_0)$. Since $\varphi : X \to \mathbb{P}^1_K$ is surjective, it suffices to show that the fibers over every closed point $z \in \mathbb{A}^1_K - S$ under the first projection $p_1 : Y \to \mathbb{A}^1_K$ are contained in the image of ψ where S is a finite set. If we choose S as the set where φ is ramified or the projection

$p_1 : Y \to \mathbb{A}^1_K$ is ramified or not finite, the assertion is true. Indeed, on such a fiber h takes n different values.

Now let $\widetilde{Y} \to Y$ be the normalization which is again projective, as Y is projective. Then the map $\psi : X \to Y$ factorizes through $\widetilde{Y} \to Y$, because the local rings of X are normal. Thus it induces an isomorphism $X \to \tilde{Y}$.

(b) In the proof of (a) we have shown that the map $K(X^{\mathrm{alg}}) \to \mathcal{M}(X)$ is bijective.

(c) By (a) we can identify $\Omega^1_{X/K}$ with the analytification of the module of the regular differential forms on X^{alg}. By Theorem 1.6.11 the canonical maps $\Gamma(X^{\mathrm{alg}}, \Omega^1_{X^{\mathrm{alg}}/K}) \to \Gamma(X, \Omega^1_{X/K})$ and $H^1(X^{\mathrm{alg}}, \mathcal{O}_{X^{\mathrm{alg}}}) \to H^1(X, \mathcal{O}_X)$ are bijective. Thus, each global analytic differential form is an algebraic one of the same degree. We know from the algebraic case that the degree of $\mathrm{div}(\omega)$ equals $2g - 2$ where $g = \dim H^1(X^{\mathrm{alg}}, \mathcal{O}_{X^{\mathrm{alg}}}) = \dim H^1(X, \mathcal{O}_X)$. □

Now as we know that X is algebraic, we can rewrite the formula of Riemann-Roch by using global differential forms. Note that $\Omega^1_{X/K}$ equals the analytification of $\Omega^1_{X^{\mathrm{alg}}/K}$.

Corollary 1.8.2. *Let X be a smooth proper rigid analytic variety of dimension 1 assumed to be geometrically connected. Then for every divisor D of X we have*

$$\dim_K H^0\big(X, \mathcal{O}_X(D)\big) = \deg D + 1 - g + \dim_K H^0\big(X, \Omega^1_{X/K}(-D)\big),$$

where $g := \dim_K H^1(X, \mathcal{O}_X)$ is the genus of X.

Chapter 2
Mumford Curves

The incentive to create the theory of holomorphic functions over a non-Archimedean field was Tate's elliptic curve. By means of rigid geometry one can explain Tate's elliptic curve from the geometric point of view, whereas Tate originally formulated it in terms of function fields; cf. Sect. 2.1.

In the following sections we study Mumford's generalization of Tate's curve to curves of higher genus in the context of rigid geometry. We introduce discontinuous actions of certain subgroups Γ of $\mathrm{PGL}(2, K)$ on the projective line in the style of Schottky. The structure of these groups Γ was found by Ihara; cf. Sect. 2.2.

Mumford curves will be introduced as orbit spaces $\Gamma\backslash\Omega$, where $\Omega \subset \mathbb{P}^1_K$ is the largest subdomain of $\mathbb{P}^1_K$ on which Γ acts in an ordinary way. The construction of the quotient $\Gamma\backslash\Omega$ can be carried out in the framework of classical rigid geometry. Note that Mumford achieves much more general results in his ground braking article [75] which deals exclusively with formal schemes. The concept here follows geometric constructions in order to explain the ideas behind Mumford's construction.

Chapter 2 is somehow a counterpart of Riemann surfaces and their Jacobians. We provide the full picture of Mumford curves and their Jacobians which are rigid analytic tori. We show the duality theory of rigid analytic tori, Riemann's period relations and, moreover, Riemann's vanishing theorem.

Our approach is a refined version of the work of Drinfeld and Manin [64] and [65], where they work over a p-adic field; i.e., a finite extension of $\mathbb{Q}_p$. Here we consider a general non-Archimedean field as defined in Definition 1.1.1; notably we mention the work of Gerritzen [31–33].

2.1 Tate's Elliptic Curve

The following statements can be found in [85, §3, VI and VII] or [93], where they are stated for non-Archimedean fields; cf. Definition 1.1.1.

Theorem 2.1.1 (Tate). *Let K be a non-Archimedean field of arbitrary characteristic and let $q \in K^\times$ with $0 < |q| < 1$. Then the field of meromorphic q-periodic*

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functions on $\mathbb{G}_{m,K}$ *is an elliptic function field* $F(q)$; *i.e.,* $F(q)$ *is finitely generated field of transcendence degree* 1 *over* K *and of genus* 1. *More precisely,* $F(q) = K(\wp, \tilde{\wp})$, *where*

$$\wp(\xi) = \sum_{n\in\mathbb{Z}} \frac{q^n\xi}{(1-q^n\xi)^2} - 2\cdot s_1,$$
$$\tilde{\wp}(\xi) = \sum_{n\in\mathbb{Z}} \frac{q^{2n}\xi^2}{(1-q^n\xi)^3} + s_1$$

with

$$s_\ell := \sum_{m\geq 1} \frac{m^\ell q^m}{1-q^m} \quad \textit{for } \ell \in \mathbb{N}.$$

The associated projective curve $E(q)$ *is given by the inhomogeneous equation*

$$\tilde{\wp}^2 + \wp\cdot\tilde{\wp} = \wp^3 + B\cdot\wp + C$$

for $B := -5\cdot s_3$, $C := \frac{1}{12}(5\cdot s_3 + 7\cdot s_5)$, *which actually lie in* $q\mathbb{Z}[[q]]$.

Its j*-invariant is*

$$j(q) = \frac{(1-48\cdot B)^3}{\Delta} = \frac{1}{q} + R(q),$$

where

$$R(q) = 744 + 196884\cdot q + \cdots \in \mathbb{Z}[[q]],$$
$$\Delta(q) = B^2 - C - 64\cdot B^3 + 72\cdot BC - 432\cdot C^2 = q\cdot\prod_{n\geq 1}\left(1-q^n\right)^{24} \in \mathbb{Z}[[q]].$$

For every element $j \in K$ *with* $|j| > 1$ *there exists a unique* $q \in K$ *with* $0 < |q| < 1$ *such that* $j = j(q)$.

In the above statement, the ring $\mathbb{Z}[[q]]$ is viewed as a subring of K. Actually, the Tate curve can also be defined over the power series ring $\mathbb{Z}[[Q]]$, where Q is a variable.

Note that the theorem is entirely stated in terms of function fields. The associated elliptic curve $E(q)$ of the function field $F(q)$ is defined via the equivalence of categories between the category of function fields and the one of normal projective curves. In terms of rigid geometry $E(q)$ is a 1-dimensional rigid analytic torus $\mathbb{G}_{m,K}/q^{\mathbb{Z}}$.

Up to a finite separable extension of the ground field there are two types of elliptic curves over a non-Archimedean field; cf. [15, §1.5]:

Theorem 2.1.2. *Let* E *be an elliptic curve over a non-Archimedean field* K. *After a suitable finite separable extension of the ground field there are two possibilities*:

(i) *If* $|j(E)| \le 1$, *then* E *has good reduction.*
(ii) *If* $|j(E)| > 1$, *then* E *is isomorphic to the rigid analytic torus* $\mathbb{G}_{m,K}/q^{\mathbb{Z}}$ *for a unique* $q \in K^\times$ *with* $0 < |q| < 1$. *The* j*-invariant bijectively depends on* q *by a series*

$$j(q) = \frac{1}{q} + f(q)$$

with $f(q) \in R[[q]] \subset K$. *Thus,* $j(q)$ *converges on the open punctured disc and* j *gives rise to a biholomorphic map*

$$j : \left\{q \in K^\times; 0 < |q| < 1\right\} \xrightarrow{\sim} \left\{j \in K^\times; |j| > 1\right\}.$$

Over an algebraically closed field, an elliptic curve is uniquely determined up to isomorphism by its j-invariant. If the characteristic of the ground field is unequal 2, every elliptic curve E can be defined by a Legendre equation

$$E \cong E_\lambda := V\big(Y^2 \cdot Z - X \cdot (X - Z) \cdot (X - \lambda Z)\big) \subset \mathbb{P}^2_K$$

in the projective plane with $\lambda \in K - \{0, 1\}$. Its j-invariant is

$$j(E) = 2^8 \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2}$$

which is invariant under the substitution

$$\lambda \longmapsto \lambda, \quad 1 - \lambda, \quad \frac{1}{\lambda}, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda - 1}{\lambda}, \quad \frac{\lambda}{\lambda - 1}.$$

This reflects the isomorphism $E_\lambda \cong E_{\lambda'}$ for λ' equal to one of these values. In view of these isomorphisms, we may assume that $|\lambda| \le 1$, when K is a non-Archimedean field. Then

E_λ has good reduction if and only if $|\lambda| = 1$ *and* $|1 - \lambda| = 1$.

E_λ has multiplicative reduction if and only if $0 < |\lambda| < 1$ *or* $|1 - \lambda| < 1$.

Good reduction means that the polynomial $P(X) := X(X - 1)(X - \lambda)$ has three distinct roots in the reduction, whereas multiplicative reduction means that two roots of $P(X)$ collapse in the reduction.

The relationship between the modulus q and the Legendre parameter $\lambda \in K$ with $0 < |\lambda| < 1$ is the following:

$$q(\lambda) = c_2\lambda^2 + c_3\lambda^3 + \cdots$$

with $|c_2| = 1$ and $|c_i| \le 1$ for $i \in \mathbb{N}$. There are always exactly two values λ_1, λ_2 such that $|\lambda_i| < 1$ for $i = 1, 2$ with $j(\lambda_1) = j(\lambda_2)$; cf. [10, §9.7].

Using rigid geometry, one can construct Tate's curve in a geometric way. Consider an element $q \in K$ of a non-Archimedean field K with absolute value

$0 < |q| < 1$. Then $M := \{q^n; n \in \mathbb{Z}\}$ is a multiplicative lattice in the multiplicative group $\mathbb{G}_{m,K}$ in the sense of Sect. 2.7 and hence one can construct the quotient

$$E(q) := \mathbb{G}_{m,K}/M,$$

which is a proper smooth rigid analytic curve. We know from Theorem 1.8.1 that $E(q)$ is the analytification of a smooth projective curve. It is easy to show that $E(q)$ is an elliptic curve. Moreover, in the situation of Theorem 2.1.1, the field of rational functions on $E(q)$ is the field $F(q)$.

In the next section we will study more general group actions than Tate's action $M \times \mathbb{P}^1_K \to \mathbb{P}^1_K$; $(q, z) \longmapsto q \cdot z$, on the projective line. The group M is only a special case of a Schottky group; cf. Example 2.2.3, and so Tate's curves are a special case of Mumford curves; cf. Theorem 2.3.1. In the following sections we will present much more general results.

2.2 Schottky Groups

From now on, let K be a non-Archimedean field as defined in Definition 1.1.1. In this section we will study the structure of those finitely generated subgroups of the projective linear group $\mathrm{PGL}(2, K)$, which are free of torsion and act discontinuously on a non-empty open subdomain of the projective line.

For this, we cannot make use of the tree presentation of p-adic numbers as Drinfeld and Manin do. Instead we follow the classical method of isometric circles as invented by Ford [28]; see also the article of Gerritzen [31], which was slightly generalized by Kotissek [57].

In the following we consider the projective line $\mathbb{P}^1_K$ and equip the set $\mathbb{P}^1_K(K)$ of its K-rational points with the topology induced by the absolute value of K. The points in $\mathbb{P}^1_K(K) = K \cup \{\infty\}$ can be written in the form $[x, y]$ for $(x, y) \in K^2 - \{0\}$ if we want to mention their homogeneous coordinates. Hereby two symbols $[x, y]$ and $[x', y']$ are identified if there exists a $\lambda \in K^\times$ with $(x', y') = \lambda(x, y)$. A point $z \in K$ corresponds to $[z, 1]$ and ∞ to $[1, 0]$.

Each matrix $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, K)$ gives rise to an automorphism

$$\gamma_A : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K,\; z = [x, y] \longmapsto \frac{az+b}{cz+d} := [ax+by, cx+dy],$$

with the usual convention for ∞:

$$\frac{az+b}{cz+d} := \begin{cases} \dfrac{az+b}{cz+d} & \text{if } cz+d \neq 0 \text{ and } z \neq \infty, \\ \infty & \text{if } cz+d = 0 \text{ and } z \neq \infty, \\ a/c & \text{if } z = \infty \text{ and } c \neq 0, \\ \infty & \text{if } z = \infty \text{ and } c = 0. \end{cases}$$

Such a map is called a *Möbius transformation*. The map γ_A equals id if and only if $A = \lambda \cdot I_2$ for some $\lambda \in K^\times$. For all $A, B \in \mathrm{GL}(2, K)$ we have that $\gamma_{A \circ B} = \gamma_A \circ \gamma_B$. The group

$$\mathrm{PGL}(2, K) := \mathrm{GL}(2, K)/K^\times$$

is called the *projective linear group*. It is easy to see that $\mathrm{PGL}(2, K)$ is the group of K-rational automorphisms $\mathrm{Aut}(\mathbb{P}^1_K)$ of the projective line. It is generated by the elements $(z \mapsto z + b, z \mapsto a \cdot z, z \mapsto 1/z)$, because

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2}\left(z + \frac{d}{c}\right)^{-1} + \frac{a}{c}$$

for $c \neq 0$.

Definition 2.2.1. Let $z_1, z_2, z_3, z_4 \in \mathbb{P}^1_K(K)$ with $\{z_1, z_2\} \cap \{z_3, z_4\} = \emptyset$, then

$$\mathrm{CR}(z_1, z_2, z_3, z_4) := \begin{cases} \dfrac{z_1 - z_3}{z_1 - z_4} \cdot \dfrac{z_2 - z_4}{z_2 - z_3} & \text{for } z_1, z_2, z_3, z_4 \in K \\ \dfrac{z_1 - z_3}{z_2 - z_3} & \text{for } z_1, z_2, z_3 \neq \infty,\ z_4 = \infty \\ \dfrac{z_2 - z_4}{z_1 - z_4} & \text{for } z_1, z_2, z_4 \neq \infty,\ z_3 = \infty \\ \dfrac{z_1 - z_3}{z_1 - z_4} & \text{for } z_1, z_3, z_4 \neq \infty,\ z_2 = \infty \\ \dfrac{z_2 - z_4}{z_2 - z_3} & \text{for } z_2, z_3, z_4 \neq \infty,\ z_1 = \infty \\ 1 & \text{for } z_1 = z_2 = \infty \text{ oder } z_3 = z_4 = \infty \end{cases}$$

is called the *cross ratio* of z_1, z_2, z_3, z_4. The cross ratio is invariant under Möbius transformations.

Each matrix $A \in \mathrm{GL}(2, K)$ has two eigenvalues $\lambda_1, \lambda_2 \in L^\times$, where L/K is a field extension of degree $[L : K] \leq 2$. If $\lambda_1 \neq \lambda_2$, the map γ_A has two fixed points $z_1, z_2 \in \mathbb{P}^1_K(L)$. One can choose the coordinate in such a way that $z_1 = 0$ and $z_2 = \infty$. Then the map γ_A is just the multiplication by $q := \lambda_1/\lambda_2$. If $|\lambda_1| < |\lambda_2|$, then z_1 is the *attractive fixed point* and ∞ is the *repelling fixed point*. This means that, for every $w \in \mathbb{P}^1_K - \{0, \infty\}$, the sequence $(\gamma_A^n(w); n \in \mathbb{N})$ converges to z_1. In particular, z_1 and z_2 are K-rational in this case as K is complete. Such transformations are called *hyperbolic transformations*. The element $q_\gamma := q$ is called the *multiplier* of the hyperbolic transformation γ; it is uniquely determined by γ due to the requirement that $|q_\gamma| < 1$. If $|\lambda_1| = |\lambda_2|$, then the transformation is called *elliptic*. In this case, the transformation is of type $z \mapsto a \cdot z$ with $|a| = 1$.

If $\lambda_1 = \lambda_2$, then the transformation is called *parabolic*. In this case, the transformation looks like $z \mapsto z + b$. Thus, we have the following:

$$\gamma_A \text{ elliptic or parabolic} \iff \left|\frac{(a+d)^2}{ad-bc}\right| \leq 1,$$

$$\gamma_A \text{ hyperbolic} \iff \left|\frac{(a+d)^2}{ad-bc}\right| > 1.$$

Consider a subgroup $\Gamma \subset \mathrm{PGL}(2, K)$ and a K-rational point w of $\mathbb{P}^1_K(K)$. Put

$$L_\Gamma(w) := \left\{ z \in \mathbb{P}^1_K(K); \begin{array}{l} \text{there exists pairwise distinct } \gamma_n \in \Gamma \\ \text{for } n \in \mathbb{N} \text{ with } \gamma_n(w) \to z \end{array} \right\}.$$

Even, if one allows limit points z with values in some field extension, then they are K-rational, because K is complete and w is K-rational. Put

$$L_\Gamma := \bigcup_{w \in \mathbb{P}^1_K(K)} L_\Gamma(w), \quad \text{the set of } \textit{limit points of } \Gamma,$$
$$\Omega_\Gamma := \mathbb{P}^1_K - L_\Gamma, \quad \text{the set of } \textit{ordinary points of } \Gamma.$$

These group are named *discontinuous* if $L_\Gamma \neq \mathbb{P}^1_K(K)$ and if for every point w of $\mathbb{P}^1_K(K)$ the topological closure $\overline{\Gamma w}$ of the orbit of w is compact with respect to the metric topology of $\mathbb{P}^1_K(K)$. If K is locally compact, the latter hypothesis is always fulfilled.

Definition 2.2.2. A subgroup $\Gamma \subset \mathrm{PGL}(2, K)$ is called a *Schottky group* if Γ is finitely generated, free of torsion and discontinuous.

The group is named after Friedrich Schottky (1851–1935) who worked with similar group actions in complex analysis [86].

Example 2.2.3. If $\gamma \in \mathrm{PGL}(2, K)$ is hyperbolic, then $\Gamma := \langle \gamma \rangle$ is a Schottky group. The set L_Γ of the limit points of Γ consists of the two fixed points of γ.

Proof. We may assume that 0 and ∞ are the fixed points. Then γ acts via multiplication with an element $\lambda \in K^\times$ with $|\lambda| \neq 1$. Then it is clear that Γ is free of torsion and $\overline{\Gamma w}$ is compact for all $w \in \mathbb{P}^1_K(K)$ and $L_\Gamma = \{0, \infty\}$. □

This example corresponds to Tate's elliptic curve. The most general example is presented in Example 2.2.13. In the following let Γ be a Schottky group. For a $\gamma = \gamma_A$ with $c \neq 0$ set

$$r_\gamma := \frac{\sqrt{|ad-bc|}}{|c|}.$$

In the following we collect some properties of Schottky groups.

Proposition 2.2.4. *If $\Gamma \subset \mathrm{PGL}(2, K)$ is a Schottky group, then we have:*

(a) *Γ is discrete.*
(b) *Every $\gamma \in \Gamma - \{\mathrm{id}\}$ is hyperbolic.*
(c) *If $\infty \notin L_\Gamma$, then for $\gamma = \gamma_A \in \Gamma$ with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, K)$ we have:*

$$\gamma = \mathrm{id} \quad \Longleftrightarrow \quad \gamma(\infty) = \infty \quad \Longleftrightarrow \quad c = 0.$$

(d) *If $\infty \notin L_\Gamma$, then $R(\Gamma, \delta) := \{\gamma \in \Gamma - \{\mathrm{id}\}; r_\gamma \geq \delta\}$ is finite for every $\delta > 0$.*
(e) *$L_\Gamma(w)$ is closed for every $w \in \mathbb{P}^1_K(K)$.*
(f) *$L_\Gamma = L_\Gamma(\infty)$ if $\infty \notin L_\Gamma$.*

Proof. (a) The unit element of Γ is not an accumulation point of Γ, because otherwise $L_\Gamma = \mathbb{P}^1_K(K)$.

(b) Assume that $\gamma \in \Gamma - \{\mathrm{id}\}$ is elliptic or parabolic. For a suitable choice of the coordinate γ is given by a matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or by $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, respectively. Since γ is not of finite order, the elements $(\gamma^n; n \in \mathbb{N})$ are pairwise distinct. Since the closure of the orbit $\Gamma[1, 1]$ or of $\Gamma[1, 0]$, respectively, is compact, we may assume that the limit $\lim_{n\to\infty} \gamma^n[1, 1]$ or $\lim_{n\to\infty} \gamma^n[0, 1]$, respectively, exists and is a limit point. Then it is easy to see that the identity is an accumulation point of Γ. This contradicts (a).

(c) Only the assertion that $c = 0$ implies $\gamma = \mathrm{id}$ requires a proof. In this case γ is of type $z \mapsto az + d$. By (b) the map γ equals id or is hyperbolic. The latter means that $|a| \neq 1$. Then we may assume $|a| > 1$ and hence ∞ is the limit of the sequence $\gamma^n[1, 1]$. Since $[1, 0] = \infty \notin L_\Gamma$, we get $\gamma = \mathrm{id}$.

(d) By (c) the value r_γ is defined for every $\gamma \in \Gamma - \{\mathrm{id}\}$. If $R(\Gamma, \delta)$ has infinitely many elements, then there exist matrices $A_n := \left(\begin{smallmatrix} a_n & b_n \\ 1 & d_n \end{smallmatrix}\right)$ in $\mathrm{PGL}(2, K)$ such that $(\gamma_n := \gamma_{A_n} \in R(\Gamma, \delta); n \in \mathbb{N})$ are pairwise distinct. Since Γ is discontinuous, we may assume that the sequences $a_n = \gamma_n(\infty)$, $d_n = -\gamma_n^{-1}(\infty)$ and $b_n = -\gamma_n(0) \cdot \gamma_n^{-1}(\infty)$ converge to elements a, b, c of $\mathbb{P}^1_K(K)$. Since $\infty \notin L_\Gamma$, the points a, b, c lie in K, and hence

$$\lim_{n\to\infty} \begin{pmatrix} a_n & b_n \\ 1 & d_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix} =: A \in M(2 \times 2, K).$$

The determinant of A is $\lim_{n\to\infty} |a_n d_n - b_n| = \lim_{n\to\infty} r^2_{\gamma_n} \geq \delta^2$. Thus, the limit $\lim_{n\to\infty} \gamma_n = \gamma$ belongs to $\mathrm{PGL}(2, K)$. This contradicts (a).

(e) We may assume that $w \in K$ by choosing a suitable coordinate of $\mathbb{P}^1_K$. Let $z_m \in L_\Gamma(w)$ for $m \in \mathbb{N}$ with $z_0 := \lim_{m\to\infty} z_m$. There exists an element $\gamma_m \in \Gamma$ such that $|\gamma_m(w) - z_m| < 1/m$ and $\gamma_m \notin \{\gamma_1, \ldots, \gamma_{m-1}\}$. Then $\lim_{m\to\infty} \gamma_m(w) = z_0$ and, hence, $z_0 \in L_\Gamma(w)$.

(f) Consider an element $z \in L_\Gamma(w)$ for some $w \in \mathbb{P}^1_K(K)$ and let z be the limit $\lim_{n\to\infty} \gamma_n(w)$. If there exists a $c > 0$ with $|w - \gamma_n^{-1}(\infty)| \geq c$ for all $n \in \mathbb{N}$, then

$$\left|\gamma_n(\infty) - \gamma_n(w)\right| = \left|\frac{a_n}{c_n} - \frac{a_n w + b_n}{c_n w + d_n}\right| = \frac{r^2_{\gamma_n}}{|w - \gamma_n^{-1}(\infty)|} \leq \frac{r^2_{\gamma_n}}{c},$$

which converges to 0 by (d). If for every $c > 0$ there exists an $n \in \mathbb{N}$ such that $|w - \gamma_n^{-1}(\infty)| \le c$, then $\lim_{n\to\infty} \gamma_n(w) \in L_\Gamma(\infty)$. Since $L_\Gamma(\infty)$ is closed by (e), it follows that $z \in L_\Gamma(\infty)$. □

Notation 2.2.5. For later use we provide some explicit calculations when $\infty \notin L_\Gamma$. Consider a Möbius transformation γ given by an invertible matrix $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}(2, K)$. The inverse of A is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let Γ be a Schottky group with $\infty \notin L_\Gamma$. For $\gamma := \gamma_A \in \Gamma - \{\mathrm{id}\}$ put

$$\begin{aligned}
r_\gamma &:= \frac{\sqrt{|ad - bc|}}{|c|}, & r_{\gamma^{-1}} &= r_\gamma, \\
m_\gamma &:= \gamma^{-1}(\infty) = -d/c, & m_{\gamma^{-1}} &= \gamma(\infty) = a/c, \\
v_\gamma(z) &:= \gamma'(z) = \frac{ad - bc}{(cz + d)^2}, & v_{\gamma^{-1}}(z) &= \left(\gamma^{-1}\right)'(z) = \frac{ad - bc}{(-cz + a)^2}, \\
\left|v_\gamma(z)\right| &= \frac{r_\gamma^2}{|z - m_\gamma|^2}, \\
V_\gamma^- &:= \left\{z \in \mathbb{P}_K^1; \left|v_\gamma(z)\right| > 1\right\} = \left\{z \in \mathbb{P}_K^1; |m_\gamma - z| < r_\gamma\right\}, \\
V_\gamma^+ &:= \left\{z \in \mathbb{P}_K^1; \left|v_\gamma(z)\right| \ge 1\right\} = \left\{z \in \mathbb{P}_K^1; |m_\gamma - z| \le r_\gamma\right\}.
\end{aligned}$$

Note that $c \neq 0$ due to Proposition 2.2.4(c). For $\gamma = \mathrm{id}$ set $v_{\mathrm{id}} = 1$.

The domains of type V_γ^- and V_γ^+ are called *open rational discs* and *closed rational discs*, respectively; cf. Definition 2.4.1 below. They are isomorphic to the open unit disc and to the closed unit disc, respectively.

As an example, consider a matrix $A := \left(\begin{smallmatrix} 0 & q^2 \\ 1 & 1 \end{smallmatrix}\right)$ with $0 < |q| < 1$. Thus, we have that $\gamma z = q^2/(z + 1)$ for $\gamma := \gamma_A$. Then $m_\gamma = -1$, $m_{\gamma^{-1}} = 0$ and $r_\gamma = |q|$. In this case we have

$$\begin{aligned}
V_\gamma^+ &= \left\{z \in \mathbb{P}_K^1; |z + 1| \le |q|\right\}, \\
V_{\gamma^{-1}}^+ &= \left\{z \in \mathbb{P}_K^1; |z| \le |q|\right\}.
\end{aligned}$$

In particular, one computes $\gamma(\mathbb{P}_K^1 - V_\gamma^-) = V_{\gamma^{-1}}^+$ and $V_\gamma^+ \cap V_{\gamma^{-1}}^+ = \emptyset$.

Lemma 2.2.6. *Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a Schottky group with $\infty \notin L_\Gamma$. Then with the above notations we have*:

(a) $v_{\alpha\beta}(z) = v_\alpha(\beta z) \cdot v_\beta(z)$ *for* $\alpha, \beta \in \Gamma$.
(b) $\gamma(\mathbb{P}_K^1 - V_\gamma^-) = V_{\gamma^{-1}}^+$ *and* $\gamma(\mathbb{P}_K^1 - V_\gamma^+) = V_{\gamma^{-1}}^-$ *for* $\gamma \in \Gamma - \{\mathrm{id}\}$.
(c) $|m_\alpha - m_\beta| = \dfrac{r_\alpha \cdot r_\beta}{r_{\alpha\beta^{-1}}}$ *for elements* $\alpha, \beta \in \Gamma$ *with* $\alpha \neq \beta$.
(d) $V_\gamma^+ \cap V_{\gamma^{-1}}^+ = \emptyset$ *for* $\gamma \in \Gamma - \{\mathrm{id}\}$.

Proof. (a) follows from the chain rule for derivatives.

(b) follows from $v_{\gamma^{-1}}(\gamma z) = 1/v_\gamma(z)$ as was shown in (a).

(c) follows by a direct computation.

(d) follows from (c). Indeed, every element $\gamma \in \Gamma - \{\mathrm{id}\}$ is hyperbolic and $\gamma \neq \gamma^{-1}$. Therefore, $|m_\gamma - m_{\gamma^{-1}}| = \frac{|a+d|}{\sqrt{|ad-bc|}} \cdot r_\gamma > r_\gamma$. □

Definition 2.2.7. Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a Schottky group. A *fundamental domain* for Γ is a complement

$$E := \mathbb{P}^1_K - \left(B_1^- \cup \cdots \cup B_n^-\right)$$

of a finite union of open discs satisfying the following properties:

(i) $\bigcup_{\gamma \in \Gamma} \gamma(E)$ equals the set Ω_Γ of ordinary points of Γ,
(ii) $E \cap \gamma(E) = \emptyset$ for almost all $\gamma \in \Gamma$,
(iii) $E \cap \gamma(E^-) = \emptyset$ for all $\gamma \in \Gamma - \{\mathrm{id}\}$ where

$$E^- := \mathbb{P}^1_K - \left(B_1^+ \cup \cdots \cup B_n^+\right)$$

is the complement of the union of the associated closed discs.

In this context a disc is a rational disc; i.e., it is given by a center which is a point in K and a radius belonging to $|K^\times|$ so that B^- and B^+ are defined.

Lemma 2.2.8. *In the situation of Lemma 2.2.6, let V_γ^- be defined as in Notation 2.2.5 and let $R(\Gamma, \delta) \subset \Gamma$ be defined as in Proposition 2.2.4. Let $\delta > 0$ be small enough such that $R(\Gamma, \delta)$ is a system of generators of Γ. If $\alpha = \gamma_1 \cdot \ldots \cdot \gamma_n$ is a minimal representation with elements $\gamma_i \in R(\Gamma, \delta)$, then $V_\alpha^- \subset V_{\gamma_n}^-$.*

Proof. We proceed by induction on n. For $n = 1$ there is nothing to prove. Thus, let $n \geq 2$, and set $\beta := \gamma_2 \cdot \ldots \cdot \gamma_n$. Due to the induction hypothesis $V_\beta^- \subset V_{\gamma_n}^-$. Since $\alpha \notin R(\Gamma, \delta)$, we have $r_\alpha < \delta \leq r_{\gamma_1}$. From Lemma 2.2.6 it follows that $|m_\alpha - m_\beta| = r_\alpha r_\beta / r_{\gamma_1} < r_\beta$, and hence $V_\alpha^- \subset V_\beta^- \subset V_{\gamma_n}^-$ due to the ultrametric inequality. □

Lemma 2.2.9. *In the situation of Lemma 2.2.6, set $\mathfrak{V}(\Gamma) := \{V_\gamma^-; \gamma \in \Gamma - \{\mathrm{id}\}\}$ and*

$$R(\Gamma) := \left\{\gamma \in \Gamma;\, V_\gamma^- \textit{ maximal in } \mathfrak{V}(\Gamma)\right\}.$$

Then $R(\Gamma)$ is finite and $R(\Gamma)$ is stable under the map $\gamma \longmapsto \gamma^{-1}$.

Proof. The finiteness follows from Lemma 2.2.8 and Proposition 2.2.4(d). The second claim follows from

$$\begin{aligned} V_\alpha^- \subsetneqq V_\beta^- \quad &\Longleftrightarrow \quad r_\beta > |m_\alpha - m_\beta| = \frac{r_\alpha r_\beta}{r_{\alpha\beta^{-1}}} \text{ and } r_\alpha < r_\beta \\ &\Longleftrightarrow \quad r_\alpha < r_{\alpha\beta^{-1}} \text{ and } r_\alpha < r_\beta \\ &\Longleftrightarrow \quad r_{\alpha^{-1}} < r_{\beta\alpha^{-1}} \text{ and } r_{\alpha^{-1}} < r_{\beta^{-1}} \quad \Longleftrightarrow \quad V_{\alpha^{-1}}^- \subsetneqq V_{\beta^{-1}}^-. \end{aligned}$$ □

Proposition 2.2.10. *In the situation of Lemma 2.2.6, we have:*

(a) $E := \mathbb{P}^1_K - \bigcup_{\alpha \in R(\Gamma)} V^-_\alpha$ *is a fundamental domain for* Γ.
(b) $R(\Gamma)$ *generates* Γ.

Proof. (a) From Lemma 2.2.8 it follows

$$E = \{z \in \mathbb{P}^1_K;\ |v_\gamma(z)| \le 1 \text{ for all } \gamma \in \Gamma - \{\text{id}\}\},$$
$$E^- = \{z \in \mathbb{P}^1_K;\ |v_\gamma(z)| < 1 \text{ for all } \gamma \in \Gamma - \{\text{id}\}\}.$$

$R(\Gamma)$ is finite due to Lemma 2.2.9, so E is an affinoid domain of $\mathbb{P}^1_K$. Due to Lemma 2.2.8 the set E contains only ordinary points and $E \cap \gamma(E) \neq \emptyset$ only when $\gamma \in R(\Gamma) \cup \{\text{id}\}$, which is a finite set. Thus, $\gamma(E)$ contains only ordinary points, and hence $\bigcup_{\gamma \in \Gamma} \gamma(E) \subset \Omega_\Gamma$.

To show the converse inclusion, consider $z \in \Omega_\Gamma$. Such a point is contained only in a finite number of discs $V \in \mathfrak{V}(\Gamma)$. Indeed, assuming the contrary, there would exist an infinite number of $\gamma_n \in \Gamma$ such that $z \in V^-_{\gamma_n}$. Since $r_{\gamma_n} \to 0$ due to Proposition 2.2.4(d), the point z would be the limit of the points $m_{\gamma_n} = \gamma_n^{-1}(\infty)$ and hence z would belong to $L_\Gamma(\infty)$. That would be a contradiction to $z \in \Omega_\Gamma$.

For $z \in \Omega_\Gamma - E$ put

$$a := \sup_{\gamma \in \Gamma} |v_\gamma(z)| > 1.$$

Since $|v_\gamma(z)| > 1$ for only finitely many $\gamma \in \Gamma$, the supremum is attained, say $a = |v_\alpha(z)|$. Then we see by Lemma 2.2.6(a) that

$$\bigl|v_\gamma\bigl(\alpha(z)\bigr)\bigr| = \frac{|v_{\gamma\alpha}(z)|}{|v_\alpha(z)|} \le 1 \quad \text{for all } \gamma \in \Gamma,$$

and hence $\alpha(z) \in E$. Thus, we obtain $\Omega_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma(E)$.

It remains to show that $E^- \cap \gamma(E) = \emptyset$. For every point $z \in E^-$ we have $|v_\gamma(z)| < 1$. Since

$$1 = v_{\text{id}}(z) = v_{\gamma^{-1}\gamma}(z) = v_{\gamma^{-1}}\bigl(\gamma(z)\bigr) \cdot v_\gamma(z),$$

it follows that $|v_{\gamma^{-1}}(\gamma(z))| > 1$, and hence $\gamma(z) \notin E$ for all $\gamma \in \Gamma - \{\text{id}\}$.

(b) Let $\Gamma' := \langle R(\Gamma) \rangle \subset \Gamma$ be the subgroup generated by $R(\Gamma)$. Then Γ and Γ' have the same fundamental domain, as follows from (a). Thus, $\mathbb{P}^1_K - \bigcup_{\alpha \in R(\Gamma)} V^-_\alpha$ is a fundamental domain for Γ' as well. From (a) it follows $\Omega_{\Gamma'} \subset \Omega_\Gamma$. Obviously we have that $L_{\Gamma'} \subset L_\Gamma$, and hence $\Omega_{\Gamma'} = \Omega_\Gamma$. Consider now some $\gamma \in \Gamma$. Then $\gamma(\infty) \in \Omega_{\Gamma'}$. Thus, there exists $\gamma' \in \Gamma'$ such that $\gamma'\gamma(\infty) \in E$. Since $\infty \in E^-$ and $\gamma'\gamma(\infty) \in E$, we see that $\gamma'\gamma = \text{id}$ by (a) and thus $\gamma \in \Gamma'$. □

Later on, we want to construct the quotient $X_\Gamma := \Gamma \backslash \Omega_\Gamma$ explicitly. For this construction, it is useful to have a suitable fundamental domain. However the fundamental domain of Proposition 2.2.10(a) can be quite complicated. For example, the system $(V_\alpha^- \alpha \in R(\Gamma))$ has the disadvantage that it can happen that $V_\alpha^- = V_\beta^-$ and $V_{\alpha^{-1}}^- \neq V_{\beta^{-1}}^-$ or $V_\alpha^- \subsetneqq V_\beta^+ - V_\beta^-$. In the following we want to improve the system by varying the radii which are chosen equal to 1 for the above system. We will do it in a constructive way.

Notation 2.2.11. Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a Schottky group with $\infty \notin L_\Gamma$ and let $\rho : \Gamma \to \overline{K}^\times$ be a group homomorphism into the multiplicative group of the complete algebraic closure $\overline{K}$ of K. For $\gamma \in \Gamma - \{\mathrm{id}\}$ set

$$w_\gamma(z) := \rho(\gamma) \cdot v_\gamma(z) = \rho(\gamma) \cdot \gamma'(z) \quad \text{so that } w_{\alpha\beta}(z) = w_\alpha\big(\beta(z)\big) \cdot w_\beta(z),$$
$$W_\gamma^- := \big\{z \in \mathbb{P}_K^1;\, \big|w_\gamma(z)\big| > 1\big\} = \big\{z \in \mathbb{P}_K^1;\, |z - m_\gamma| < \sqrt{\big|\rho(\gamma)\big|} \cdot r_\gamma\big\},$$
$$W_\gamma^+ := \big\{z \in \mathbb{P}_K^1;\, \big|w_\gamma(z)\big| \geq 1\big\} = \big\{z \in \mathbb{P}_K^1;\, |z - m_\gamma| \leq \sqrt{\big|\rho(\gamma)\big|} \cdot r_\gamma\big\}.$$

Note that $w_\gamma(z)$ is the derivative of $\overline{\gamma}(z) := \rho(\gamma) \cdot \gamma(z)$ and $m_{\overline{\gamma}} = m_\gamma$ and $r_{\overline{\gamma}} = \sqrt{|\rho(\gamma)|} \cdot r_\gamma$. If $|\rho(\gamma)| < 1$ then $W_\gamma^+ \subset V_\gamma^-$ and $V_{\gamma^{-1}}^+ \subset W_{\gamma^{-1}}^-$.

The product formula implies $\gamma(\mathbb{P}_K^1 - W_\gamma^\pm) = W_{\gamma^{-1}}^\mp$ as in Lemma 2.2.6(b). Put

$$F := \mathbb{P}_K^1 - \bigcup_{\gamma \in \Gamma - \{\mathrm{id}\}} W_\gamma^- = \big\{z \in \mathbb{P}_K^1;\, \big|w_\gamma(z)\big| \leq 1 \text{ for all } \gamma \in \Gamma - \{\mathrm{id}\}\big\},$$
$$F^- := \mathbb{P}_K^1 - \bigcup_{\gamma \in \Gamma - \{\mathrm{id}\}} W_\gamma^+ = \big\{z \in \mathbb{P}_K^1;\, \big|w_\gamma(z)\big| < 1 \text{ for all } \gamma \in \Gamma - \{\mathrm{id}\}\big\}.$$

ρ is called *separating* if there exists a finite system of generators $\alpha_1, \ldots, \alpha_g$ of Γ such that the closed discs $W_{\alpha_1}^+, \ldots, W_{\alpha_g}^+, W_{\alpha_1^{-1}}^+, \ldots, W_{\alpha_g^{-1}}^+$ are pairwise disjoint. In this case, we use the numbering $\alpha_{g+j} := \alpha_j^{-1}$ for $j = 1, \ldots, g$.

Proposition 2.2.12. *In the situation of Notation 2.2.11, assume that the homomorphism $\rho : \Gamma \to \overline{K}^\times$ is separating. Then we have the following*:

(a) *There exists an element $q \in \sqrt{|K^\times|}$, $q < 1$ such $|w_{\alpha_i}(z)| < q$ for all $z \in W_{\alpha_j}^+$ and $j \in \{1, \ldots, 2g\}$ with $j \neq i$.*

(b) *Let $\gamma = \alpha_{j(1)} \cdot \ldots \cdot \alpha_{j(n)}$ be a reduced representation with $\alpha_{j(\nu)}$ in $\{\alpha_1, \ldots, \alpha_{2g}\}$ and $n \geq 1$. Consider a point $z \in \mathbb{P}_K^1 - W_{\alpha_{j(n)}}^-$. Then $\gamma(z)$ belongs to $W_{\alpha_{j(1)}^{-1}}^+$ and $|w_\gamma(z)| < q^{n-1}$; in particular, this is true for $z \in F$.*

If $n \geq 2$, then we even have that $\gamma(z) \in W_{\alpha_{j(1)}^{-1}}^-$.

(c) *We have that $W_\gamma^- \subset W_{\alpha_{j(n)}}^-$ for every $\gamma \in \Gamma - \{\mathrm{id}\}$ as in* (b). *In particular,*

$$F = \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^- \quad \textit{and} \quad F^- = \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^+.$$

(d) *The system $(W_{\alpha_1}^-, \ldots, W_{\alpha_{2g}}^-)$ of discs has the properties*:

(d.1) $\gamma(F) \cap F \neq \emptyset \Longleftrightarrow \gamma \in \{\mathrm{id}, \alpha_1, \ldots, \alpha_{2g}\}$,
(d.2) $\alpha_i(F) \cap F = W^+_{\alpha_i^{-1}} - W^-_{\alpha_i^{-1}}$.

(e) *Γ is a free group and $(\alpha_1, \ldots, \alpha_g)$ is a free system of generators.*

Proof. (a) This follows from the maximum principle, but it can also be seen by explicit calculations as follows. Since $W^+_{\alpha_i} \cap W^+_{\alpha_j} = \emptyset$ for $i \neq j$, it follows that

$$|m_{\alpha_i} - m_{\alpha_j}| > \max\left\{r_{\alpha_i} \cdot \sqrt{|\rho(\alpha_i)|}, r_{\alpha_j} \cdot \sqrt{|\rho(\alpha_j)|}\right\}.$$

For $z \in W^+_{\alpha_j}$ and $i \neq j$ it follows that $|z - m_{\alpha_i}| = |m_{\alpha_j} - m_{\alpha_i}|$ by the ultrametric inequality, and hence

$$\left|w_{\alpha_i}(z)\right| = \frac{|\rho(\alpha_i)| \cdot r^2_{\alpha_i}}{|z - m_{\alpha_i}|^2} = \frac{|\rho(\alpha_i)| \cdot r^2_{\alpha_i}}{|m_{\alpha_j} - m_{\alpha_i}|^2} < 1;$$

cf. Notation 2.2.5. Choose $q \in \sqrt{|K^\times|}$ with

$$1 > q > \max\left\{\frac{|\rho(\alpha_i)| \cdot r^2_{\alpha_i}}{|m_{\alpha_j} - m_{\alpha_i}|^2}; 1 \leq i, j \leq 2g, i \neq j\right\}.$$

Then q satisfies the assertion.

(b) We proceed by induction on n. For $n = 1$ is $\gamma = \alpha_{j(1)}$. For every z in $\mathbb{P}^1_K - W^-_{\alpha_{j(1)}}$ we have $\alpha_{j(1)}(z) \in W^+_{\alpha^{-1}_{j(1)}} \subset \mathbb{P}^1_K - W^+_{\alpha_{j(1)}}$, because $W^+_{\alpha_1}, \ldots, W^+_{\alpha_g}$, $W^+_{\alpha_1^{-1}}, \ldots, W^+_{\alpha_g^{-1}}$ are pairwise disjoint. Thus, we obtain for the absolute value $|w_{\alpha_{j(1)}}(z)| < 1 = q^0$.

Now assume $n \geq 2$. Set $\beta := \alpha_{j(2)} \cdot \ldots \cdot \alpha_{j(n)}$. From the induction hypothesis we obtain that $|w_\beta(z)| < q^{n-2}$ and $\beta(z) \in W^+_{\alpha^{-1}_{j(2)}} \subset \mathbb{P}^1_K - W^+_{\alpha_{j(1)}}$ and $|w_{\alpha_{j(1)}}(\beta(z))| < q$ for $z \in \mathbb{P}^1_K - W^-_{\alpha_{j(n)}}$, because $\alpha_{j(1)} \neq \alpha^{-1}_{j(2)}$. Then it follows that $\gamma(z) = \alpha_{j(1)}(\beta(z)) \in W^-_{\alpha^{-1}_{j(1)}}$. The inequality follows from the chain rule

$$\left|w_\gamma(z)\right| = \left|w_{\alpha_{j(1)}}\big(\beta(z)\big) \cdot w_\beta(z)\right| = \left|w_{\alpha_{j(1)}}\big(\beta(z)\big)\right| \cdot \left|w_\beta(z)\right| < q \cdot q^{n-2} = q^{n-1}.$$

(c) One has $m_\gamma = \alpha^{-1}_{j(n)} \cdot \ldots \cdot \alpha^{-1}_{j(1)}(\infty) \in W^-_{\alpha_{j(n)}}$ and $|w_\gamma(z)| < q^{n-1}$ for z in $\mathbb{P}^1_K - W^-_{\alpha_{j(n)}}$ due to (b). Thus, we see that $W^-_\gamma \subset W^-_{\alpha_{j(n)}}$ and, hence, $W^+_\gamma \subset W^+_{\alpha_{j(n)}}$. The latter implies the assertion on F and F^-.

(d.1) The implication "$\to$" follows from (b) and "$\leftarrow$" follows from (d.2).

(d.2) For $\alpha \in \{\alpha_1, \ldots, \alpha_{2g}\}$ we have that $\alpha(\mathbb{P}^1_K - W^\pm_\alpha) = W^\mp_{\alpha^{-1}}$ by the chain rule.

(e) If Γ were not free, then there would be a relation $\mathrm{id} = \alpha_{j(1)} \cdot \ldots \cdot \alpha_{j(n)}$ with $n \geq 1$. So, one obtains $z \in W^-_{\alpha^{-1}_{j(1)}}$ for all $z \in F$ due to (b). This is impossible. □

Using Proposition 2.2.12 one can construct examples of Schottky groups.

Example 2.2.13. Let $g \geq 1$ be an integer. Consider g pairs (B_i^-, B_{g+i}^-) of open discs and let (B_i^+, B_{g+i}^+) be the associated closed affinoid discs in $\mathbb{P}_K^1$ for $i = 1, \dots, g$. Assume that the $2g$ closed discs are pairwise disjoint and that ∞ does not belong to any of the closed discs.

Let $\alpha_1, \dots, \alpha_g$ be Möbius transformations in $\mathrm{PGL}(2, K)$ such that $\alpha_i(\mathbb{P}_K^1 - B_i^+) = B_{g+i}^-$ for $i = 1, \dots, g$. Then $\Gamma := \langle \alpha_1, \dots, \alpha_g \rangle$ is a Schottky group and $(\alpha_1, \dots, \alpha_g)$ is a free system of generators.

A fundamental domain for Γ is given by $F := \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} B_i^-$ and the set of its associated ordinary points is given by $\Omega_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma(F)$.

Such transformations $\alpha_i \in \mathrm{PGL}(2, K)$ with $\alpha_i(\mathbb{P}_K^1 - B_i^+) = B_{g+i}^-$ exist for $i = 1, \dots, g$ if the centers and the radii of $B_1, \dots, B_{2g}$ are K-rational.

Proof. Using the geometric configuration, one verifies that all the statements of Proposition 2.2.12 hold. First one shows that Γ is free and that $(\alpha_1, \dots, \alpha_g)$ is a free system of generators. In fact, put $\alpha_{g+i} := \alpha_i^{-1}$ for $i = 1, \dots, g$, as before. If $\gamma = \alpha_{j(1)} \dots \alpha_{j(n)} \in \Gamma$ is a reduced representation with $j(i) \in \{1, \dots, 2g\}$, then $\gamma(\infty) \in B^-_{\alpha^{-1}_{j(1)}}$. Therefore $\gamma \neq \mathrm{id}$. Then one can construct a group homomorphism $\rho : \Gamma \to \overline{K}^\times$ by choosing the images of ρ on the generating system $(\alpha_1, \dots, \alpha_g)$ arbitrarily. So one can define ρ such that $B_i^\pm = W_{\alpha_i}^\pm$ for $i = 1, \dots, 2g$. As in Proposition 2.2.12 all assertions on F resp. Ω_Γ follow.

It remains to show that Γ acts discontinuously. We only have to explain that the closure of the orbit Γw is compact for every $w \in \mathbb{P}_K^1(K)$. Since $\infty \notin L_\Gamma$, it suffices to show that every sequence $(\gamma_i(w); i \in \mathbb{N})$ with pairwise distinct $\gamma_i \in \Gamma$ admits a convergent subsequence. Put

$$F(n) := \bigcup_{\ell(\gamma) \leq n} \gamma(F) = \mathbb{P}_K^1 - (B_{n,1} \cup \dots \cup B_{n,r(n)}),$$

where $B_{n,j}$ are open discs contained in a large disc $B_0 := \{z \in \mathbb{A}_K^1; |z| \leq c\}$ because of $\infty \in F$. The symbol $\ell(\gamma)$ indicates the number of elements used in a reduced representation of γ as a product of the α_i. Note that there are only finitely many $\gamma \in \Gamma$ with $\ell(\gamma) \leq n$. Since F is a fundamental domain for Γ, almost all $(\gamma_i(w); i \in \mathbb{N})$ are contained in $\mathbb{P}_K^1 - F_n$. So almost all $(\gamma_i(w); i \in \mathbb{N})$ are contained in $(B_{n,1} \cup \dots \cup B_{n,r(n)})$. Then there exists a sequence $(B_{n,k(n)}; n \in \mathbb{N})$ such that infinitely many of the elements $(\gamma_i(w); i \in \mathbb{N})$ are contained in $B_{n,k(n)}$. Moreover, we can arrange the sequence in such a way that

$$B_{n+1,k(n+1)} \subset B_{n,k(n)} \quad \text{for all } n \in \mathbb{N}.$$

The radii $\rho(n)$ of $B_{n,k(n)}$ tend to 0 for $n \to \infty$. In fact, the set of the heights (cf. Definition 1.3.3) of the annuli $B_{n,k(n)} - B_{n+1,k(n+1)}$ is finite, because they are related under certain elements of Γ. Now we can choose elements $i(n) \in \mathbb{N}$ such that $\gamma_{i(n)}(w) \in B_{n,k(n)}$ for all $n \in \mathbb{N}$. Since the radii $\rho(n)$ tend to 0, the sequence $(\gamma_{i(n)}(w); n \in \mathbb{N})$ is a Cauchy sequence. Since K is complete, the sequence $(\gamma_{i(n)}(w); n \in \mathbb{N})$ converges. $\square$

Now we come to the main theorem of this section. The key result here is that every Schottky group can be obtained by the method of Example 2.2.13.

Theorem 2.2.14. *Let Γ be a Schottky group with $\infty \notin L_\Gamma$. Let $R(\Gamma)$ be the subset of Γ indexing the maximal discs V_γ^-; cf. Lemma 2.2.9.*

Then for every $q \in \mathbb{R}$ with $q < 1$ there exists a group homomorphism $\rho : \Gamma \to \overline{K}^\times$ and a separating system of generators $\alpha_1, \ldots, \alpha_g$ of Γ with respect to ρ with the following properties:

(i) $\alpha_i \in R := R(\Gamma)$ *for* $i = 1, \ldots, g$.
(ii) $q < \sqrt{|\rho(\alpha_i)|} < 1$ *for* $i = 1, \ldots, g$.

In particular, $(\alpha_1, \ldots, \alpha_g)$ is a free system of generators of Γ.

Proof. We proceed by induction on the number of elements of $R := R(\Gamma)$.

If R consists of ≤ 2 elements, then $\Gamma = \langle \gamma \rangle$ is the free group generated by one element; cf. Proposition 2.2.10.

The induction step will be done in several steps depending on the geometry of the fundamental domain given by the discs $(V_\alpha^-; \alpha \in R)$ defined in Lemma 2.2.9. Now we define

$$\begin{aligned} t &:= \min\{r_\alpha; \alpha \in R\}, \\ R' &:= \{\alpha \in R; r_\alpha > t\}, \\ \Gamma' &:= \langle R' \rangle, \\ q' &:= \max\left\{q, \frac{r_\alpha}{|m_\alpha - m_\beta|}; \alpha, \beta \in R, |m_\alpha - m_\beta| > r_\alpha\right\}. \end{aligned}$$

By the induction hypothesis we may assume that there exist a group homomorphism $\rho' : \Gamma' \to \overline{K}^\times$ and a separating system $\alpha_1, \ldots, \alpha_g$ for the subgroup $\Gamma' := \langle R' \rangle \subset \Gamma$ with respect to ρ', where $q' < \sqrt{|\rho'(\alpha_i)|} < 1$ for $i = 1, \ldots, g$. Here q' is chosen in such a way that for the enlarged discs $W_{\alpha_i^{-1}}^+$ we have $V_\gamma^- \cap W_{\alpha_i^{-1}}^+ = \emptyset$ if $|m_\gamma - m_{\alpha_i^{-1}}| > r_{\alpha_i^{-1}} = r_{\alpha_i}$ and $\gamma \in R - R'$ Due to Proposition 2.2.12(c), the group Γ' has the fundamental domain

$$F' := \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^- = \mathbb{P}_K^1 - \bigcup_{\gamma \in \Gamma' - \{\mathrm{id}\}} W_\gamma^-.$$

Now we have to consider the elements $\gamma \in R$ with $V_\gamma^- \subset F'$. This set is given by

$$T' := \left\{\alpha \in R; r_\alpha = t, m_\alpha \in F'\right\} \subset R - R'.$$

Moreover, we know that $R(\Gamma') = R'$. In fact, the inclusion $R' \subset R(\Gamma')$ is trivial because of $R' \subset R(\Gamma)$. Conversely, by Lemma 2.2.8 for every $\alpha \in R(\Gamma')$ there exists an element $\beta \in R'$ with $V_\alpha^- \subset V_\beta^-$, because R' generates Γ'. Since V_α^- is maximal and $\beta \in \Gamma'$, we obtain that $V_\alpha^- = V_\beta^-$ and hence $\alpha \in R'$ as R' is a subset of $R(\Gamma)$. In particular, we see that $\alpha_1, \ldots, \alpha_g \in R'$.

Due to the induction principle, we may assume that $\langle \Gamma', T' \rangle = \Gamma$. The problem now is how to vary the radii of the discs V_α^- for $\alpha \in R - R'$ via a suitable extension $\rho : \Gamma \to \overline{K}^\times$ of $\rho' : \Gamma' \to \overline{K}^\times$ in order to obtain a separating system for Γ. For this we have to analyze the configuration of the discs $(V_\alpha^-; \alpha \in T')$.

Let us first deduce some properties of T':

(1) If $\alpha \in T'$ and $\beta \in \Gamma$ with $r_\beta = t$ and $|m_\alpha - m_\beta| = t$, then $\beta \in T'$.

Since V_α^- is maximal and $V_\beta^+ = V_\alpha^+ \subset F'$, the disc V_β^- is maximal and $m_\beta \in F'$.

(2) For every $\gamma \in R - R'$ with $m_\gamma \notin F'$ there exists $\alpha \in \{\alpha_1, \ldots, \alpha_g\}$ such that $|m_\gamma - m_{\alpha^{-1}}| = r_\alpha$.

Since $m_\gamma \notin F'$, it follows that $m_\gamma \in W_\alpha^-$ for some $\alpha \in \{\alpha_1, \ldots, \alpha_{2g}\}$; i.e.

$$|m_\gamma - m_\alpha| < \sqrt{|\rho'(\alpha)|} \cdot r_\alpha.$$

Since V_γ^- is maximal, we have that $|m_\gamma - m_\alpha| \geq r_\alpha$, and so $\sqrt{|\rho'(\alpha)|} > 1$. Thus, it follows that $\alpha^{-1} \in \{\alpha_1, \ldots, \alpha_g\}$. If $|m_\gamma - m_\alpha| > r_\alpha$ then, due to the choice of q', we have that

$$r_\alpha \leq q' \cdot |m_\gamma - m_\alpha| < q' \cdot \sqrt{|\rho'(\alpha)|} \cdot r_\alpha < r_\alpha,$$

since $q'\sqrt{|\rho'(\alpha)|} < 1$. This is impossible, and hence we see $|m_\gamma - m_\alpha| = r_\alpha$.

(3) For $\gamma \in R - R'$ there exists a unique $\beta \in R$ with $|m_\gamma - m_\beta| < t$, $r_\beta = t$ and $\beta^{-1} \in T'$. In particular, $\beta\gamma^{-1} \in \Gamma'$. If $\gamma \in T'$, then $\beta \in T'$.

For every $\gamma \in R - R'$ we have $r_\gamma = t$. If $\gamma^{-1} \in T'$, then we choose $\beta := \gamma$. Otherwise, we have that $m_{\gamma^{-1}} \notin F'$ because of the very definition of T'. So due to (2) there exists $\alpha \in \{\alpha_1, \ldots, \alpha_g\}$ such that $|m_{\gamma^{-1}} - m_{\alpha^{-1}}| = r_\alpha$. Then set $\gamma_1 := \alpha^{-1}\gamma$. In particular, by Lemma 2.2.6(c) we have

$$r_\alpha = |m_{\gamma^{-1}} - m_{\alpha^{-1}}| = \frac{r_\alpha \cdot r_\gamma}{r_{\alpha^{-1}\gamma}} = \frac{r_\alpha \cdot r_\gamma}{r_{\gamma_1}},$$

and hence $r_{\gamma_1} = r_\gamma = t$. So we obtain that $\gamma_1 \in R - R'$. Moreover, it follows that

$$|m_\gamma - m_{\gamma_1}| = \frac{r_\gamma \cdot r_{\gamma_1}}{r_{\gamma\gamma_1^{-1}}} = \frac{t^2}{r_\alpha} < t.$$

If $m_{\gamma_1^{-1}} \notin F'$, then repeat the procedure with γ_1 instead of γ. Thus, we can construct a sequence $\gamma_n = \alpha_{j(n)}^{-1} \cdot \ldots \cdot \alpha_{j(1)}^{-1} \cdot \gamma$ with $j(\nu) \in \{1, \ldots, g\}$ and $r_{\gamma_n} = t$ such that

$$|m_{\gamma_{n-1}} - m_{\gamma_n}| = \frac{r_{\gamma_n} \cdot r_{\gamma_{n-1}}}{r_{\alpha_{j(n)}}} < t \quad \text{and also} \quad |m_\gamma - m_{\gamma_n}| < t,$$

because of the ultrametric inequality. In particular, we have $\gamma_n \in R$. Since R has only finitely many elements, the procedure stops after finitely many steps. Indeed, if not, then we would arrive at a non-trivial relation $\mathrm{id} = \alpha_{j(n)} \cdot \ldots \cdot \alpha_{j(m+1)}$, which is impossible as $(\alpha_1, \ldots, \alpha_g)$ is a free system of generators. Thus, we finally arrive at the situation that $\beta = \gamma_n \in R$ with $m_{\beta^{-1}} \in F'$; i.e., $\beta^{-1} \in T'$ and $\beta\gamma^{-1} = \alpha_{j(1)}^{-1} \ldots \alpha_{j(n)}^{-1} \in \varGamma'$.

If $\gamma \in T'$, then $m_\gamma \in F'$. Since $|m_\gamma - m_\beta| < t$, we have $m_\beta \in F'$, and then $\beta \in T'$ because of $r_\beta = t$.

Next we turn to the proof of the uniqueness. Assume that there are α, β in R with the asserted properties. Now put $\delta := \alpha\beta^{-1} \in \varGamma'$ and assume $\delta \neq \mathrm{id}$. Then we have that

$$\delta(m_{\beta^{-1}}) = \alpha\beta^{-1}\big(\beta(\infty)\big) = \alpha(\infty) = m_{\alpha^{-1}};$$

cf. Notation 2.2.5. Because $\alpha^{-1}, \beta^{-1} \in T'$, we have $m_{\alpha^{-1}}, m_{\beta^{-1}} \in F'$ and hence $F' \cap \delta(F') \neq \emptyset$. Thus, we see that $\delta \in \{\alpha_1, \ldots, \alpha_{2g}\}$ due to Proposition 2.2.12(d.1), and hence $|\rho'(\delta)| \neq 1$. Moreover, we have that

$$|m_{\delta^{-1}} - m_{\alpha^{-1}}| = \frac{r_\delta \cdot r_\alpha}{r_{\delta^{-1}\alpha}} = \frac{r_\delta \cdot r_\alpha}{r_\beta} = r_\delta,$$
$$|m_\delta - m_{\beta^{-1}}| = \frac{r_\delta \cdot r_\beta}{r_{\delta\beta}} = \frac{r_\delta \cdot r_\beta}{r_\alpha} = r_\delta,$$

because $r_\alpha = t = r_\beta$, and hence $m_{\beta^{-1}} \in V_\delta^+$ and $m_{\alpha^{-1}} \in V_{\delta^{-1}}^+$. Since $|\rho'(\delta)| \neq 1$, we have that $V_{\delta^{-1}}^+ \subset W_{\delta^{-1}}^-$ or $V_\delta^+ \subset W_\delta^-$ and so $m_{\alpha^{-1}} \in W_{\delta^{-1}}^-$ or $m_{\beta^{-1}} \in W_\delta^-$. This contradicts the fact that $m_{\beta^{-1}} \in F'$ and $m_{\alpha^{-1}} \in F'$ because of $\alpha^{-1}, \beta^{-1} \in T'$.

(4) $\varGamma = \langle \varGamma', T \rangle$ where $T := \{\alpha \in T'; \alpha^{-1} \in T'\}$.

It suffices to show that $\gamma \in \langle \varGamma', T \rangle$ for all $\gamma \in R - R'$. From (3) it follows that $\gamma \in \langle \varGamma', T' \rangle$, and also $T' \subset \langle \varGamma', T \rangle$. In fact, for $\alpha \in T'$ there exists $\beta \in R$ with $\beta\alpha^{-1} = \gamma \in \varGamma'$ and $\beta^{-1} \in T'$ by (3). Since $\alpha \in T'$, we also have that $\beta \in T'$ by (3). Thus, β and β^{-1} belong to T', and hence $\beta \in T$. Thus, we obtain that $\alpha = \beta\gamma^{-1}$ lies in $\langle \varGamma', T \rangle$.

For $\alpha \in T$ put

$$T_\alpha := \big\{\beta \in T; |m_\alpha - m_\beta| \leq t\big\}.$$

Because of the uniqueness in (3), we even have

$$T_\alpha := \big\{\beta \in T; |m_\alpha - m_\beta| = t\big\} \cup \{\alpha\}.$$

(5) If $\alpha \in T$ and $\beta \in T_\alpha$, then $T_\alpha = T_\beta$.

This follows from the ultrametric inequality.

(6) If $\alpha \in T$, then $T_{\alpha^{-1}} = \{\beta\alpha^{-1}; \beta \in T_\alpha, \beta \neq \alpha\} \cup \{\alpha^{-1}\}$.

Indeed, if $\beta \in T_\alpha$ and $\alpha \neq \beta$, then

$$t = |m_\alpha - m_\beta| = \frac{r_\alpha \cdot r_\beta}{r_{\beta\alpha^{-1}}} = \frac{t \cdot t}{r_{\beta\alpha^{-1}}},$$

and hence $r_{\beta\alpha^{-1}} = t$. Thus, by the same formula $|m_{\beta\alpha^{-1}} - m_{\alpha^{-1}}| = t$ and so $\beta\alpha^{-1} \in T'$ due to (1) because of $\alpha^{-1} \in T'$. Similarly, one shows that $\alpha\beta^{-1} \in T'$, because one can use the fact that $T_\beta = T_\alpha$; cf. (5). Thus, we see $\beta\alpha^{-1} \in T$. The computation showed $\beta\alpha^{-1} \in T_{\alpha^{-1}}$. Conversely, consider an element $\beta \in T_{\alpha^{-1}}$ with $\beta \neq \alpha^{-1}$. From what we have proved above, it follows that $\beta(\alpha^{-1})^{-1} \in T_\alpha$. Obviously we have that $\beta(\alpha^{-1})^{-1} \neq \alpha$ because of $\beta \neq \mathrm{id}$, and hence the element $\beta = (\beta(\alpha^{-1})^{-1})\alpha^{-1}$ belongs to the right-hand side.

(7) If $\alpha \in T$ and $\beta \in T_\alpha$ with $\alpha \neq \beta$, then $T_{\alpha^{-1}} \cap T_{\beta^{-1}} = \emptyset$.

Indeed, if $T_{\alpha^{-1}} \cap T_{\beta^{-1}} \neq \emptyset$, then $T_{\alpha^{-1}} = T_{\beta^{-1}}$ by (5), and hence $T_\alpha = T_\beta$ as follows from Lemma 2.2.6(c). From (6) we obtain that $\beta^{-1} = \delta\alpha^{-1}$ for some $\delta \in T_\alpha$. Similarly one has that $\alpha^{-1} = \delta^{-1}\beta^{-1} \in T_{\beta^{-1}}$ with $\delta^{-1} \in T_\beta$ by (6). Since $T_\alpha = T_\beta$, we see $\delta, \delta^{-1} \in T_\alpha$. But this is impossible due to Lemma 2.2.6(d), since $|m_\delta - m_{\delta^{-1}}| > r_\delta = t$.

For $\alpha \in T$ put

$$S_\alpha := T_\alpha \cup \bigcup_{\beta \in T_\alpha} T_{\beta^{-1}} \quad \text{and} \quad \Gamma_\alpha := \langle S_\alpha \rangle.$$

The union over $\beta \in T_\alpha$ is disjoint due to (7), because by (5) we know that $T_\alpha = T_\beta = T_\gamma$ for $\beta, \gamma \in T_\alpha$. Furthermore, $T_\alpha \cap T_{\beta^{-1}} = T_\beta \cap T_{\beta^{-1}} = \emptyset$ due to Lemma 2.2.6(c). Then (6) implies that

$$(8) \quad \begin{aligned} S_\alpha &= T_\alpha \cup T_{\alpha^{-1}} \cup \textstyle\bigcup_{\beta \in T_\alpha} \{\delta\beta^{-1}; \delta \in T_\alpha, \delta \neq \beta\} \\ &= T_\alpha \cup T_{\alpha^{-1}} \cup \{\delta\varepsilon^{-1}; \delta, \varepsilon \in T_\alpha, \delta \neq \varepsilon\}. \end{aligned}$$

In particular, S_α is invariant under the inverse map $\gamma \mapsto \gamma^{-1}$. In a special case the arrangement of the discs is shown in Fig. 2.1.

If $\gamma \in T_\alpha$, then $T_\gamma = T_\alpha$ by (5) and hence $S_\gamma = S_\alpha$.

If $\gamma \in T_{\beta^{-1}}$ for some $\beta \in T_\alpha$, then $T_\alpha = T_\beta$ and $T_\gamma = T_{\beta^{-1}}$ by (5). Thus, we see $T_\gamma = T_{\alpha^{-1}}$. Therefore, we may assume $\gamma = \alpha^{-1}$. Since S_α is invariant under the inverse map, we obtain $S_\alpha = S_{\alpha^{-1}}$. Thus, (6) implies:

(9) We have that $S_\alpha = S_\gamma$ for every $\gamma \in S_\alpha$. Therefore,

$$T = S_{\tau_1} \dot{\cup} \cdots \dot{\cup} S_{\tau_r}$$

is a disjoint union of sets S_{τ_i} for suitable $\tau_1, \ldots, \tau_r \in T$.

(10) T_α is a free system of generators of Γ_α and is separating with respect to a suitable group homomorphism $\rho_\alpha : \Gamma_\alpha \to \overline{K}^\times$ which can be chosen arbitrarily close to 1.

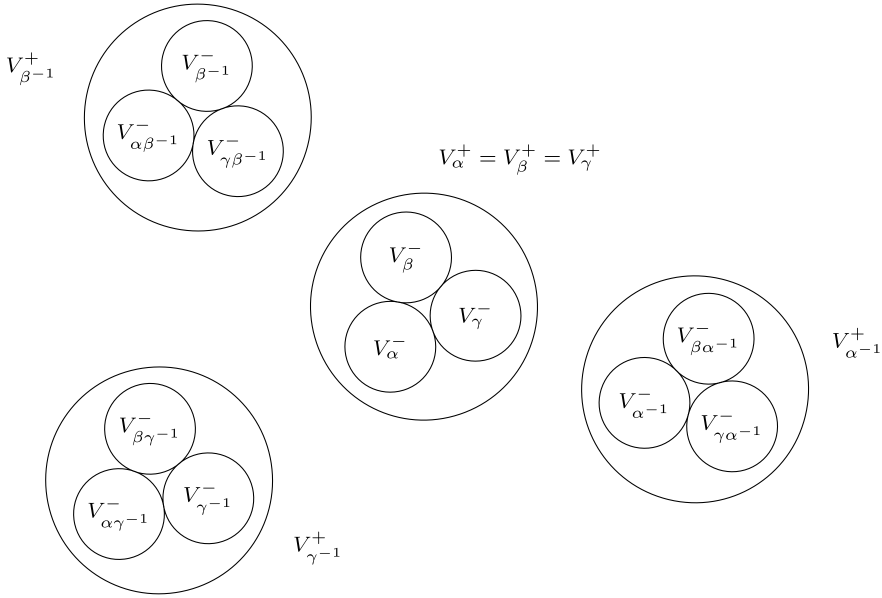


Fig. 2.1 Position of the discs S_α for $T_\alpha = \{\alpha, \beta, \gamma\}$

Indeed, from (8) it follows $\langle T_\alpha \rangle = \Gamma_\alpha$ and from (7) that $V^+_{\beta_1^{-1}} \cap V^+_{\beta_2^{-1}} = \emptyset$ for $\beta_1, \beta_2 \in T_\alpha$ with $\beta_1 \neq \beta_2$. For $q_\alpha \in |\overline{K}^\times|$ with $q_\alpha < 1$ put

$$B^+_\beta \quad := \left\{ z \in \mathbb{P}^1_K;\, |m_\beta - z| \le q_\alpha r_\beta \right\},$$
$$B^+_{\beta^{-1}} := \left\{ z \in \mathbb{P}^1_K;\, |m_{\beta^{-1}} - z| \le r_\beta / q_\alpha \right\},$$

and define B^-_β and $B^-_{\beta^{-1}}$ similarly. In this way we shrink $V^\pm_\beta$ and enlarge $V^\pm_{\beta^{-1}}$ for $\beta \in T_\alpha$. If we choose q_α close to 1, then the discs B^+_β for $\beta \in T_\alpha \cup T_\alpha^{-1}$ are pairwise disjoint. Furthermore, $\beta(\mathbb{P}^1_K - B^\pm_\beta) = B^\mp_{\beta^{-1}}$ for $\beta \in T_\alpha$. Thus, we arrive at the situation of Example 2.2.13, and hence T_α is a free system of generators of Γ_α.

Then we can define $\rho_\alpha : \Gamma_\alpha \to \overline{K}^\times$ by choosing values $\rho_\alpha(\beta)$ for $\beta \in T_\alpha$ with $q' < \sqrt{|\rho_\alpha(\beta)|} < 1$ and extending the map by linearity. This means that we shrink the discs V^-_β for $\beta \in T_\alpha$. Due to (7) we can do this in such a way that the intersection $W^+_{\beta_1^{-1}} \cap W^+_{\beta_2^{-1}} = \emptyset$ is empty for $\beta_1, \beta_2 \in T_\alpha$ with $\beta_1 \neq \beta_2$.

(11) $R(\Gamma_\alpha) = S_\alpha$.

By definition we have that $S_\alpha \subset R(\Gamma_\alpha)$. Any $\gamma \in R(\Gamma_\alpha)$ has a unique representation as a finite product by elements of $T_\alpha \cup T_\alpha^{-1}$. Thus, for every $\gamma \in R(\Gamma_\alpha)$ we have by Proposition 2.2.12(c) that $W^-_\gamma \subset W^-_\beta$ for some $\beta \in T_\alpha \cup T_\alpha^{-1}$. Since we can choose ρ_α such that $|\rho_\alpha|$ is arbitrarily close to 1, we obtain that $V^+_\gamma \subset V^+_\beta$ for some β in

$T_\alpha \cup T_\alpha^{-1}$. Thus, we see that $|m_\gamma - m_\beta| \le t$ and $r_\gamma = t$, because t is the maximal possible radius. If $|m_\gamma - m_\beta| < t$, then $\gamma = \beta$ due to (3). If $|m_\gamma - m_\beta| = t$, then $\gamma \in T_\beta \subset S_\alpha$, and hence the assertion follows.

In (4) we defined the set T and showed that $\Gamma = \langle \Gamma', T\rangle$. In (9) we saw that T decomposes into a disjoint union

$$T = S_{\tau_1} \dot\cup \cdots \dot\cup S_{\tau_r}.$$

Put $T_i := T_{\tau_i}$ and $\Gamma_i := \Gamma_{\tau_i} := \langle S_{\tau_i}\rangle = \langle T_{\tau_i}\rangle$ for $i = 1, \dots, r$. In (10) we saw that Γ_i is a free group and, moreover, we constructed a separating group homomorphism $\rho_i : \Gamma_i \to \overline{K}^\times$. From the induction hypothesis we know that the subgroup $\Gamma_0 := \Gamma'$ is free and that there exists a separating group homomorphism $\rho_0 := \rho'$. Then let $T_0 := \{\alpha_1, \dots, \alpha_g\}$ be the separating basis of Γ_0. Then set

$$\mathcal{T} = T_0 \cup T_1 \cup \cdots \cup T_r.$$

Due to our construction we have $T_i \cap T_j = \emptyset$ for $i \ne j$, and hence $V_\alpha^+ \cap V_\beta^+ = \emptyset$ for $\alpha^{\pm 1} \in T_i$ and $\beta^{\pm 1} \in T_j$ for $i \ne j$. With respect to $\rho_0, \dots, \rho_r$ we define the subdomains $W_{\alpha^{\pm 1}}^{\pm}$ for $\alpha \in \mathcal{T}$ as in Notation 2.2.11. We can choose the ρ_i in such a way that $W_\alpha^+ \cap W_\beta^+ = \emptyset$ for all $\alpha, \beta \in \mathcal{T} \cup \mathcal{T}^{-1}$ with $\alpha \ne \beta$. Then we claim

(12) $\Gamma = \Gamma_0 \amalg \cdots \amalg \Gamma_r$

is a coproduct in the category of groups. Let $\rho : \Gamma \to \overline{K}^\times$ be the group homomorphism induced by $\rho_0, \dots, \rho_r$ given on the factors.

In fact, due to (4) the group Γ is generated by Γ_0 and T. Since T is contained in $\langle \Gamma_1, \dots, \Gamma_r\rangle$, we have that $\Gamma \subset \langle \Gamma', \Gamma_1, \dots, \Gamma_r\rangle$, and hence that Γ coincides with $\langle \Gamma', \Gamma_1, \dots, \Gamma_r\rangle$.

The direct decomposition $\coprod_{i=0}^r \Gamma_i$ of Γ follows as in (10). Indeed, if there is a reduced product $\gamma := \delta_{j(1)}^{\pm 1} \cdot \ldots \cdot \delta_{j(n)}^{\pm 1}$ with $\delta_{j(\nu)} \in \mathcal{T}$, then $\gamma(z) \in W^+_{\delta^{-1}_{j(1)}}$ or $\gamma(z) \in W^+_{\delta_{j(1)}}$ for every point z in $F := \mathbb{P}^1_K - \bigcup_{\delta^{\pm 1} \in \mathcal{T}} W_\delta^-$. Thus, we see $\gamma \ne \mathrm{id}$. So, the coproduct is direct and ρ is a separating morphism.

Then $(\mathcal{T}, \rho)$ satisfies the assertion of the theorem. Indeed, each factor in (12) is a free group due to the induction hypothesis and because of (10).

This finishes the proof of Theorem 2.2.14. □

Remark 2.2.15. The radii r_{α_i} of the discs V_{α_i} belong to $\sqrt[2]{|K^\times|}$.

If the valuation of K is not discrete, the homomorphism ρ can be chosen in such a way that $\rho \in \mathrm{Hom}(\Gamma, \overline{K}^\times)$ and the radii $\sqrt[2]{|\rho(\alpha_i)|} \cdot r_{\alpha_i}$ of the discs W_{α_i} belong to $|K^\times|$.

If the valuation is discrete and π is a uniformizer of the valuation, then

$$\max\left\{ \frac{r_\alpha}{|m_\alpha - m_\beta|}; \alpha, \beta \in R(\Gamma), r_\alpha < |m_\alpha - m_\beta| \right\} \le \sqrt[2]{|\pi|}.$$

If one puts $q = \sqrt[2]{|\pi|}$, then there exists a separating homomorphism ρ in $\mathrm{Hom}(\Gamma, \overline{K}^\times)$ with $|\pi| < |\rho(\alpha_i)| < 1$ for $i = 1, \dots, g$. Without loss of generality one can choose $|\rho(\alpha_i)| = \sqrt[2]{|\pi|}$ for $i = 1, \dots, g$. Then the radii of the discs W_{α_i} belong to $\sqrt[2]{|K^\times|}$. This is the best possibility.

Corollary 2.2.16 (Ihara). *Schottky groups are free.*

This result was shown by Ihara in the case of a discrete valuation; cf. [48]. See also the book of Serre [90, II, §1.5].

For later use we add a result on the geometry of the fundamental domain.

Corollary 2.2.17. *Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a non-trivial Schottky group with $\infty \notin L_\Gamma$. Then there exists a separating system of generators $(\alpha_1, \dots, \alpha_g)$ with respect to a suitable homomorphism $\rho : \Gamma \to \overline{K}^\times$ which can be chosen in such a way that $|\rho(\alpha_i)| < 1$ is arbitrarily close to 1 for $i = 1, \dots, g$. Put $\alpha_{g+i} = \alpha_i^{-1}$ for $i = 1, \dots, g$, and put*

$$E^\circ := \mathbb{P}^1_K - \left[V^-_{\alpha_1} \cup \dots \cup V^-_{\alpha_g} \cup V^+_{\alpha_{g+1}} \cup \dots \cup V^+_{\alpha_{2g}}\right].$$

Then E° is a complete system of representative of $\Gamma \backslash \Omega_\Gamma$.

Let $z_1, z_2 \in \Omega_\Gamma$. Assume that $z_1 \in E^\circ$ and $\beta z_2 \in E^\circ$ for some $\beta \in \Gamma$. Then, for $\gamma \in \Gamma$, the following is true:

If $\beta = \mathrm{id}$, then $|\gamma z_1 - \gamma z_2| \le r_\gamma$.

If $\beta \ne \mathrm{id}$, then $|\gamma z_1 - \gamma z_2| \le \max\{\frac{r_\gamma \cdot r_{\gamma\beta^{-1}}}{r_\beta}, r_\gamma, r_{\beta\gamma^{-1}}\}$.

If $\ell(\beta)$ is bounded, then the distance $|\gamma z_1 - \gamma z_2| \to 0$ converges uniformly to 0 if $\ell(\gamma) \to \infty$ tends to ∞.

If $V \subset \Omega_\Gamma$ is an affinoid subdomain, there exists an $N \in \mathbb{N}$ such that the intersection $V \cap \gamma(V) = \emptyset$ is empty for all $\gamma \in \Gamma$ with $\ell(\gamma) \ge N$.

Proof. The first assertion follows from Theorem 2.2.14. If we approach $\rho(\alpha_i)$ to 1 from below for $i = 1, \dots, g$, then we see that the discs $V^+_{\alpha_1}, \dots, V^+_{\alpha_{2g}}$ are the maximal closed discs in the family $(V^+_\gamma; \gamma \in \Gamma)$. Due to Lemma 2.2.6(b) we know that $\gamma z_1 \in V^+_{\gamma^{-1}}$ and $\gamma z_2 = \gamma\beta^{-1}(\beta z_2) \in V^+_{\beta\gamma^{-1}}$.

If $\beta = \mathrm{id}$, then $\gamma z_1, \gamma z_2 \in V^+_{\gamma^{-1}}$, and hence, $|\gamma z_1 - \gamma z_2| \le r_{\gamma^{-1}} = r_\gamma$.

If $\beta \ne \mathrm{id}$, then $|\gamma z_1 - m_{\gamma^{-1}}| \le r_\gamma$ and $|\gamma z_2 - m_{\beta\gamma^{-1}}| \le r_{\beta\gamma^{-1}}$; cf. Notation 2.2.5. Then it follows from Lemma 2.2.6(c) that

$$|m_{\gamma^{-1}} - m_{\beta\gamma^{-1}}| = \frac{r_{\gamma^{-1}} \cdot r_{\beta\gamma^{-1}}}{r_\beta}.$$

Then the asserted estimate follows by the ultrametric inequality.

The bound tends to 0 if $\ell(\gamma)$ tends to ∞ due to Proposition 2.2.4(d).

Let $E \subset \Omega_\Gamma$ be a fundamental domain. If $V \subset \Omega$ is affinoid, then there exists finitely many $\beta_1, \dots, \beta_r$ with $V \subset \beta_1(E) \cup \dots \cup \beta_r(E)$. Therefore, it suffices to show that $E \cap \gamma(E) \neq \emptyset$ for only finitely many $\gamma \in \Gamma$. This is true due to the very definition of a fundamental domain. Since Γ is a free group, there exists an $N \in \mathbb{N}$ such that, for a finite subset $\Gamma_0 \subset \Gamma$, every $\gamma \in \Gamma$ with $\ell(\gamma) \geq N$ does not belong to Γ_0. □

2.3 Definition and Properties

For the following we keep the notations and hypotheses of Notations 2.2.5 and 2.2.11. Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a non-trivial Schottky group and assume $\infty \notin L_\Gamma$.

Due to Theorem 2.2.14 the group Γ is free with g generators $\alpha_1, \ldots, \alpha_g$; we set $\alpha_{g+i} := \alpha_i^{-1}$ for $i = 1, \ldots, g$. Let

$$F := \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^- = \left\{ z \in \mathbb{P}_K^1 ; \left| w_{\alpha_i}(z) \right| \leq 1 \text{ for } i = 1, \ldots, 2g \right\}$$

be the fundamental domain as constructed in Proposition 2.2.12 by Theorem 2.2.14. Then

$$\Omega_\Gamma := \bigcup_{\gamma \in \Gamma} \gamma(F) \subset \mathbb{P}_K^1$$

is the maximal open set where Γ acts discontinuously. In the following we will equip the orbit space $X_\Gamma := \Gamma \backslash \Omega_\Gamma$ with a rigid analytic structure in a canonical way such that the residue map $p : \Omega_\Gamma \to X_\Gamma$ is a rigid analytic morphism. Moreover, we will see that X_Γ is a smooth proper rigid analytic space of dimension 1 and that p is an unramified covering in the topological sense; i.e., there is an admissible covering $\{V_i ; i \in I\}$ of X_Γ such that each $p^{-1}(V_i) = \bigcup_{j \in I_i} U_{i,j}$ is a disjoint union and the restriction $p|_{U_{i,j}} : U_{i,j} \xrightarrow{\sim} V_i$ of p to each $U_{i,j}$ is an isomorphism.

Theorem 2.3.1. *In the above situation we have*:

(a) *There is a unique structure of a rigid-analytic variety on* X_Γ *such that the residue map* $p : \Omega_\Gamma \to X_\Gamma$ *is an unramified covering in the topological sense.*
(b) X_Γ *is a smooth proper rigid analytic curve of genus* g.
(c) X_Γ *is the analytification of a smooth projective algebraic curve* X_Γ^{alg}.
(d) *Every meromorphic function on* X_Γ *is a rational function on* X_Γ^{alg}.

The curves X_Γ *as defined above are called* Mumford curves.

Proof. (a) Any $\gamma \in \Gamma$ has a unique reduced representation

$$\gamma = \alpha_{j(1)} \cdot \ldots \cdot \alpha_{j(n)} \quad \text{with } j(i) \in \{\alpha_1, \ldots, \alpha_{2g}\}.$$

Let $\ell(\gamma) := n$ be the length of γ. Then

$$\Omega_\Gamma(n) := \bigcup_{\gamma\in\Gamma,\ell(\gamma)\le n} \gamma(F)$$

is the complement of finitely many open discs in $\mathbb{P}^1_K$ and hence carries a unique structure of a smooth affinoid domain of dimension 1. The inclusion map $\Omega_\Gamma(m) \hookrightarrow \Omega_\Gamma(n)$ for $m \le n$ is an open immersion of affinoid domains. Thus, $\Omega_\Gamma = \bigcup_{n\in\mathbb{N}} \Omega_\Gamma(n)$ inherits a unique structure of a smooth rigid analytic domain such that all $\Omega_\Gamma(n)$ are open subdomains. Since all $\Omega_\Gamma(n)$ are separated as affinoid domains, Ω_Γ is separated.

As a set of points we define X_Γ as the orbit space of the action of Γ on Ω_Γ. The rigid analytic structure of X_Γ will be defined as a geometric quotient of Ω_Γ; i.e., a set $V \subset X_\Gamma$ and a covering $\mathfrak{V}$ of $V \subset X_\Gamma$, respectively, is admissible if $p^{-1}(V) \subset \Omega_\Gamma$ and $p^*\mathfrak{V}$, respectively, are admissible. Due to Proposition 2.2.12(a) there exists $q \in \sqrt{|K^\times|}$ with $q < 1$ such that $|w_{\alpha_i}(z)| < q$ for all $z \in W^+_{\alpha_j}$, for $j = 1,\dots,2g$ with $j \ne i$. Now choose $q' \in \sqrt{|K^\times|}$ with $q < q' < 1$, and put

$$U_i := \left\{z \in \mathbb{P}^1_K; q' \le \left|w_{\alpha_i}(z)\right| \le 1 \text{ for } i = 1,\dots,2g\right\},$$
$$U_0 := \left\{z \in \mathbb{P}^1_K; \left|w_{\alpha_i}(z)\right| \le q' \text{ for } i = 1,\dots,2g\right\}.$$

Then $\{U_0,\dots,U_{2g}\}$ is an admissible covering of F with the properties

$$U_0 \cap U_i = \left\{z \in \mathbb{P}^1_K; \left|w_{\alpha_i}(z)\right| = q'\right\} \quad \text{for } i = 1,\dots,2g,$$
$$U_i \cap U_j = \emptyset \quad \text{for } 1 \le i < j \le 2g.$$

The map $p : U_i \to X_\Gamma$ is injective. We will endow X_Γ with the structure of a rigid analytic space in the following way. We view $p : U_i \to X_\Gamma$ as an open immersion and we equip the image $V_i := p(U_i)$ with the holomorphic structure given by U_i. The family

$$\left\{V_i := p(U_i); i = 0,\dots,2g\right\}$$

is regarded as an admissible covering of X_Γ. Thus, we obtain the structure of a rigid analytic variety on X_Γ which is smooth and 1-dimensional.

The map $p : \Omega_\Gamma \to X_\Gamma$ is a covering in the topological sense and is a quotient in the categorical sense; i.e., every Γ-invariant morphism $\Omega_\Gamma \to Z$ factorizes through $p : \Omega_\Gamma \to X_\Gamma$.

(b) X_Γ is separated, because Ω_Γ is separated; cf. [10, 9.6.1/5]. To show that X_Γ is proper, we have to construct a further covering $\{V'_0,\dots,V'_{2g}\}$ by affinoid domains of X_Γ such that $V_i \Subset V'_i$ for $i = 0,\dots,2g$; cf. Definition 1.6.3. Thus, we choose two absolute values $q_1, q_2 \in \sqrt{|K^\times|}$ with $q < q_1 < q' < q_2 < 1$, where q_1 is close to q and q_2 is close to 1. Put

$$U'_i := \left\{z \in \mathbb{P}^1_K; q_1 \le \left|w_{\alpha_i}(z)\right| \le 1/q_2\right\} \quad \text{for } i = 1,\dots,2g,$$
$$U'_0 := \left\{z \in \mathbb{P}^1_K; \left|w_{\alpha_i}(z)\right| \le q_2 \text{ for } i = 1,\dots,2g\right\}.$$

Then $p : U_i' \to X_\Gamma$ is an open immersion for $i = 0, \ldots, 2g$. Hence, by setting $V_i' := p(U_i')$ for $i = 0, \ldots, 2g$, the covering $\{V_0', \ldots, V_{2g}'\}$ satisfies the requirement of Definition 1.6.3.

Next we want to determine the genus of X_Γ. For this we will construct a non-trivial differential form ω on X_Γ and calculate the degree of its divisor. Let $w_\gamma(z)$ be as defined in Notation 2.2.11. Then look at the formal series

$$g_1(z) := \sum_{\gamma \in \Gamma} w_\gamma(z) \quad \text{and} \quad g_2(z) := \sum_{\gamma \in \Gamma} \rho(\gamma) w_\gamma(z).$$

We may assume here without loss of generality that $(\alpha_1, \ldots, \alpha_g)$ is separating for ρ and ρ^2. By Proposition 2.2.12(b) these series converge on F. By the product formula $w_{\gamma\alpha}(z) = w_\gamma(\alpha(z)) \cdot w_\alpha(z)$, the series also converges on $\alpha(F)$ for $\alpha \in \Gamma$, and hence on Ω_Γ. Moreover, we have

$$w_\alpha(z) \cdot g_1(\alpha z) = g_1(z) \quad \text{and} \quad \rho(\alpha) \cdot w_\alpha(z) \cdot g_2(\alpha z) = g_2(z).$$

Now look at the meromorphic differential form on Ω_Γ

$$\omega := \frac{g_1^2(z)}{g_2(z)} dz.$$

Since $\rho(\alpha) \cdot \alpha'(z) = w_\alpha(z)$, we obtain

$$\alpha^* \omega = \frac{g_1^2(\alpha(z))}{g_2(\alpha(z))} \cdot \alpha'(z) \cdot dz = \frac{g_1^2(z) \cdot \rho(\alpha) w_\alpha(z)}{w_\alpha(z)^2 \cdot g_2(z)} \frac{w_\alpha(z)}{\rho(\alpha)} \cdot dz = \omega.$$

Thus, ω is invariant under Γ, and so it induces a differential form on X_Γ.

It remains to determine the degree of $\operatorname{div}(\omega)$. Equivalently, we can consider the divisor associated to g_1 and g_2 on Ω_Γ which consists of Γ-orbits and count the number of the orbits with multiplicity. It suffices to do it for g_1, because the arguments for g_2 are analogous.

By Proposition 2.2.12(b) we know that $|w_\gamma(z)| < 1$, for all $z \in F^-$ and all transformations $\gamma \in \Gamma - \{\mathrm{id}\}$. Since $w_{\mathrm{id}}(z) = 1$, the ultrametric inequality yields $|g_1(z)| = 1$ for all $z \in F^- - \{\infty\}$.

Next we compute the degree of $g_1|(W_{\alpha_i}^+ - W_{\alpha_i}^-)$. Here we have two dominating terms, namely $w_{\mathrm{id}}(z)$ and $w_{\alpha_i}(z)$. Their absolute values are equal to 1, whereas those of all the others terms are less than 1. Consider the Laurent series with respect to the coordinate $\zeta := (z - m_{\alpha_i})/\pi_i$, where $\pi_i \in \overline{K}^\times$ is a constant to normalize the coordinate to absolute value equal to 1. On $(W_{\alpha_i}^+ - W_{\alpha_i}^-)$ it is given by $1 + \zeta^2$ up to terms of absolute value less than 1. Now it is an elementary fact about such functions that their number of zeros is 2. Thus, we see that each $i \in \{1, \ldots, g\}$ gives rise to a Γ-orbit of degree 2. Furthermore, there is exactly one orbit for the poles of dz; namely $\{m_\gamma; \gamma \in \Gamma\} = \Gamma\infty$ of order 2. Thus, we see that the degree of ω is $2g - 2$, and hence the genus of X_Γ is g.

(c) and (d) Follow from Theorem 1.8.1. □

2.4 Skeletons

In their article [64] Manin and Drinfeld make fundamental use of the tree representation of the p-adic numbers. In order to deal with the case of arbitrary non-Archimedean valuations, we have to generalize the approach slightly.

Let us start with some definitions. For the following, we fix a non-Archimedean field K.

Definition 2.4.1. The *closed unit disc* $\mathbb{D}_K$ is the affinoid space $\operatorname{Sp} K\langle\xi\rangle$. The *open unit disc* is the rigid analytic space $\mathbb{D}_K^- := \{z \in \mathbb{D}_K;\, |\xi(z)| < 1\}$. A *closed rational disc* or an *open rational disc* is a rigid analytic space isomorphic to $\mathbb{D}_K$ or $\mathbb{D}_K^-$, respectively.

A *closed rational annulus* or an *open rational annuls* is a rigid analytic space isomorphic to

$$A(r,1)^+ = \left\{z \in \mathbb{D}_K;\, r \le \left|\xi(z)\right| \le 1\right\},$$

respectively to

$$A(r,1)^- = \left\{z \in \mathbb{D}_K;\, r < \left|\xi(z)\right| < 1\right\},$$

for an element r of the value group $|K^\times|$. The number $r \le 1$ is called the *height* of the annulus $A(r,1)^\pm$; cf. Definition 1.3.3.

A subdomain Ω of the projective line $\mathbb{P}_K^1$ is a closed rational disc if and only if there exists a coordinate function ξ of $\mathbb{P}_K^1$ with a zero in $\mathbb{P}_K^1(K)$ and a number $r \in |K^\times|$ such that $\Omega = \{z \in \mathbb{P}_K^1;\, |\xi(z)| \le r\}$.

A subdomain Ω of the projective line $\mathbb{P}_K^1$ is a closed rational annulus if and only if Ω is the complement of a closed rational disc by an open rational disc. In fact, there exists a coordinate function ξ on $\mathbb{P}_K^1$ which has a zero in the open disc and a pole outside the closed disc. Then ξ yields the description of the complement as an annulus. This can easily be seen by the description of the invertible functions on a disc in Proposition 1.2.1(b).

Definition 2.4.2. The *standard reduction map* $\rho: \mathbb{P}_K^1 \to \mathbb{P}_k^1$ of the projective line is associated to the choice of a coordinate function on $\mathbb{P}_K^1$ which also serves as a coordinate function on $\mathbb{P}_R^1$. Then ρ is the specialization map $\mathbb{P}_K^1 \to \mathbb{P}_k^1$ on $\mathbb{P}_R^1$.

The *canonical reduction map* $\rho: \mathbb{D}_K \to \mathbb{A}_k^1$ of the unit disc $\mathbb{D}_K$ is the map which, as above, associates to a K'-rational point x of $\mathbb{D}_K$ its reduction $\tilde{x}$ which is defined as the closed point $x_R \otimes_R k$ of the extension $x_R: \operatorname{Spec}(R') \to \mathbb{A}_R^1$ of x.

If $D \subset \mathbb{D}_K$ is the unit disc punctured by finitely many maximal open discs $D(a_1)^-, \ldots, D(a_n)^-$ of $\mathbb{D}_K$, then the reduction map $\rho: \mathbb{D}_K \to \mathbb{A}_k^1$ restricts to a reduction map $\rho: D \to \mathbb{A}_k^1 - \{\rho(a_1), \ldots, \rho(a_n)\}$. We refer to this as *canonical reduction* as well.

In the following we make use of some notion about graphs; these are explained in Sect. A.1. Note that we here identify an edge e of a graph with its inverse $\overline{e}$; i.e., we consider only "geometric edges". In Definition 2.4.3 we do not need an orientation on the graph. Such graphs are called *geometric*.

Definition 2.4.3. Let Z be a rigid analytic space which is geometrically connected and locally planar. The latter means that, locally with respect to the holomorphic topology, Z is isomorphic to affinoid subdomains of the projective line.

A *semi-stable skeleton* of Z is a surjective map $\rho : Z \to S$ from Z to a geometric graph S with the following properties:

(i) The inverse image $\rho^{-1}(v)$ of a vertex $v \in V(S)$ is either the whole $\mathbb{P}^1_K$ or a domain in $\mathbb{P}^1_K$ which is isomorphic to the closed unit disc $\mathbb{D}_K$ punctured by finitely many maximal open discs $D_1^- \cup \cdots \cup D_n^-$ of $\mathbb{D}_K$.
(ii) The inverse image $\rho^{-1}(e)$ of an edge $e \in E(S)$ is isomorphic to an open rational annulus $A(\varepsilon(e), 1)^-$ of a certain height $\varepsilon(e) \in |K^\times|$.
(iii) ρ is continuous; i.e., the inverse image $\rho^{-1}(\{v_1, e, v_2\})$ of an edge e with its two extremities v_1, v_2 is an affinoid subdomain of Z or the whole Z.

Since the reduction of $\rho^{-1}(v)$ for a vertex $v \in S$ is isomorphic to a projective line minus finitely many closed points, it is irreducible and, hence, the sup-norm is multiplicative on $\rho^{-1}(v)$; cf. Remark 1.4.6.

A *semi-stable skeleton* $\rho : Z \to S$ of Z is said to *separate the points* $a_1, \ldots, a_n$ of Z if these points are mapped to vertices such that for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ either the points a_i, a_j are mapped to different vertices of S or, if mapped to the same vertex $v \in V(S)$, the points a_i, a_j have different reductions under the canonical reduction map.

A semi-stable skeleton of Z is called *stable with respect to given points* $a_1, \ldots, a_n$ for $n \geq 3$ if it separates the points and if, for each vertex v, the sum of the number of points of $\rho(a_1), \ldots, \rho(a_n)$ equal to v and of the index of v is at least 3; cf. Definition A.1.7.

Note that, for the definition of the index, in Definition A.1.7 one distinguishes between e and its inverse $\overline{e}$. One can also define the index of a vertex by the number of geometric edges which have v as an extremity.

Example 2.4.4. Let $r \in |K^\times|$ with $r < 1$. Consider the rational annulus $A(r, 1) := \{x \in \mathbb{D}_K;\, |r| \leq |\xi(x)|\}$, where ξ is a coordinate of the disc $\mathbb{D}_K$. We define a skeleton $\rho : A(r, 1) \to S$ in the following way. The image S consists of two vertices v_r, v_1 which are connected by one edge e. The map $\rho : A(r, 1) \to S$ sends the subset where ξ takes absolute values r to v_r, and the subset where ξ takes absolute values 1 to v_1, and the subset where ξ takes absolute values $r < |\xi(x)| < 1$ to e.

More precisely, the reduction of $A(r, 1)$ consists of two lines $\tilde{L}_r \cup \tilde{L}_1$, where $\tilde{L}_r = \mathbb{P}^1_k - \{0\}$ with coordinate $\widetilde{\xi/c}$, $c \in K$ with $|c| = r$, and $\tilde{L}_1 = \mathbb{P}^1_k - \{\infty\}$ with coordinate $\tilde{\xi}$. The point $\infty \in \tilde{L}_r$ is identified with the point $0 \in \tilde{L}_1$.

If f is a holomorphic function on $A(r, 1)$, then $f|_{A(r,r)}$ has a sup-norm $|c_r|$ and on $A(1, 1)$ a sup-norm $|c_1|$. Via the reduction one gets two functions $\tilde{f}_r := \widetilde{f/c_r}$ on L_r and $\tilde{f}_1 := \widetilde{f/c_1}$ on L_1. Both functions have a certain order m_r of $\tilde{f}_r$ at $\infty \in \tilde{L}_r$ and m_1 at $0 \in \tilde{L}_1$. Then one easily shows that $m_1 + m_r$ equals the number of zeros of f on the open annulus $A(r,1)^-$. If there are no zeros, then $|c_r| = r^{\pm m_1} \cdot |c_1|$. Such a situation will be discussed in Sect. 4.3 in a more general context.

Lemma 2.4.5. *Let $n \geq 3$ and let $a_1, \ldots, a_n \in \mathbb{P}^1_K(K)$ be pairwise distinct K-rational points. Then there exists a stable skeleton of $\mathbb{P}^1_K$ which separates the points $a_1, \ldots, a_n$. Furthermore, the graph of this skeleton is a tree.*

A stable skeleton separating $a_1, \ldots, a_n$ is uniquely determined by these points.

Proof. To prove the existence, we proceed by induction on n. For $n = 3$ we can choose a coordinate such that the points are $0, 1, \infty$. Then the standard reduction separates these points. In this case the skeleton consists of a single vertex without any edges.

Now let $n \geq 3$ and assume that we have already constructed a stable skeleton $\rho : \mathbb{P}^1_K \to S$ which separates the points $a_1, \ldots, a_n$. Consider an additional point $b \in \mathbb{P}^1_K(K)$ which is not equal to any of the $a_1, \ldots, a_n$.

If $\rho(b) = v$ is a vertex, then we have to distinguish two cases. If the reduction of b under the canonical reduction map of the vertex is different from the reduction of the points $a_1, \ldots, a_n$ which are also mapped to v under ρ, then it is not necessary to change anything. On the other hand, if this is not the case, then there is a point a_i with $\rho(a_i) = v$ which has the same canonical reduction as b. Then we introduce a new vertex v' and a new geometric edge e' which connects v with v'. The map $\rho' : \mathbb{P}^1_K \to S'$ is defined as follows. Let ζ be a coordinate function on the maximal open disc D^- inside $\rho^{-1}(v)$ which contains a_i and normalize ζ by $\zeta(a_i) = 0$ and by the condition that sup-norm of $|\zeta|_{D^-} = 1$. Then set $\rho'(z) = v'$ for all $z \in D^-$ with $|\zeta(z)| \leq |\zeta(b)|$ and $\rho'(z) = e'$ for all $z \in D^-$ with $|\zeta(z)| > |\zeta(b)|$ and $\rho'(z) = \rho(z)$ for all points $z \in \mathbb{P}^1_K - D^-$. So $\rho'(b)$ and $\rho'(a_i)$ are equal, but b and a_i have distinct reductions. Furthermore, v' is connected to v by e'. Then it is clear that ρ' is stable.

If $\rho(b) = e$ is an edge, then $A^- := \rho^{-1}(e)$ is an open annulus of height $\varepsilon \in |K^\times|$, since A^- is rational. Let ζ be a coordinate on $A^- := \rho^{-1}(e)$ which is normalized by having sup-norm 1. We introduce a new vertex v' and two new geometric edges e'_0, e'_1 which connect v' to the extremities of e in S. Then we define $\rho' : \mathbb{P}^1_K \to S'$ by the following formula. If $z \in A^-$, then put

$$\rho'(z) := \begin{cases} e'_1 & \text{if } |\zeta(z)| > |\zeta(b)|, \\ v' & \text{if } |\zeta(z)| = |\zeta(b)|, \\ e'_0 & \text{if } |\zeta(z)| < |\zeta(b)|. \end{cases}$$

If $z \in \mathbb{P}^1_K - A^-$, then put $\rho'(z) = \rho(z)$. Thus, $v' = \rho'(b)$ is connected to the remaining part by two edges and hence ρ' is stable.

To prove the uniqueness, we proceed by induction on n. Consider first the case $n = 3$. If S consists of a single vertex, it is the standard one with $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$. Otherwise, S has at least two terminal vertices in Definition A.1.7, since it is a finite tree. Then there are at least four points due to the very definition of stability. This is a contradiction.

Now let us consider the case $n \geq 4$. If S consists of single point, then it is easy to see that S is unique. Thus, we may assume that there is more than one vertex; in particular, there exists a terminal vertex v of S. Then v supports at least two points; say a_1 and a_2. Choose a coordinate function ζ on $\mathbb{P}^1_K$ such that $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$ and $|a_\nu| \geq 1$ for $\nu = 4, \ldots, n$. Then $\rho^{-1}(v)$ is uniquely determined by the position of the points and is independent of the stable skeleton. Indeed, we have that

$$\rho^{-1}(v) = D^+ - \left[D_1^- \cup \cdots \cup D_m^-\right],$$

where D^+ is the unit disc at 0 and D_μ^- are maximal open discs in D^+. Furthermore, in each D_μ^- there exists at least one of the given points.

If v supports more than 2 points, then we can remove one and thereby we do not destroy the stability, and hence we are done by the induction hypothesis. Consider now the case, where v supports exactly 2 points.

If v has only one neighbor w supporting also a point, then we contract the vertex and the edge e leading from v to w and remove a_1. In this way we get again a stable skeleton separating the remaining points as w supports now two points. Note that in this case $\rho^{-1}\{v, e\}$ is also uniquely determined. Therefore, we are also done by the induction hypothesis. If w does not support any of the points, then the index of w is at least 3. Then, by removing v, the edge e and a_1, one obtains a skeleton with vertex w which supports one point and its index is at least 2. So, we end up with a stable skeleton and are done by the induction hypothesis. □

Proposition 2.4.6. *Let Ω be a connected affinoid subdomain of $\mathbb{P}^1_K$. Then the following assertions are equivalent*:

(a) *Ω admits a semi-stable skeleton.*
(b) *Ω is the projective line punctured by finitely many open rational discs.*

If, in the representation $\Omega := \mathbb{P}^1_K - (D_1^- \cup \cdots \cup D_n^-)$, even the closed discs $D_1^+, \ldots, D_n^+$ are pairwise disjoint, the graph of the skeleton can be chosen in such a way that its terminal vertices correspond to $D_j^+ - D_j^-$ for $j = 1, \ldots, n$.

Proof. (a) → (b): Since Ω is affinoid, it is quasi-compact and not equal to the whole projective line. Thus, the skeleton is finite and there are terminal vertices. Here, a vertex v is called *terminal* if the reduction of $\rho^{-1}(v)$ is a projective line $\mathbb{P}^1_k$ punctured by more points than the index of v. All these points correspond to open rational discs in the complement of Ω in $\mathbb{P}^1_K$.

(b) → (a): Let $\Omega := \mathbb{P}^1_K - (D_1^- \cup \cdots \cup D_n^-)$ be a complement of $\mathbb{P}^1_K$ of the union of open rational discs $D_1^-, \ldots, D_n^-$ which are pairwise disjoint. In particular, there are K-rational points $a_i \in D_i^-$ for $i = 1, \ldots, n$.

If $n \leq 2$, then Ω is an annulus, and the assertion was explained in Example 2.4.4.

If $n \geq 3$, then there exists a stable skeleton $\rho : \mathbb{P}^1_K \to S$, which separates the points $\{a_1, \ldots, a_n\}$ by Lemma 2.4.5. First we refine the skeleton in the following way. Each a_j is supported by a vertex v_j. Then, for each $j = 1, \ldots, n$, we introduce a new vertex v'_j, which is associated to a small closed rational disc $D'_j \subset D_j^-$

around a_j, and a new edge e_j which is associated to the open rational annulus $D_j^- - D_j'$. Now, there exists a coordinate function ξ on $\mathbb{P}_K^1$ with a zero at a_1 and a pole at a_2. Since D_1^- is a rational disc, we can adjust ξ such that the sup-norm of ξ on D_1^- is equal to 1. By Proposition 1.3.4 we have that the absolute value function $|\xi|$ behaves like the one of a power of the coordinate functions on the annuli associated to the edges of this skeleton. Then we subdivide every edge at the subannulus of height 1 where $|\xi|$ takes the value 1. Thus, we obtain a skeleton such that ρ restricts to a skeleton $\rho_1 : \mathbb{P}_K^1 - D_1^- \to S_1$, where S_1 is obtained from S by removing all the vertices and edges where $|\xi|$ takes values less than 1. Likewise we proceed with all the other discs, and hence we obtain a semi-stable skeleton $\rho_n : \Omega \to S_n$ as required.

The additional assertion follows easily from the proof of "b → a" as well, because the subset $\{z; |\xi(z)| = 1\}$ is exactly $D_1^+ - D_1^-$ and $|\xi|$ takes only values greater than 1 on all the other closed discs D_j^+ for $j \geq 2$. Eventually one has to contract the part where $|\xi|$ is equal to 1. □

Corollary 2.4.7. *Let K be algebraically closed. If $\Omega \subset \mathbb{P}_K^1$ is a connected affinoid subdomain, then Ω has a semi-stable skeleton, and hence Ω is a closed rational disc punctured by finitely many open rational discs.*

Proof. Since Ω is affinoid, it is strictly contained in $\mathbb{P}_K^1$. So, there exists a coordinate function ξ on $\mathbb{P}_K^1$ such that ξ has its pole a_0 outside Ω and its zero inside Ω. Moreover, one can adjust ξ such that the sup-norm of $\xi|_\Omega$ is equal to 1. Thus, Ω is contained in the closed unit disc $\mathbb{D}_K$. By Theorem 1.3.7 we know that Ω is a union of finitely many rational domains. It suffices to consider the case $\Omega = X(f_1/f_0, \ldots, f_N/f_0)$, where $f_0, \ldots, f_N$ are holomorphic functions on $X := \mathbb{D}_K$ without common zeros. Due to Theorem 1.2.5 we may assume that $f_0, \ldots, f_N$ are polynomials, because invertible functions on $\mathbb{D}_K$ have constant absolute value functions by Proposition 1.2.1. Since $f_0, \ldots, f_N$ have no common zeros, there exists some $r' \in |K^\times|$ such that Ω is contained in $\Omega' := \{x \in \mathbb{D}_K; |f_0(x)| \geq r'\}$. Note that Ω' is equal to $\mathbb{D}_K$ minus finitely many open discs around the zeros of f_0.

Now it suffices to analyze the structure of $\Omega'(f_i/f_0)$. It follows from Proposition 2.4.6 that there exists a semi-stable skeleton of Ω'. Then it is an easy combinatorial game to show how to obtain a skeleton of $\Omega'(f_i/f_0)$ from the skeleton of Ω'. In more detail, the absolute value function $|f_0|$ of f_0 is constant on the pre-image of vertices and behaves like a power of the absolute value of the coordinate on the pre-image of an edge by Proposition 1.3.4. The function $|f_1|$ behaves similarly after removing small discs around the zeros of f_1. Thus, by subdividing some annuli associated to the skeleton of Ω', we construct a new skeleton such that $\Omega'(f_i/f_0)$ can be viewed as the preimage of a subgraph of this new skeleton. □

The number $n + 1$ of holes used in the representation of Ω as a subset of $\mathbb{P}_K^1$ can be characterized in terms of the structure of the group of invertible holomorphic functions on Ω. There is the following result.

Proposition 2.4.8. *Let $\Omega \subset \mathbb{P}^1_K$ be a closed rational disc which is punctured by finitely many open rational discs*

$$\Omega := \mathbb{P}^1_K - \big(D(a_0, r_0)^- \cup \cdots \cup D(a_n, r_n)^-\big).$$

Let ξ be a coordinate function on $\mathbb{P}^1_K$ with a pole at a_0 and a zero outside of $D(a_0, r_0)^-$ such that $D(a_0, r_0)^- := \{x \in \mathbb{P}^1; |\xi(x)| > 1\}$.

Put $\xi_\nu := c_\nu/(\xi - \xi(a_\nu))$, where $c_\nu \in K^\times$ has absolute value $|c_\nu|$ equal to the sup-norm of $\xi - \xi(a_\nu)$ on the disc $D(a_\nu, r_\nu)^-$. Then we have:

(a) *Every holomorphic function on Ω has a unique representation*

$$f = \sum_{i=0}^{\infty} c_{0,i}\xi^i + \sum_{\nu=1}^{n}\sum_{i=1}^{\infty} c_{\nu,i}\xi_\nu^i$$

with coefficients $c_{\nu,i} \in K$. For the sup-norm we have that

$$|f|_\Omega = \max\big\{|c_{\nu,i}|; i \in \mathbb{N}, \nu = 0, \ldots, n\big\}.$$

(b) *If f has no zeros on Ω, then f has a unique representation*

$$f = c \cdot (\xi - a_1)^{m_1} \ldots (\xi - a_n)^{m_n} \cdot (1 + h),$$

where $c \in K^\times$ is a unit, h is a holomorphic function on Ω with sup-norm $|h|_\Omega < 1$, and where $m_1, \ldots, m_n$ are integers.

Proof. (a) Since f can be approximated by rational functions which have poles only in $\{a_0, \ldots, a_n\}$, it suffices to verify the assertion for such rational functions. In that case we have a unique partial fraction decomposition. Such a decomposition is of the same form as in the assertion, but there are only finitely many coefficients unequal 0.

To verify the assertion on the sup-norm, we first assume that the closed discs $D(a_\nu, r_\nu)^+$ are pairwise disjoint. In that case the assertion follows by the ultrametric inequality, because the sup-norm of ξ_μ on $D(a_\nu, r_\nu)^+$ is less than 1 for all μ, ν with $\mu \neq \nu$. By a limit argument (enlarging the discs) and the maximum principle this implies the assertion in the general case.

The assertion about the uniqueness follows from the formula for $|f|_\Omega$.

(b) As in the proof of (a) it suffices to verify the assertion for rational functions which have poles only in $\{a_0, \ldots, a_n\}$. Moreover, we may assume that the closed discs $D(a_\nu, r_\nu)^+$ are pairwise disjoint. By Proposition 1.3.4 we know that the restriction of f to the annulus $D(a_\nu, r_\nu)^+ - D(a_\nu, r_\nu)^-$ can be written as $c_\nu\xi_\nu^{m_\nu} \cdot (1 + h)$, where the sup-norm of h is less than 1. After replacing f by $f \cdot \prod_{\nu=1}^n \xi_\nu^{-m_\nu}$ we can assume that all the exponents m_ν are zero for $\nu = 1, \ldots, n$. In that case we will verify that m_0 is also zero, and that f equals $c \cdot (1 + h)$ with a constant $c \in K^\times$ and with a holomorphic function h on Ω with sup-norm less that 1.

Indeed, by (a) we may assume that $|f|_\Omega = 1$ and that $|f|$ takes a maximum on an annulus $D(a_\nu, r_\nu)^+ - D(a_\nu, r_\nu)^-$ for some $\nu \in \{1, \ldots, n\}$; otherwise consider $1/f$.

Then by (a) it follows $|c_{0,0}| = 1$ and $|c_{\nu,i}| \leq 1$ for all ν, i. Since f has no zeros on $D(a_0, r_0)^+ - D(a_0, r_0)^-$, by Proposition 1.3.4 we see that $m_0 = 0$. Thus, the assertion is proved. □

Remark 2.4.9. In the situation of Proposition 2.4.8 the group of invertible holomorphic functions on Ω can be represented in the following way:

$$\mathcal{O}_\Omega(\Omega)^\times \cong K^\times \times \mathbb{Z}^n \times \left\{1 + h, h \in \mathcal{O}_\Omega(\Omega) \text{ with } |h|_\Omega < 1\right\}.$$

Here $n + 1$ is the number of holes of Ω in $\mathbb{P}^1_K$.

Proof. Put $H := \{m \in \mathbb{Z}^{n+1}; m_0 + \cdots + m_n = 0\} \cong \mathbb{Z}^n$. The map

$$\varphi : K \longrightarrow \mathcal{O}_\Omega(\Omega)^\times,\ (m_0, \ldots, m_n) \longmapsto \xi^{m_0} \cdot (\xi - a_1)^{m_1} \cdot \ldots \cdot (\xi - a_n)^{m_n}$$

is injective, and its image is a direct summand by Proposition 2.4.8. Thus, we see that the assertion is true. □

Proposition 2.4.10. *In the situation of Proposition 2.4.8 consider a meromorphic function f on Ω which is not identically zero. Assume, in addition, that the annulus $A_\nu := D_\nu^+ - D_\nu^-$ belongs to Ω, and that $f|_{A_\nu}$ has neither zeros nor poles for $\nu = 1, \ldots, n$. Then the degree of the divisor of f on Ω is given by the formula*

$$\deg \operatorname{div}(f) = -\sum_{\nu=0}^{n} \operatorname{ord}_{A_\nu} f.$$

Here $\operatorname{ord}_{A_\nu} f$ is the exponent of the dominating term in the Laurent expansion of $f|_{A_\nu}$ with respect to the coordinate function ξ_ν on D_ν^+; cf. Proposition 1.3.4.

In Proposition 4.3.1 there is a more general formula than the given one.

Proof. The support of the divisor of f is finite. Thus there exists a rational function g on $\mathbb{P}^1_K$ such that $\operatorname{div}(g|_\Omega) = \operatorname{div}(f)$. It we put $u := f/g$, then u is an invertible holomorphic function on Ω. Using an approximation as in the proof of Proposition 2.4.8 we can also assume that $u = 1 + h$ with a holomorphic function h on Ω with $|h|_\Omega < 1$. Thus, we can replace f by g, because f and g have the same order on A_ν. Since $\deg \operatorname{div} g = 0$, it remains to see that $\operatorname{ord}_{A_\nu} g = \deg \operatorname{div}(g|_{D_\nu^-})$ for $\nu = 0, \ldots, n$. The latter follows easily from Theorem 1.2.5, because the degree of a Weierstraß polynomial equals the number of its zeros in the unit disc. □

Now let us return to the Schottky groups.

Proposition 2.4.11. *Let Γ be a Schottky group. Consider the situation of Notation 2.2.11 with respect to a separating homomorphism $\rho : \Gamma \to \overline{K}^\times$. Assume that the radii $\sqrt{|\rho(\gamma)|} \cdot r_\gamma$ belong to $|K^\times|$ for all $\gamma \in \Gamma - \{\mathrm{id}\}$. Let $F \subset \Omega_\Gamma$ be the fundamental domain of Γ as in Notation 2.2.11. Then we have:*

(a) *There exists a semi-stable skeleton* $\rho_F : F \to S_F$ *of* F *with terminal vertices* $v_1, \ldots, v_{2g}$ *such that* $\rho_F^{-1}(v_i) = W_i^+ - W_i^-$ *for all* $i = 1, \ldots, 2g$, *where* $W_i^\pm := W_{\alpha_i}^\pm$ *for the system* $(\alpha_1, \ldots, \alpha_{2g})$. *In particular,* S_F *is a tree.*
(b) *There exists a semi-stable skeleton* $\rho_\Omega : \Omega_\Gamma \to S_\Omega$ *of* Ω_Γ *which extends* ρ_F *such that* Γ *acts on* S_Ω *canonically. In particular,* S_Ω *is a tree.*
(c) *There exists a semi-stable skeleton* $\rho_X : X_\Gamma \to S_X$ *of* X_Γ *which is the quotient of* ρ_Ω *with respect to the action of* Γ. *In particular, the quotient map* $p_S : S_\Omega \to S_X$ *is the universal covering in the category of graphs. There is a commutative diagram*

$$\begin{array}{ccc} \Omega_\Gamma & \xrightarrow{p_X} & X_\Gamma \\ \downarrow{\scriptstyle \rho_\Omega} & & \downarrow{\scriptstyle \rho_X} \\ S_\Omega & \xrightarrow{p_S} & S_X. \end{array}$$

Proof. (a) Consider the fundamental domain

$$F = \mathbb{P}_K^1 - \bigcup_{i=1}^{2g} W_{\alpha_i}^-$$

associated to a separating basis $\alpha_1, \ldots, \alpha_g$ of Γ. Set $\alpha_{i+g} := \alpha_i^{-1}$ for $i = 1, \ldots, g$. By Proposition 2.4.6 there exists a semi-stable skeleton $\rho_F : F \to S_F$. It is clear that S_F satisfies the claim.

(b) The skeleton $S_0 := S_F$ constructed in (a) has $2g$ terminal vertices $v_1, \ldots, v_{2g}$. Then each α_i maps the domains $\rho^{-1}(v_i)$ bijectively to $\rho^{-1}(v_{i+g})$ for all $i = 1, \ldots, g$. Thus we obtain a skeleton

$$S_1 := S_0 \cup \bigcup_{i=1}^{2g} \alpha_i(S_0)$$

by gluing the skeleton S_0 with the skeleton $\alpha_i(S_0)$ of $\alpha_i(F)$ along v_{i+g} for $i = 1, \ldots, 2g$. Then S_1 is a skeleton of

$$\Omega_\Gamma(n) := \bigcup_{\gamma \in \Gamma; \ell(\gamma) \le n} \gamma(F)$$

for $n = 1$. Continuing in this way, one obtains skeletons S_n of $\Omega_\Gamma(n)$ for all $n \in \mathbb{N}$ and hence in the limit a skeleton S_Ω of Ω_Γ. The group Γ acts on S_Ω by translation and is compatible with action on Ω_Γ.

(c) The skeleton S_X is obtained from S_F by identifying the terminal vertices v_i and v_{i+g} for $i = 1, \ldots, g$. This is compatible with the group action of Γ on Ω_Γ as was explained in (a). $\square$

Corollary 2.4.12. *In the situation of Proposition 2.4.11, let e be an edge of the skeleton S_Ω. If $\gamma \in \Gamma$ fixes e, then $\gamma = \mathrm{id}$.*

Proof. The skeleton S_Ω is the universal covering of S_X. Thus, Γ is canonically isomorphic to the deck transformation group of S_Ω/S_X. If a deck transformation fixes one point, then it is equal to the identity. □

Remark 2.4.13. A *ray* $(e_i; i \in \mathbb{N})$ in S_Ω is an infinite path without backtracking; cf. Definition A.1.3. Note that a ray has an origin and no target in S_Ω. Two rays are called *equivalent* if they are equal after removing finite parts at their origins.

An *axis* $(e_i; i \in \mathbb{Z})$ in S_Ω is an infinite path without backtracking. Note that an axis has neither an origin nor a target in S_Ω.

However we have:

(a) Each ray in S_Ω defines a unique point in the set L_Γ of the limit points of Γ.
(b) The equivalence classes of rays correspond bijectively to the limits points of Γ.
(c) If $\alpha \in \Gamma - \{\mathrm{id}\}$, then let $x_\alpha \subset S_\Omega$ be the axis leading from z_α^- to z_α^+, where z_α^+ is the attractive fixed point and z_α^- is the repelling fixed point of α. Then α acts on x_α by shifting.

Proof. (a) We choose the coordinate function on $\mathbb{P}_K^1$ such that $\infty \in \rho^{-1}(v_0)$. Thus, for $i \in \mathbb{N}$ we have

$$\rho^{-1}(v_i) = D(a_{v_i}, r_i)^+ - \left[D(a_{v_i,1}, r_i)^- \cup \cdots \cup D(a_{v_i,k_i}, r_i)^-\right],$$

where $D(a_{v_i,j}, r_i)^-$ are maximal open rational discs in the closed rational disc $D(a_{v_i}, r_i)^+$. Let $(e_i; i \in \mathbb{N})$ be a ray. Then the edge e_i satisfies

$$\rho^{-1}(e_i) = D(a_{v_i,j}, r_i)^- - D(a_{v_{i+1}}, r_{i+1})^+$$

for a suitable $j \in \{1, \ldots, k_i\}$. Only finitely many heights r_{i+1}/r_i can occur, because they are related under Γ. Thus, the limit $\lim a_i$ exists and is a limit point of Γ; cf. the proof of Example 2.2.13.

(b) If two rays define the same limit point, then they are equivalent, because a ray induces a filter of neighborhoods of the limit point. Conversely, every limit point induces a ray, because $L_\Gamma = L_\Gamma(\infty)$ by Proposition 2.2.4(f).

(c) The subtree x_α of S_Ω contains the vertices associated to the sets $W_{\alpha^n}^+ - W_{\alpha^n}^-$ for $n \in \mathbb{Z}$. Then it is clear that α acts by shifting. □

Definition 2.4.14. Let $\rho : Z \to S$ be a semi-stable skeleton.

(a) A *path* $c := e_1 + \cdots + e_n$ in Z is a path $(e_1, \ldots, e_n)$ in S; cf. Definition A.1.3. The *length of a path* $c := e_1 + \cdots + e_n$ in Z is defined by

$$\ell(c) := -\sum_{\nu=1}^{n} \log \varepsilon(e_\nu),$$

where $\varepsilon(e_\nu)$ is the height of the annulus $\rho^{-1}(e_\nu)$.

(b) For two paths $p = \sum_{\mu=1}^{m} e_\mu$ and $p' = \sum_{\nu=1}^{n} e'_\nu$ in Z their *pairing* is defined by

$$[p, p'] := \sum_{\mu,\nu} [e_\mu, e'_\nu],$$

where

$$[e_1, e_2] = \begin{cases} -\log \varepsilon(e) & \text{if } e_1 = e_2 = e, \\ \log \varepsilon(e) & \text{if } e_1 = -e_2 = e, \\ 0 & \text{if } e_1 \neq \pm e_2, \end{cases}$$

for edges $e_1, e_2 \in E(S)$, where $\varepsilon(e)$ is the height of the annulus $\rho_\Omega^{-1}(e)$. Here $-e$ means the edge e with the opposite orientation.

Lemma 2.4.15. *In the situation of Definition 2.4.14, let $\rho' : Z \to S'$ be a second semi-stable skeleton. Assume that there is a map $\varrho : S' \to S$ which contracts subtrees satisfying $\varrho \circ \rho' = \rho$. Thus, one has two notions of length according to the chosen skeletons. Let $v'_1, v'_2 \in S'$ be two vertices which are mapped to vertices v_1, v_2 in S, respectively. Then for each path c' leading from v'_1 to v'_2, the image $\varrho(c')$ has the same length as the path c leading from v_1 to v_2.*

Proof. We may assume that v_1 and v_2 are connected by a single edge e. Let $c := (e'_1, e'_2, \ldots, e'_n)$ be the path in S' leading from v'_1 to v'_2. Then $\rho'^{-1}(c) = \rho^{-1}(\{e\})$. Since the height of an annulus $A(r_1 r_2, 1)$ is the product of the heights of $A(r_1, 1)$ and $A(r_2, 1)$, the assertion follows. □

Notation 2.4.16. Let Γ be a Schottky group.

For every $\alpha \in \Gamma$ with $\overline{\alpha} \neq 1$ in $H := \Gamma/[\Gamma, \Gamma] = \Gamma_{\text{ab}}$ there is an axis x_α in S_Ω which we orient from the attractive fixed point z_α^+ to the repulsive z_α^-. Let c_α be the part of x_α which belongs to a fundamental domain of $\alpha^{\mathbb{Z}}$. Note that c_α is a finite path in S_Ω. Let $e_1, \ldots, e_r$ be the consecutive edges with the induced orientation of c_α. Then, with the notation of Proposition 2.4.11,

$$\overline{c}_\alpha := \overline{p_S(c_\alpha)} := \sum_{i=1}^{r} p_S(e_i) \in Z_1(S_X, \mathbb{Z})$$

is a 1-chain. Its homology class in $H_1(S_X, \mathbb{Z})$ can be identified with the image of $\alpha \in \Gamma$ in the maximal abelian quotient

$$\pi_1(S_X) = \Gamma \longrightarrow H := \Gamma/[\Gamma, \Gamma].$$

More precisely, here one has to consider the realization $\text{real}(S_X)$ of the graph S_X; cf. Definition A.1.2.

Remark 2.4.17. The pairing of Definition 2.4.14 induces a scalar product on $H_1(S_X, \mathbb{Z})$. This bilinear form is symmetric and positive definite.

The notion of skeletons is useful for the interpretation of $H_1(X, \mathbb{Z})$.

Remark 2.4.18. Affinoid spaces with smooth reduction are simply connected in the sense that they admit only trivial coverings in the topological sense, as easily follows from Proposition 3.1.12. Furthermore, every affinoid subdomain Ω of the projective line $\mathbb{P}^1_K$ is also simply connected. Indeed, one can assume that K is algebraically closed. Thus, Ω admits a semi-stable skeleton which is tree due to Corollary 2.4.7, and so it is simply connected.

If a rigid analytic space X admits a semi-stable skeleton S_X, then every finite topological covering $Y \to X$ inherits a semi-stable skeleton S_Y such that $S_Y \to S_X$ is a topological covering as well and is compatible with the map $Y \to X$. Therefore, one can view $H_1(S_X, \mathbb{Z})$ as a replacement of "$H_1(X, \mathbb{Z})$".

2.5 Automorphic Functions

In this section we return to Mumford curves. Thus, let us consider a Schottky group $\Gamma \subset \mathrm{PGL}(2, K)$ of rank $g \geq 1$; i.e., it is a free group over g generators. Let $\Omega := \Omega_\Gamma \subset \mathbb{P}^1_K$ be the set of ordinary points of Γ and let X_Γ be the associated Mumford curve. As was explained in Sect. 2.3, the canonical map $p : \Omega_\Gamma \to X := X_\Gamma$ can be viewed as the universal covering in the rigid analytic sense. Equivalently, the universal covering can be identified with the associated skeletons $p_S : S_\Omega \to S_X$. Moreover, the rigid analytic deck transformation group

$$\Gamma \cong \pi_1(X) \cong \pi_1(S_X)$$

can be identified with the deck transformation group of the skeleton S_X. Its maximal abelian quotient

$$H := \Gamma_{\mathrm{ab}} := \Gamma/[\Gamma, \Gamma] \cong H_1(X, \mathbb{Z}) \cong H_1(S_X, \mathbb{Z})$$

is the group of closed cycles by Notation 2.4.16. In the following we review some results taken from [64] and adapt them to the case of a non-Archimedean field with valuation which is not necessarily discrete.

Definition 2.5.1. A *K-divisor* on Ω is a function $\Omega(\overline{K}) \to \mathbb{Z}$, $n \mapsto n_z$, with the following properties:

(i) $n_{z_1} = n_{z_2}$ if z_1 and z_2 are conjugate over K.
(ii) There is a finite extension L/K such that every $z \in \Omega(\overline{K})$ with $n_z \neq 0$ belongs to $\Omega(L)$.
(iii) The set $\{z; n_z \neq 0\}$ has no accumulation points in Ω with respect to the holomorphic topology.

We denote by $\mathcal{D}$ the set of K-divisors. The group Γ acts on $\mathcal{D}$.

We write a divisor in the form $d = \sum n_z \cdot z$. The set

$$\operatorname{supp}(d) := \{z; n_z \neq 0\}$$

is called its *support*. A divisor is *finite* if $\operatorname{supp}(d)$ is finite.

Lemma 2.5.2.

(a) *The group of Γ-invariant K-divisors consists exactly of the divisors of the form $\sum_{\gamma\in\Gamma} \gamma(d)$, where d is a finite K-divisor.*
(b) $\sum_{\gamma\in\Gamma} \gamma(d) = 0$ *if and only if there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ and finite K-divisors $d_1, \ldots, d_n$ such that $d = \sum_{i=1}^{n} (\mathrm{id} - \gamma_i) d_i$.*

Proof. (a) Let $E \subset \Omega$ be a fundamental domain of Γ; cf. Definition 2.2.7. Let D be a Γ-invariant divisor on Ω. Then $D|_E$ is a finite divisor. Let $d := D|_{E^\circ}$, where $E^\circ \subset E$ is a system of representatives of X; cf. Corollary 2.2.17. Thus, we have that $D = \sum_{\gamma\in\Gamma} \gamma d$.

(b) We proceed by induction on the number of closed points in the support of d. We begin the induction with the empty divisor; here is nothing to show. For the induction step, consider a point x in $\operatorname{supp}(d)$ and let n be its order in d. Since $\sum_{\gamma\in\Gamma} \gamma d = 0$ and since d is finite, there exist finitely many points $y_1, \ldots, y_r$ in the support of d and elements $\gamma_1, \ldots, \gamma_r \in \Gamma - \{\mathrm{id}\}$ such that $x = \gamma_i y_i$, for $i = 1, \ldots, r$ and $n = n_1 + \cdots + n_r$, where n_i is the order of d at y_i. Put $d_i := -n_i \cdot y_i$. Then $(\mathrm{id} - \gamma_i)(d_i) = -n_i y_i + n_i x$ has support contained in $\operatorname{supp}(d)$ and $d - \sum_{i=1}^{r} d_i$ has support in $\operatorname{supp}(d) - \{x\}$. Thus, the assertion follows by induction. The converse implication is clear due to $\Gamma = \Gamma\gamma_i$ for every γ_i, for $i = 1, \ldots, r$. □

Corollary 2.5.3. *Let $\mathcal{D}^\Gamma$ be the set of Γ-invariant divisors. Then the map*

$$\deg : \mathcal{D}^\Gamma \longrightarrow \mathbb{Z},\ D = \sum_{\gamma\in\Gamma} \gamma(d) \longmapsto \deg(d) = \sum_{i=1}^{r} n_i \cdot \big[K(x_i) : K\big],$$

is well-defined if $d = n_1 x_1 + \cdots + n_r x_r$ is a finite divisor as in Lemma 2.5.2.

Proof. The assertion follows from Lemma 2.5.2(b) because of $\deg(d) = \deg(\gamma d)$ for all elements $\gamma \in \Gamma$. □

Definition 2.5.4. For a finite K-divisor $d = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ on Ω_Γ put

$$\theta(d; z) := \frac{(z - a_1) \ldots (z - a_m)}{(z - b_1) \ldots (z - b_n)} \in K(z).$$

Assume, in addition, that $\deg(d) = 0$ and fix a rational base point

$$z_0 \in \Omega - \bigcup_{\gamma\in\Gamma} \operatorname{supp}(\gamma d).$$

If $D := \sum_{\gamma\in\Gamma} \gamma d$ is the associated Γ-invariant divisor, then we consider the formal *Weierstraß product*

$$\Theta(d;z) := \prod_{\gamma\in\Gamma} \frac{\theta(d;\gamma z)}{\theta(d;\gamma z_0)} \quad \text{for } z \in \Omega_\Gamma - \operatorname{supp}(D).$$

Note that we do not include the dependence of z_0 in our notations.

If d is a finite K-divisor, then the points a_i, b_j are not necessarily K-rational. Nevertheless $\theta(d;z)$ is a K-meromorphic function, as can be seen by elementary Galois arguments.

Proposition 2.5.5. *Let d be a finite K-divisor of degree 0. On every affinoid subdomain $V \subset \Omega$, the product $\Theta(d;z)$ can be written as a product of a finite number of factors having zeros or poles on V and a convergent infinite factor which converges uniformly to a holomorphic function on V without zeros. Therefore, $\Theta(d;z)$ is a K-meromorphic function on Ω and has the following properties*:

(a) *For every $\alpha \in \Gamma$ we have that*

$$\Theta(d;\alpha z) = c(d)(\alpha) \cdot \Theta(d;z),$$

where

$$c : \mathcal{D}_0^f \times \Gamma \longrightarrow K^\times,\ (d,\alpha) \longmapsto c(d)(\alpha) = \prod_{\gamma\in\Gamma} \frac{\theta(d;\gamma z_0)}{\theta(d;\gamma\alpha^{-1} z_0)},$$

is bilinear and independent of the base point z_0. Here $\mathcal{D}_0^f$ denotes the set of finite K-divisors of degree 0.

(b) *In the case that $D := \sum_{\gamma\in\Gamma} \gamma d = 0$, the product $\Theta(d;z)$ is an invertible holomorphic function on Ω.*

Proof. There are only finitely many $\gamma \in \Gamma$ such that $V \cap \gamma(V) \neq \emptyset$, as follows from Corollary 2.2.17. To verify the convergence we first show $|\gamma z - \gamma z_0| \to 0$ if $\ell(\gamma) \to \infty$, where $\ell(\gamma)$ is the number of elements in a reduced representation of γ in a separating basis of Γ. In fact, for $z, z_0 \in V$ we can write $z = \alpha w$ and $z_0 = \alpha\beta w_0$ for suitable $w, w_0 \in E^\circ$. There are only finitely many $\alpha, \beta \in \Gamma$ involved as V is affinoid. Then Corollary 2.2.17 yields that $|\gamma z - \gamma z_0| \to 0$, where the distance can be bounded by $\ell(\gamma\alpha)$ and $\ell(\gamma)$. Since $\deg d = 0$, we have to study the behavior of single products

$$\frac{\theta(a;\gamma z)}{\theta(b;\gamma z)} = \frac{\gamma z - a}{\gamma z_0 - a}\frac{\gamma z_0 - b}{\gamma z - b}.$$

Thus, we have to look at the growth of the absolute value of

$$\frac{\gamma z - a}{\gamma z_0 - a} = 1 - \frac{\gamma z - \gamma z_0}{a - \gamma z_0} \quad \text{for } \ell(\gamma) \to \infty.$$

We may assume that $\infty \in \Omega$. Then $|a - \gamma z_0|$ is bounded from below, because, in an affinoid neighborhood U of a, there are only finitely many $\gamma \in \Gamma$ with $\gamma(z_0) \in U$ due to Corollary 2.2.17. Moreover, it follows from Corollary 2.2.17 that the distance $|\gamma z - \gamma z_0|$ tends to 0 uniformly. Thus, the Weierstraß product converges uniformly on every affinoid subdomain V of Ω.

(a) One has

$$\Theta(d; \alpha z) = \prod_{\gamma \in \Gamma} \frac{\theta(d; \gamma \alpha z)}{\theta(d; \gamma z_0)} = \prod_{\gamma \in \Gamma} \frac{\theta(d; \gamma z)}{\theta(d; \gamma \alpha^{-1} z_0)}.$$

Thus, the automorphy factor is

$$c(d)(\alpha) = \prod_{\gamma \in \Gamma} \frac{\theta(d; \gamma z_0)}{\theta(d; \gamma \alpha^{-1} z_0)}.$$

If z_1 is a second base point, then we obtain a new $\Theta_1(d; z)$ (cf. the definition of $\Theta(d; z)$), but they differ only by a constant

$$\Theta(d; z) = \prod_{\gamma \in \Gamma} \frac{\theta(d; \gamma z_1)}{\theta(d; \gamma z_0)} \cdot \Theta_1(d; z).$$

So they have the same automorphy factor. Thus, we see that $c(d)(\alpha)$ is independent of z_0. Moreover, $c(d)(\alpha)$ belongs to $K^\times$ even if $z_0 \in \Omega(L)$ for some finite separable field extension L/K. Indeed, $c(d)(\gamma) \in L^\times$ and is $\mathrm{Gal}(\overline{K}/K)$-invariant as it is independent of z_0. Obviously, we have that

$$c(d_1 + d_2)(\alpha) = c(d_1)(\alpha) \cdot c(d_2)(\alpha),$$
$$c(d)(\alpha_1 \alpha_2) = c(d)(\alpha_1) \cdot c(d)(\alpha_2).$$

(b) By Lemma 2.5.2, it suffices to look at the case $d = \alpha z_1 - z_1$ for some $\alpha \in \Gamma$ and $z_1 \in \Omega$. Then, it is clear that $\Theta(d; z)$ has no zeros and poles. □

Definition 2.5.6. A meromorphic function f on Ω satisfying

$$f(\gamma z) = c(\gamma) \cdot f(z) \quad \text{for } \gamma \in \Gamma$$

with a constant $c(\gamma) \in K^\times$ is called an *automorphic function* with respect to the Γ-action on Ω. The constants satisfy $c(\alpha\beta) = c(\alpha) \cdot c(\beta)$ for all $\alpha, \beta \in \Gamma$. Thus, they give rise to a group homomorphism $c : \Gamma \to K^\times$. The latter is called the *automorphy factor* of f.

By Proposition 2.5.5(a), the function $\Theta(d; z)$ is an automorphic function with automorphy factor $c(d)$.

Remark 2.5.7. Let $z_1 \in \Omega(K)$ and $\alpha \in \Gamma$. Then put

$$\Theta(\alpha z_1 - z_1; z) := \prod_{\gamma \in \Gamma} \frac{\theta(\alpha z_1 - z_1; \gamma z)}{\theta(\alpha z_1 - z_1; \gamma z_0)}.$$

Note that $\Theta(\alpha z_1 - z_1; z)$ depends on z_0 and z_1. It is an automorphic function without zeros and poles and with the automorphy factor $c(\alpha z_1 - z_1)$ which is defined for $\beta \in \Gamma$ by

$$c(\alpha z_1 - z_1)(\beta) = \prod_{\gamma \in \Gamma} \frac{\theta(\alpha z_1 - z_1; \gamma z_0)}{\theta(\alpha z_1 - z_1; \gamma \beta^{-1} z_0)} = \prod_{\gamma \in \Gamma} \frac{\theta(\alpha z_1 - z_1; \gamma \beta z_0)}{\theta(\alpha z_1 - z_1; \gamma z_0)}.$$

The automorphy factor $c(\alpha z_1 - z_1)$ is independent of z_0 due to Proposition 2.5.5.

Proposition 2.5.8. *Let $H := \Gamma/[\Gamma, \Gamma]$ be the maximal abelian quotient of Γ. For $\alpha, \beta \in \Gamma$ let $\overline{\alpha}, \overline{\beta} \in H$ be their residue classes. Moreover, for a point $z_1 \in \Omega(K)$ put*

$$\langle \alpha, \beta \rangle := c(\beta z_1 - z_1)(\alpha).$$

Then $\langle \alpha, \beta \rangle$ depends only on $\overline{\alpha}, \overline{\beta} \in H$, but not on z_0 and z_1. The mapping

$$\langle \_, \_ \rangle : H \times H \longrightarrow K^\times, \ (\alpha, \beta) \longmapsto \langle \alpha, \beta \rangle,$$

is a symmetric bimultiplicative pairing.

Proof. If $z_0 \in \Omega - (\Gamma\infty \cup \Gamma z_1)$, then we obtain from Proposition 2.5.5 that

$$c(\beta z_1 - z_1)(\alpha) = \prod_{\gamma \in \Gamma} \frac{\gamma z_0 - \beta z_1}{\gamma z_0 - z_1} \frac{\gamma \alpha^{-1} z_0 - z_1}{\gamma \alpha^{-1} z_0 - \beta z_1}.$$

Using the projective invariance of the cross-ratio, we can rewrite this formula in the form

$$\begin{aligned} c(\beta z_1 - z_1)(\alpha) &= \prod_{\gamma \in \Gamma} \frac{\gamma^{-1} \beta z_1 - z_0}{\gamma^{-1} \beta z_1 - \alpha^{-1} z_0} \frac{\gamma^{-1} z_1 - \alpha^{-1} z_0}{\gamma^{-1} z_1 - z_0} \\ &= \prod_{\gamma \in \Gamma} \frac{\theta(\alpha^{-1} z_0 - z_0; \gamma^{-1} z_1)}{\theta(\alpha^{-1} z_0 - z_0; \gamma^{-1} \beta z_1)} = c\big(\alpha^{-1} z_0 - z_0\big)\big(\beta^{-1}\big). \end{aligned}$$

The left-hand side is multiplicative in α and does not depend on z_0. The right-hand side is multiplicative in β and does not depend on z_1. Both sides are independent of the variable z_1 and z_0. Thus, they depend only on α and β. Therefore, $\langle \alpha, \beta \rangle = \langle \beta^{-1}, \alpha^{-1} \rangle = \langle \beta, \alpha \rangle$; the last equality follows from the bilinearity; cf. Proposition 2.5.5(a). The form depends only on the residue class in H, because the codomain of the bilinear map is a commutative group. □

Definition 2.5.9. Let $\gamma \in \Gamma - \{\mathrm{id}\}$. Then

$$z_\gamma^+ := \lim_{n \to \infty} \gamma^n(z_0) \quad \text{and} \quad z_\gamma^- := \lim_{n \to -\infty} \gamma^n(z_0)$$

denote the *attractive* and the *repelling fixed point* of γ, respectively. Here $z_0 \in \mathbb{P}_K^1$ is any point which is not a fixed point of γ. If the coordinate function z is chosen in such a way that $z(z_\gamma^+) = 0$ and $z(z_\gamma^-) = \infty$, then $\gamma(z) = q_\gamma \cdot z$ for some $q_\gamma \in K^\times$ with $0 < |q_\gamma| < 1$. The element q_γ is called the *multiplier* of γ.

The formula in Proposition 2.5.8 for $\langle\alpha, \beta\rangle$ is not very useful in practice, because it involves the points z_0 and z_1. We next show how to remedy this drawback. We begin with a simple lemma which examines the meaning of the restrictions imposed in the following proposition.

Lemma 2.5.10. *Let Γ be a Schottky group.*

(a) *Let $\alpha, \beta \in \Gamma$ be elements such that their images in $H := \Gamma/[\Gamma, \Gamma]$ are independent; i.e., they generate a free abelian group of rank 2. If $C(\alpha|\beta)$ denotes a system of representatives for the set of double cosets of $\alpha^{\mathbb{Z}}\backslash\Gamma/\beta^{\mathbb{Z}}$, then every element of Γ has a unique representation in the form $\alpha^m\gamma\beta^n$ with $m, n \in \mathbb{Z}$ and $\gamma \in C(\alpha|\beta)$.*

(b) *Let $\alpha \in \Gamma$ be an element whose image in H is not divisible and set $C_0(\alpha|\alpha) = C(\alpha|\alpha) - \{\mathrm{id}\}$ where $\{\mathrm{id}\}$ represents the unitary class. Then every element of Γ has a unique representation which either has the form α^m with $m \in \mathbb{Z}$ or the form $\alpha^m\gamma\alpha^n$ with $m, n \in \mathbb{Z}$ and $\gamma \in C_0(\alpha|\alpha)$.*

Proof. (a) If $\alpha^m\gamma\beta^n = \gamma'$ for $\gamma, \gamma' \in C(\alpha|\beta)$, then $\gamma = \gamma'$, because $C(\alpha|\beta)$ is a set of representatives of double cosets. So there is a relation between α and β modulo the commutator subgroup. This would be a contradiction.

(b) As above, if $\alpha^m\gamma\alpha^n = \gamma'$, then $\gamma = \gamma'$. Since Γ is free and hence H is a free abelian group, the order the image of α in H is infinite. Thus, we see that $m+n = 0$, and hence that $\alpha^m\gamma = \gamma\alpha^m$. Now one shows by a combinatorial computation that α^m and γ belong to a common cyclic subgroup of Γ, because Γ is free. This follows also from Schreier's theorem that every subgroup of a free group is free; cf. [90, §3.4, Theorem 5]. Since α is not divisible in H, we see that γ represents the unitary class. □

Proposition 2.5.11. *Let Γ be a Schottky group.*

(a) *If $\alpha, \beta \in \Gamma$ are as in Lemma 2.5.10(a), then*

$$\langle\alpha, \beta\rangle = \prod_{\gamma \in C(\beta|\alpha)} \frac{\theta(z_\beta^+ - z_\beta^-; \gamma z_\alpha^+)}{\theta(z_\beta^+ - z_\beta^-; \gamma z_\alpha^-)}.$$

(b) *If $\alpha \in \Gamma$ is as in Lemma 2.5.10(b), then*

$$\langle\alpha, \alpha\rangle = q_\alpha \prod_{\gamma \in C_0(\alpha|\alpha)} \frac{\theta(z_\alpha^+ - z_\alpha^-; \gamma z_\alpha^+)}{\theta(z_\alpha^+ - z_\alpha^-; \gamma z_\alpha^-)},$$

where q_α is the multiplier of α; cf. Definition 2.5.9.

Proof. (a) By the assumption on α and β, every element in $C(\beta|\alpha)$ has a unique representation $\beta^m\gamma\alpha^n$ by Lemma 2.5.10. From Remark 2.5.7 we know that

$$\langle\alpha,\beta\rangle = \prod_{\gamma\in\Gamma} \frac{\beta z_1-\gamma z_0}{\beta z_1-\gamma\alpha^{-1}z_0}\frac{z_1-\gamma\alpha^{-1}z_0}{z_1-\gamma z_0}.$$

We group all members of a coset of $\beta^{\mathbb{Z}}\backslash\Gamma/\alpha^{\mathbb{Z}}$ and re-sort the product

$$\langle\alpha,\beta\rangle = \prod_{\gamma\in C(\beta|\alpha)}\prod_{m\in\mathbb{Z}}\prod_{n\in\mathbb{Z}} \frac{\beta z_1-\beta^{-m}\gamma\alpha^n z_0}{\beta z_1-\beta^{-m}\gamma\alpha^{n-1}z_0}\frac{z_1-\beta^{-m}\gamma\alpha^{n-1}z_0}{z_1-\beta^{-m}\gamma\alpha^n z_0}.$$

For a fixed $m\in\mathbb{Z}$ and $\gamma\in C(\beta|\alpha)$ the interior product $\prod_{n\in\mathbb{Z}}$ is easily computed as follows:

$$\prod_{n\in\mathbb{Z}} \frac{\beta z_1-\beta^{-m}\gamma\alpha^n z_0}{\beta z_1-\beta^{-m}\gamma\alpha^{n-1}z_0}\frac{z_1-\beta^{-m}\gamma\alpha^{n-1}z_0}{z_1-\beta^{-m}\gamma\alpha^n z_0} = \frac{\beta^{m+1}z_1-\gamma z_\alpha^+}{\beta^{m+1}z_1-\gamma z_\alpha^-}\frac{\beta^m z_1-\gamma z_\alpha^-}{\beta^m z_1-\gamma z_\alpha^+}$$

because consecutive terms simplify and $\alpha^n z_0\to z_\alpha^+$ for $n\to\infty$ whereas $\alpha^n z_0\to z_\alpha^-$ for $n\to-\infty$. In addition, we make use of the projective invariance of the cross ratio under β^m. The same trick can be applied to the product $\prod_{m\in\mathbb{Z}}$ over the last results. Thus,

$$\prod_{m\in\mathbb{Z}} \frac{\beta^{m+1}z_1-\gamma z_\alpha^+}{\beta^m z_1-\gamma z_\alpha^+}\frac{\beta^m z_1-\gamma z_\alpha^-}{\beta^{m+1}z_1-\gamma z_\alpha^-} = \frac{z_\beta^+-\gamma z_\alpha^+}{z_\beta^--\gamma z_\alpha^+}\frac{z_\beta^--\gamma z_\alpha^-}{z_\beta^+-\gamma z_\alpha^-}.$$

These are exactly the factors of the desired product we asserted.

(b) Since α is not divisible in H, every element in Γ either belongs to $\alpha^{\mathbb{Z}}$ or can be written uniquely in the form $\alpha^m\gamma\alpha^n$, where $m,n\in\mathbb{Z}$ and $\gamma\in C_0(\alpha|\alpha)$ due to Lemma 2.5.10. Now the proof of (b) is completely analogous to the one of (a). The contribution of the coset $\alpha^{\mathbb{Z}}$ to the product is given by

$$\frac{\theta(z_\alpha^+-z_\alpha^-;\alpha z_1)}{\theta(z_\alpha^+-z_\alpha^-;z_1)} = \frac{\alpha z_1-z_\alpha^+}{\alpha z_1-z_\alpha^-}\frac{z_1-z_\alpha^-}{z_1-z_\alpha^+} = q_\alpha.$$

Indeed, the cross-ratio is invariant under $\mathrm{PGL}(2,K)$, so we may assume that $z_\alpha^-=\infty$ and $z_\alpha^+=0$. In this case the cross-ratio is $\alpha z_1/z_1=q_\alpha$. □

Example 2.5.12. If the coordinate function is chosen in such a way that α is induced by the matrix $\begin{pmatrix} q & 1\\ 0 & 1\end{pmatrix}$ with $|q|<1$, then $z_\alpha^+=0$ and $z_\alpha^-=\infty$ and hence $\theta(z_\alpha^+-z_\alpha^-;z)=z$. Furthermore, if γ is associated to $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, then $\gamma(z_\alpha^+)=b/d$ and $\gamma(z_\alpha^-)=a/c$. Thus, the formula of Proposition 2.5.11(b) reduces to

$$\langle\alpha,\alpha\rangle = q\prod_{i\in I}\frac{b_i c_i}{a_i d_i},$$

where $C_0(\alpha|\alpha)=\{\gamma_i;\, i\in I\}$ and γ_i is induced by $\begin{pmatrix} a_i & b_i\\ c_i & d_i\end{pmatrix}$ for $i\in I$.

Proposition 2.5.13. *For $\alpha \in \Gamma - \{\mathrm{id}\}$ we have that*

$$\Theta(\alpha z_1 - z_1; z) = \prod_{\gamma \in C(\mathrm{id}\,|\alpha)} \frac{\theta(\gamma z_\alpha^+ - \gamma z_\alpha^-; z)}{\theta(\gamma z_\alpha^+ - \gamma z_\alpha^-; z_0)}.$$

This product has neither poles nor zeroes in Ω. Furthermore, it does not depend on z_1 and it is multiplicative in α.

Proof. This follows in the same way as Proposition 2.5.11. Indeed, on the left-hand side the product runs only over $\gamma \in C(\mathrm{id}\,|\alpha)$ and $n \in \mathbb{Z}$; cf. Remark 2.5.7. The result of the interior product over $n \in \mathbb{Z}$ yields

$$\prod_{n \in \mathbb{Z}} \frac{z - \gamma\alpha^n \alpha z_1}{z - \gamma\alpha^n z_1} \frac{z_0 - \gamma\alpha^n z_1}{z_0 - \gamma\alpha^n \alpha z_1} = \frac{z - \gamma z_\alpha^+}{z - \gamma z_\alpha^-} \frac{z_0 - \gamma z_\alpha^-}{z_0 - \gamma z_\alpha^+} = \frac{\theta(\gamma z_\alpha^+ - \gamma z_\alpha^-; z)}{\theta(\gamma z_\alpha^+ - \gamma z_\alpha^-; z_0)}.$$

In fact, the finite partial product is

$$\prod_{n=-N}^{N} \frac{z - \gamma\alpha^n \alpha z_1}{z - \gamma\alpha^n z_1} \frac{z_0 - \gamma\alpha^n z_1}{z_0 - \gamma\alpha^n \alpha z_1} = \frac{z - \gamma\alpha^{N+1} z_1}{z - \gamma\alpha^{-N} z_1} \frac{z_0 - \gamma\alpha^{-N} z_1}{z_0 - \gamma\alpha^{N+1} z_1}.$$

Thus, taking the limit yields the given formula, because $\alpha^n z_1 \to z_\alpha^+$ for $n \to \infty$ and $\alpha^n z_1 \to z_\alpha^-$ for $n \to -\infty$. The multiplicativity follows from the formula

$$(\alpha\beta - \mathrm{id})z_1 = (\alpha - \mathrm{id})(\beta z_1) + (\beta - \mathrm{id})z_1$$

and the fact that $\Theta(\alpha z_1 - z_1; z) = \Theta(\alpha\beta(z_1) - \beta(z_1); z)$.

The independence of z_1 follows from the formula, because the right-hand side does not involve z_1. □

Since $\Theta(\alpha z_1 - z_1; \beta z) = \langle \beta, \alpha \rangle \cdot \Theta(\alpha z_1 - z_1; z)$, we see that the logarithmic derivative $\mathrm{dlog}(\Theta(\alpha z_1 - z_1; z))$ is Γ-invariant. Thus, we obtain the result:

Corollary 2.5.14. *The logarithmic derivative* $\mathrm{dlog}(\Theta(\alpha z_1 - z_1; z))$ *gives rise to a Γ-invariant holomorphic differential on Ω, and hence to a holomorphic differential form on X. It depends additively on $\alpha \in \Gamma$.*

2.6 Drinfeld's Polarization

As in Sect. 2.5, let Γ be a Schottky group of rang $g \geq 1$. In the following we keep the same notation as there. After the preparations in Sect. 2.4 we are now able to show the positivity of the bilinear form in Proposition 2.5.8; this form is called *Drinfeld's polarization*. Later in Corollary 2.9.16 we will show that the scalar form $\langle \_, \_ \rangle$ is the Riemann form of the Jacobian of the Mumford curve X_Γ.

Lemma 2.6.1. *Let $\rho : \Omega \to S_\Omega$ be a semi-stable skeleton and consider K-rational points a_1, a_2, z_1, z_2 of Ω which are separated by ρ. Denote by $\overline{a_1, a_2}$, respectively, $\overline{z_1, z_2}$, the oriented paths leading from $\rho(a_1)$ to $\rho(a_2)$, respectively, from $\rho(z_1)$ to $\rho(z_2)$. Then*

$$-\log\left|\frac{\theta(a_1 - a_2; z_1)}{\theta(a_1 - a_2; z_2)}\right| = [\overline{a_1, a_2}, \overline{z_1, z_2}],$$

where $[\_, \_]$ is the pairing of Definition 2.4.14.

Proof. The left-hand side is the cross-ratio

$$\frac{\theta(a_1 - a_2; z_1)}{\theta(a_1 - a_2; z_2)} = \frac{z_1 - a_1}{z_1 - a_2}\frac{z_2 - a_2}{z_2 - a_1}.$$

Using the projective invariance of the cross-ratio, we may assume that $z_1 = 0$, $z_2 = \infty$, $a_2 = 1$ and $a_1 = a \notin \{0, 1, \infty\}$. Then the left-hand side is equal to $-\log|a|$. For computing the right-hand side, it suffices by Lemma 2.4.15 to consider the stable skeleton associated to the points $0, 1, a, \infty$. So, let us start with the standard reduction of $\mathbb{P}^1_K$.

If $|a| = 1$ and $|a - 1| = 1$, the standard skeleton is the stable one. Thus, the stable skeleton consists of one vertex and the right-hand side is equal to 0 and coincides with the left-hand side. If $|a - 1| < 1$, one has to refine the standard skeleton as was explained in the proof of Lemma 2.4.5. Thus, we see that the path $\overline{1, a}$ has no edge in common with $\overline{0, \infty}$. Thus, we see that the formula is correct if $|a| = 1$.

If $|a| < 1$, then one has to refine the standard skeleton by introducing a new vertex associated to the disc $\{z \in \mathbb{P}^1_K; |\zeta(z)| \le |a|\}$, where ζ is the standard coordinate function on $\mathbb{P}^1_K$, and a new edge e' associated to the annulus $A(|a|, 1)^-$. The two paths have the edge e' in common with the same orientation. If $|a| > 1$, then we introduce the vertex associated to $\{z \in \mathbb{P}^1_K; |\zeta(z)| \ge |a|\}$. Now the common edge e' has opposite orientation on the two paths. However, the height of $A(1, |a|)^-$ is $1/|a|$. Therefore, we obtain in both cases $[\overline{1, a}, \overline{0, \infty}] = -\log|a|$. □

Now we are ready to prove the main result of this section.

Theorem 2.6.2 (Drinfeld). *Let $H := \Gamma/[\Gamma, \Gamma]$. The bilinear form of Proposition 2.5.8 satisfies*

$$\left|\langle\alpha, \alpha\rangle\right| < 1,$$

for $\alpha \in \Gamma$ with $\overline{\alpha} \in H - \{1\}$; i.e., the bilinear form

$$-\log\left|\langle\_, \_\rangle\right| : H \times H \longrightarrow \mathbb{R},\ (\alpha, \beta) \longmapsto -\log\left|\langle\alpha, \beta\rangle\right|,$$

is positive definite.

To prove Theorem 2.6.2 we will explicitly calculate the form $\langle\alpha, \beta\rangle$ in terms of the skeleton S_X. First we introduce some notations.

Consider the semi-stable skeleton $\rho : \Omega \to S_\Omega$ of Proposition 2.4.11. In the following we use the notations of Definition 2.4.14 and Notation 2.4.16.

For a transformation $\alpha \in \Gamma - \{\mathrm{id}\}$ let z_α^+ and z_α^- be the attractive and the repelling fixed point of α, respectively; cf. Definition 2.5.9. Let $x_\alpha := \overline{z_\alpha^+, z_\alpha^-} \subset S_\Omega$ be the axis leading from z_α^+ to z_α^-; cf. Remark 2.4.13. Then α acts on x_α by shifting. Let γx_α be the image of this axis by $\gamma \in \Gamma$. Denote by c_α the section of the axis x_α contained in the fundamental domain of $\alpha^{\mathbb{Z}}$. If one has fixed a base point z_0 such that $\rho_\Omega(z_0)$ is a vertex, then c_α is conjugate to the unique path $\overline{\rho(z_0), \rho(\alpha(z_0))}$, which leads from $\rho_\Omega(z_0)$ to $\rho_\Omega(\alpha(z_0))$ without backtracking. Then

$$c_\alpha = e_1 + \cdots + e_s$$

is a finite sequence of oriented edges $e_1, \ldots, e_s$. The axis can be presented in the form

$$x_\alpha = \sum_{n\in\mathbb{Z}} \alpha^n c_\alpha = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{s} \alpha^n e_i. \qquad (*)$$

The chain c_α is cycle whose homology class in $H_1(S_X, \mathbb{Z})$, more precisely in $H_1(\mathrm{real}(S_X), \mathbb{Z})$, coincides with the image of $\alpha \in \Gamma$ under the natural map $\pi_1(S_X) \cong \Gamma \to H_1(S_X, \mathbb{Z}) \cong H$. So c_α depends only on α and is additive in α.

Theorem 2.6.2 is now a consequence of the following theorem, which is closely related to results of Grothendieck; cf. [42, Exp. IX, 12.3.7 and 12.5].

Theorem 2.6.3. *In the situation of Theorem 2.6.2, if $\alpha, \beta \in \Gamma$, then we have*

$$-\log\big|\langle\alpha, \beta\rangle\big| = [c_\alpha, c_\beta].$$

Proof. Both sides of the formula are bilinear symmetric forms with respect to α and β. It therefore suffices to show that they coincide if $\alpha = \beta$ and $\overline{\alpha}$ is not divisible in H. By Proposition 2.5.11(b) we have

$$\langle\alpha, \alpha\rangle = q_\alpha \prod_{\gamma\in C(\alpha|\alpha)-\{\mathrm{id}\}} \frac{\theta(z_\alpha^+ - z_\alpha^-; \gamma z_\alpha^+)}{\theta(z_\alpha^+ - z_\alpha^-; \gamma z_\alpha^-)}.$$

Define $r := -\log|q_\alpha|$ which is positive. Then we obtain

$$\begin{aligned}
-\log\big|\langle\alpha, \alpha\rangle\big| &= r + \sum_{\gamma\in C_0(\alpha|\alpha)} [x_\alpha, \gamma x_\alpha] \\
&= r + \sum_{\gamma\in C_0(\alpha|\alpha)} \sum_{m,n\in\mathbb{Z}} \big[\alpha^m c_\alpha, \gamma\alpha^n c_\alpha\big] \\
&= r + \sum_{\gamma\in C_0(\alpha|\alpha)} \sum_{m,n\in\mathbb{Z}} \big[c_\alpha, \alpha^{-m}\gamma\alpha^n c_\alpha\big] \\
&= r + \sum_{\gamma\in\Gamma-\alpha^{\mathbb{Z}}} [c_\alpha, \gamma c_\alpha] = \sum_{\gamma\in\Gamma} [c_\alpha, \gamma c_\alpha],
\end{aligned}$$

where $C_0(\alpha|\alpha) := C(\alpha|\alpha) - \{\text{id}\}$ is as in Proposition 2.5.11. The first equation follows from Lemma 2.6.1. The second equation follows form $(*)$. The third equation is valid, because the height is invariant under projective transformations. For the fourth equation note that every element in $\Gamma - \alpha^{\mathbb{Z}}$ has a unique representation $\alpha^m \gamma \alpha^n$ with $\gamma \in C_0(\alpha|\alpha)$ and $m, n \in \mathbb{Z}$, since $\overline{\alpha}$ is not divisible. The last equation is true, because $[c_\alpha, \alpha^n c_\alpha]$ equals r if $n = 0$, and equals 0 if $n \neq 0$. Note that x_α and γx_α have only a finite path in common for $\gamma \in C_0(\alpha|\alpha)$, because c_α and $\alpha^m \gamma \alpha^n c_\alpha$ are disjoint for almost all $\alpha^m \gamma \alpha^n$, as follows from Corollary 2.2.17. Finally, we show that

$$\sum_{\gamma \in \Gamma} [c_\alpha, \gamma c_\alpha] = \sum_{i,j=1}^{r} [e_i, \gamma e_j] = [\overline{c}_\alpha, \overline{c}_\alpha] = \sum_{j=1}^{N} -m_i^2 \log \varepsilon(\overline{e}_i) > 0,$$

where $\overline{c}_\alpha := p_S(c_\alpha)$ is the image of c_α as defined in Notation 2.4.16, and $\overline{e}_1, \ldots, \overline{e}_N$ are all the (geometric) edges of S_X and $\varepsilon(\overline{e}_j)$ is the height of its associated annulus. The integer m_j denotes the number of times the path $\overline{c}_\alpha$ passes the edge $\overline{e}_j$. We may assume that none of the edges of S_X is a loop.

Indeed, using the fact that $c_\alpha = e_1 + \cdots + e_s$, we have

$$\big[p_S(e_i), p_S(e_j)\big] = \begin{cases} [e_i, \gamma e_j] & \text{if there exists a } \gamma \in \Gamma \text{ with } \gamma e_i = \pm e_j, \\ 0 & \text{else.} \end{cases}$$

In the first case, the transformation $\gamma \in \Gamma$ is unique by Corollary 2.4.12. Thus, we obtain the above formula, because we have that $[e_i, e_i] = -\log \varepsilon(e_i) > 0$ by Definition 2.4.14 and $\varepsilon(e_i) = \varepsilon(p_S(e_i))$. This also completes the proof of Theorem 2.6.2. □

In Sect. 6.5 we will present an analog of Drinfeld's pairing in the case of a smooth projective curve with semi-stable reduction; cf. Corollary 6.5.10. In more detail, one can rephrase Theorem 2.6.3 in the following form.

Remark 2.6.4. Consider the situation of Proposition 2.4.11(c) where, in addition, we fix a K-rational base point z_0 which is sent to a vertex $\rho_\Omega(z_0) \in S_\Omega$. Let c_α be a path in S_Ω which starts at $\rho_\Omega(z_0)$ and terminates in $\rho_\Omega(\alpha z_0)$. Let $\overline{c}_\alpha = p_S(c_\alpha)$ be the image of the path c_α for $\alpha \in \Gamma$, which is cycle in S_X. If $\alpha, \beta \in \Gamma$, then

$$\log\big|\langle \alpha, \beta \rangle\big| = [\overline{c}_\alpha, \overline{c}_\beta] = \sum_{i=1}^{N} m_i n_i \log \varepsilon(\overline{e}_i).$$

Here $\overline{e}_1, \ldots, \overline{e}_N$ are all the edges of S_X equipped with an orientation. The integers m_i and n_i, respectively, are the number of times the paths $p_S(c_\alpha)$ and $p_S(c_\beta)$, respectively, pass through the edge $\overline{e}_i$; the counting takes the orientation of the edges into account.

2.7 Rigid Analytic Tori and Their Duals

In this section we present the basic theory of rigid analytic tori. This will be used in Sect. 2.8 when studying the Jacobian variety of a Mumford curve. For a systematic approach it is appropriate to study line bundles of such tori and to show the representability of the Picard functor of translation invariant line bundles on a rigid analytic torus. The latter can be used to classify the abelian varieties among the rigid analytic tori.

Let M' be a free abelian group of rank r, and denote its group law by "+". Let $T := \operatorname{Spec} K[M']$ be the associated affine torus. In the following we regard T as a rigid analytic variety. Let us start by recalling some well-known facts on tori.

Proposition 2.7.1. *In the above situation we have*:

(a) *For every field extension L/K, the set of L-valued points of T can be identified with the set of group homomorphisms $t : M' \to L^\times$.*
(b) *Let $M'_1 = \mathbb{Z}^{r_1}$ and $M'_2 := \mathbb{Z}^{r_2}$ be free abelian groups of finite rank and let $T_1 :=$ $\operatorname{Spec} K[M'_1]$ and $T_2 := \operatorname{Spec} K[M'_2]$ be the associated affine tori. Then the map*

$$\operatorname{Hom}(T_1, T_2) \xrightarrow{\sim} \operatorname{Hom}\big(M'_2, M'_1\big), \quad \varphi \longmapsto \varphi^*|_{M'_2},$$

is an isomorphism of groups. Its inverse sends $\lambda : M'_2 \to M'_1$ to the map

$$\varphi : T_1 \longrightarrow T_2,\ t_1 \longmapsto \big[\varphi(t_1) : M'_2 \longrightarrow \mathbb{G}_{m,K}, m'_2 \longmapsto t_1\big(\lambda\big(m'_2\big)\big)\big].$$

In particular, there is a canonical isomorphism

$$M' \xrightarrow{\sim} \operatorname{Hom}(T, \mathbb{G}_{m,K}), \quad m' \longmapsto \big[\chi_{m'} : T \to \mathbb{G}_{m,K}, t \longmapsto t\big(m'\big)\big].$$

A *lattice* $M \subset T$ is a discrete rigid analytic subgroup of T such that the group homomorphism

$$\ell : T(\overline{K}) \longrightarrow \mathbb{R}^r,\ z = (z_1, \ldots, z_r) \longmapsto \ell(z) := -\big(\log|z_1|, \ldots, \log|z_r|\big),$$

induces an isomorphism of $M(\overline{K})$ to a lattice $\Lambda \subset \mathbb{R}^r$, where $\overline{K}$ is a complete algebraic closure of K. It is said to have *full rank* if the rank of Λ is r. We write the group law on M additively, as in the case of M'.

Proposition 2.7.2. *Let M and M' be a free abelian group of rank r and set* $T := \operatorname{Spec} K[M']$ *and* $T' := \operatorname{Spec} K[M]$.

(a) *There is a one-to-one correspondence between* $\operatorname{Hom}(M, \operatorname{Hom}(M', K^\times))$ *and the group* $\operatorname{Bihom}(M \times M', K^\times)$ *of bilinear forms.*
(b) *There is a one-to-one correspondence between* $\operatorname{Hom}(M, \operatorname{Hom}(M', K^\times))$ *and the group of homomorphisms $h : M \to T$.*

(c) *A group homomorphism $h : M \to T$ maps M bijectively to a lattice of rank r if and only if, for the associated bilinear form b, the bilinear form*

$$-\log|b| : M \times M' \longrightarrow \mathbb{R}, \ \big(m, m'\big) \longmapsto -\log\big|b\big(m, m'\big)\big|,$$

is non-degenerate.

(d) *The group of homomorphisms $h : M \to T$ corresponds one-to-one to the group of homomorphisms $h' : M' \to T'$. Moreover, h maps M bijectively to a lattice of T if and only if the corresponding homomorphism $h' : M' \to T'$ maps M' bijectively to a lattice of T'.*

(e) *If h is an inclusion, then the inclusion h' is given by*

$$M' \longrightarrow T, \quad m' \longmapsto \big[m'|_M : M \to K^\times\big].$$

Proof. (a) is well-known.

(b) If $h \in \mathrm{Hom}(M, \mathrm{Hom}(M', K^\times))$, then the map $h(m)$ is a group homomorphism $h(m) : M' \to K^\times$ for each $m \in M$, and hence a K-valued point of T due to Proposition 2.7.1(a). Moreover, $h : M \to T$ is group homomorphism. Conversely, if $h : M \to T$ is a group homomorphism, then $h(m) : M' \to K^\times$ is group homomorphism, and hence h can be identified with an element of $\mathrm{Hom}(M, \mathrm{Hom}(M', K^\times))$. Obviously, the correspondence is one-to-one.

(c) By (a) and (b) we have that every group homomorphism $h : M \to T$ corresponds to a bilinear form $b : M \times M' \to K^\times$. Now h maps M bijectively to a lattice in T if and only if for each $m \in M - \{0\}$ there exists a character $m' \in M'$ with $|m'(h(m))| \neq 1$. The latter is equivalent to the fact that $-\log|b|$ is non-degenerate.

(d) follows from (a) and (b), because the condition (a) is a symmetric in M and M'. The assertion about lattices follows from (c).

(e) follows from the proof of (b). □

The next proposition characterizes lattices of full rank.

Proposition 2.7.3. *Let $M \subset T$ be a lattice. Then the following holds:*

(a) *The rigid analytic quotient $A := T/M$ exists and is a smooth rigid analytic group variety. The quotient map $p : T \to A$ is a unramified covering in the topological sense; cf. Definition 1.7.10.*

(b) *M has full rank if and only if T/M is a proper rigid analytic variety.*

Proof. (a) The rigid analytic structure of A is defined as the geometric quotient of T with respect to the M-action on T; i.e., a set $V \subset A$, respectively, a covering $\mathfrak{V}$ of A is admissible if $p^{-1}(V) \subset T$, respectively, $p^*\mathfrak{V}$ is admissible. An atlas of charts of A is given in the following way. Let $Q \subset \mathbb{R}^r$ be an r-dimensional polytope whose vertices take coordinates in the divisible additive value group $\log(|\overline{K}^\times|)$ such that Q is contained in a fundamental domain of $\mathbb{R}^r$ modulo Λ. Then $V := \ell^{-1}(Q) \subset T$ is a connected admissible domain which does not contain M-congruent points, where $\ell := -\log$ is as in Proposition 2.7.2, and V is affinoid if M has full rank. These subsets form an atlas of charts of A. The inverse image $W := p^{-1}(V)$ decomposes into a disjoint union $W = \bigcup_{m \in M} m \cdot V$, where $V \subset T$ is as above. The induced group law on T/M is a morphism with respect to this holomorphic structure.

(b) If M has full rank, then T/M is obviously covered by finitely many charts as was defined above. Moreover, if we choose a second polytope $Q' \Subset Q$, we obtain a chart V' with $V' \Subset V$. Obviously, A is also covered by finitely many $V'_1, \ldots, V'_n$ of such charts. Thus, there are two finite coverings $\mathfrak{V} := \{V_1, \ldots, V_n\}$ and $\mathfrak{V}' := \{V'_1, \ldots, V'_n\}$ by affinoid domains such that $V'_i \Subset V_i$ for $i = 1, \ldots, n$; cf. Definition 1.6.3. This means that A is proper as a rigid analytic variety; cf. Definition 1.6.3.

If A is proper, then A can be covered by finitely many affinoid charts. Since these are bounded subsets in T, we see that there exists a bounded subset $B \subset \mathbb{R}^r$ which covers $\mathbb{R}^r/\Lambda$. Thus, M has full rank. □

In the following we always assume that M has full rank. Let

$$p : T \longrightarrow A := T/M$$

be the canonical quotient map. Then A is called a *rigid analytic torus*.

Next we wish to give an explicit geometric description of line bundles on A. Moreover, we want to construct a universal space A' which parameterizes all translation invariant line bundles in a canonical way; this space A' will be called the *dual* of A.

Lemma 2.7.4. *Every line bundle L on the torus T is trivial. The set of trivializations of a rigidified* (*Definition* 1.7.8) *line bundle is a principal homogeneous space under the character group $M' = \mathrm{Hom}(T, \mathbb{G}_m)$.*

Proof. Let $\mathcal{L}$ be the invertible sheaf associated to L; i.e., the dual of the sheaf of sections in Remark 1.7.2. Let $c \in |K^\times|$ with $c < 1$. It suffices to show that $\mathcal{L}$ is trivial. Put

$$T(c) := \left\{(t_1, \ldots, t_r) \in \mathbb{G}^r_{m,K} ;\, c \le |t_\rho| \le c^{-1} \text{ for } \rho = 1, \ldots, r\right\}.$$

By Proposition 1.6.13 the ring $\mathcal{O}_T(T(c))$ is factorial. Thus, for $n \in \mathbb{N}$, there exists a generator ℓ_n of $\mathcal{L}|_{T(c^n)}$. Since $\ell_n|_{T(c^m)}$ is a generator of $\mathcal{L}$ over $T(c^m)$ for all $m \le n$ as well. There are relations

$$\ell_n|_{T(c^m)} = u_{n,m} \cdot \ell_m$$

with units $u_{n,m} \in T(c^m)$. By Proposition 1.3.4 the units can be written in the form

$$u_{n,m} = c_{n,m} \cdot \xi^{m'(n,m)} \cdot (1 + h_{n,m}),$$

where $c_{n,m} \in K^\times$ is a constant, $\xi^{m'(n,m)}$ is a character and $h_{n,m}$ is a holomorphic function on $T(c^m)$ with sup-norm less than 1. After having fixed ℓ_1, one can adjust the other ℓ_n by requiring that $c_{n,1} = 1$, $m'(n,1) = 0$, and $h_{n,1}(1) = 0$. Then one easily shows that the sequence ℓ_n converges to a global generator of $\mathcal{L}$. Indeed, it follows that $c_{n,m} = 1$ and $m'(n,m) = 0$ for all $m \leq n$. Furthermore, we see by Proposition 1.3.4 that there is the estimate

$$|h_{n,1}|_{T(c)}| \leq c^{n-1}.$$

Thus, the sequence converges on $T(c)$. The same argument shows that it also converges on each $T(c^m)$.

The last assertion follows from the fact that the units u on an affine torus with $u(1) = 1$ are the characters as seen by Proposition 1.3.4. □

Thus, one can present the line bundles on A in terms of M-linearizations on the trivial line bundle over T with respect to the lattice M; cf. Example 1.7.13.

Proposition 2.7.5. *Let $M \subset T$ be a lattice of full rank and $A := T/M$. Then we have the following*:

(a) *The isomorphism classes of line bundles on A correspond bijectively to the isomorphism classes of M-linearizations of the trivial line bundle $\mathbb{A}^1_T$.*
Such a linearization is determined by a couple (r, λ), where $\lambda : M \to M'$ is a group homomorphism and $r : M \to \mathbb{G}_{m,K}$ is a map satisfying the relation

$$\langle m_2, \lambda(m_1)\rangle = \frac{r(m_1 + m_2)}{r(m_1) \cdot r(m_2)} \quad \textit{for all } m_1, m_1 \in M.$$

Here $\langle \_, \_\rangle : T \times M' \longrightarrow \mathbb{G}_{m,K}$ is the evaluation of characters at points.

(b) *A line bundle $\overline{L}$ on A is trivial if and only if the homomorphism λ of the corresponding linearization (r, λ) is zero and the map r satisfies $r(m) = \langle m, m'\rangle$ for some $m' \in M'$ and for all $m \in M$.*

(c) *A line bundle $\overline{L}$ on A is translation invariant if and only if the homomorphism λ of the corresponding linearization (r, λ) is zero. In this case $r : M \to \mathbb{G}_{m,K}$ is a group homomorphism.*

Proof. (a) In view of Lemma 2.7.4, we have to consider only the M-linearizations on the trivial line bundle on T. Consider an M-linearization on the trivial line bundle $T \times \mathbb{A}^1$, say it is given by morphisms

$$c_m : T \times \mathbb{A}^1 \longrightarrow T \times \mathbb{A}^1, \ (z, \ell) \longmapsto \big(m \cdot z, c_m(z) \cdot \ell\big),$$

via a morphism $c_m : T \to \mathbb{G}_{m,K}$. Writing the global function c_m as a Laurent series

$$c_m(z) = \sum_{m' \in M'} r_{m'} \cdot \langle z, m'\rangle$$

with coefficients $r_{m'} \in K$, we see that there exists a character $m' = \lambda(m)$ in M' such that

$$c_m(z) = r_{\lambda(m)} \cdot \langle z, \lambda(m)\rangle;$$

cf. Proposition 1.3.4. The corresponding isomorphism on the trivial line bundle $T \times \mathbb{A}^1$ is necessarily of type $(z, \ell) \longmapsto (z, \langle z, m' \rangle \cdot \ell)$. Now the associativity of the action is equivalent to the condition that

$$r_{m_1} \cdot r_{m_2} \cdot \big\langle z, \lambda(m_1)\big\rangle \cdot \big\langle m_1 \cdot z, \lambda(m_2)\big\rangle = r_{m_1+m_2} \cdot \big\langle z, \lambda(m_1 + m_2)\big\rangle$$

for all $m_1, m_2 \in M$ and all $z \in T$. This in turn is equivalent to the condition that λ is a group homomorphism $\lambda : M \to M'$ and that

$$\big\langle m_1, \lambda(m_2)\big\rangle = r_{m_1+m_2} \cdot r_{m_1}^{-1} \cdot r_{m_2}^{-1}.$$

Thus, we see that $r : M \to \mathbb{G}_{m,K}, m \mapsto r(m) := r_m$, satisfies the condition.

(b) Assume that (r, λ) gives rise to a trivial line bundle on A. Then there exists an invertible function u on T such that

$$r(m) \cdot \big\langle z, \lambda(m)\big\rangle = u(mz) \cdot u(z)^{-1}$$

for all $z \in T$ and $m \in M$. Since $u(z) = c \cdot \langle z, m' \rangle$ for some $m' \in M'$ and $c \in K^\times$ by Proposition 1.3.4, it follows that $\lambda = 0$ and $r(m) = \langle m, m' \rangle$.

Conversely, for a character $m' \in M'$ the M-linearization defined by the homomorphism $r(m) := \langle m, m' \rangle$ for $m \in M$ leads to the trivial line bundle on A.

(c) If (r, λ) corresponds to a translation invariant line bundle on A, then the actions on $T \times \mathbb{A}^1$ given by

$$r(m) \cdot \big\langle z, \lambda(m)\big\rangle \quad \text{and} \quad \big\langle x, \lambda(m)\big\rangle \cdot r(m) \cdot \big\langle z, \lambda(m)\big\rangle, \quad \text{for all } m \in M, z \in T,$$

are isomorphic for every $x \in T$. Due to (b), there exists a character $m'(x)$ in M' such that $\langle m, m'(x) \rangle = \langle x, \lambda(m) \rangle$ for all $m \in M$ and $x \in T$. Thus, we see $\lambda = 0$ and $m'(x) = 0$, since M is a lattice in T.

Conversely, it is easy to see that every pair (r, λ) with $\lambda = 0$ leads to a translation invariant line bundle on A. □

One easily proves the following result.

Lemma 2.7.6. *In the above situation we have*:

(a) *Let* (r, λ) *be a linearization on* $\mathbb{A}^1_T$. *Let* a *be a* K*-rational point of* T *and let* $\tau_a : A \to A$ *be the left-translation by* a. *Then*

$$\tau_a^*\big(r(m), \lambda\big) = \big\langle a, \lambda(m)\big\rangle \cdot \big\langle r(m), \lambda\big).$$

(b) *If* (r_i, λ_i) *are linearizations for* $i = 1, 2$, *then*

$$(r_1, \lambda_1) \otimes (r_2, \lambda_2) = (r_1 \cdot r_2, \lambda_1 + \lambda_2).$$

(c) [Theorem of the Square] *If* a, b *are* K*-rational points of* T, *then*

$$\tau_a^*(r, \lambda) \otimes \tau_b^*(r, \lambda) = \tau_{a+b}^*(r, \lambda) \otimes (r, \lambda).$$

In the following we want to construct the dual variety A' of $A = T/M$; i.e., the universal space which parameterizes the isomorphism classes of translation invariant line bundles on A. Moreover, we will define a universal line bundle $P_{A\times A'}$. This line bundle is called the *Poincaré bundle*. It is rigidified along the unit section of A. For more details see Remark 2.7.8 below.

Theorem 2.7.7. *Let $M \subset T$ be a lattice of full rank. Then we have*:

(a) *The dual variety A' which parameterizes the translation invariant line bundle on A can be represented as*

$$A' = T'/M',$$

where $T' = \operatorname{Spec} K[M]$ which, as a set of closed points, can be identified with $\operatorname{Hom}(M, \mathbb{G}_{m,K})$, *and where $M' \hookrightarrow T'$ is regarded via the mapping*

$$M' \longrightarrow \operatorname{Hom}(M, \mathbb{G}_{m,K}), \quad m' \longmapsto m'\big|_M.$$

(b) *The Poincaré bundle $P_{A\times A'}$ on $A \times A'$ is given by the linearization (R, Λ) of the trivial line bundle on $T \times T'$ with respect to the lattice $M \times M'$, where*

$$\Lambda : M \times M' \longrightarrow M' \times M,\ \big(m, m'\big) \longmapsto \big(m', m\big),$$
$$R : M \times M' \longrightarrow \mathbb{G}_{m,K},\ \big(m, m'\big) \longmapsto \big\langle m, m'\big\rangle.$$

Proof. (b) The given formula defines a linearization. Indeed, on the one hand we have

$$\big\langle\big(m_2, m_2'\big), \Lambda\big(m_1, m_1'\big)\big\rangle = m_1'(m_2)\cdot m_2'(m_1)$$

and on the other hand

$$\frac{R((m_1, m_1') + (m_2, m_2'))}{R(m_1, m_1')\cdot R(m_2, m_2')} = m_1'(m_2)\cdot m_2'(m_1).$$

Obviously, $P|_{0\times A'}$ and $P|_{A\times 0'}$ are trivial. Its universal property is verified below.

(a) Let L be a translation invariant line bundle on A. By Lemma 2.7.4 the pull-back p^*L is trivial on T. Then the natural M-linearization on $p^*L \cong T \times \mathbb{A}^1_K$ is given in the manner of Proposition 2.7.5(c) by a homomorphism $r : M \to \mathbb{G}_{m,K}$ which, in turn, may be viewed as a point $x' \in T'$; namely, as the one which satisfies $r(m) = \langle x', m\rangle$. Thus, x' induces a point $p'(x') \in A'$. By Proposition 2.7.5(b) the point $p'(x')$ is uniquely determined by L. Now consider the $(M \times M')$-action on the line bundle $(T \times T') \times \mathbb{A}^1_K$ which is given by (R, Λ) and restrict it to $M \times 0'$ and to $T \times \{x'\}$. The resulting morphisms

$$T \times \mathbb{A}^1_K \longrightarrow T \times \mathbb{A}^1_K, \quad (x, \ell) \longmapsto \big(x, \big\langle m, x'\big\rangle\cdot \ell\big) \quad \text{for } m \in M,$$

coincide with the ones defining the action on p^*L, because $r(m) = \langle x', m\rangle$ for $m \in M$. Thus, it follows that $L \cong (x' \times \mathrm{id})^* P_{A\times A'}$ for the Poincaré bundle. □

Remark 2.7.8. While the theorem only states that the set of isomorphism classes of translation invariant line bundles L on $A \otimes K'$ bijectively corresponds to the K'-rational points of A', we have in fact that A' satisfies more generally the following universal property:

If S is a rigid analytic space, then, for every rigidified translation invariant line bundle (L, ℓ) on $A \times S$, the map

$$\varphi : S \longrightarrow A', \ s \longmapsto [L|_{A\times s}],$$

is holomorphic, where $[L|_{A\times s}]$ denotes the isomorphism class of the restriction $L|_{A\times s}$, and there is a canonical isomorphism $L \xrightarrow{\sim} (\mathrm{id}, \varphi)^* P_{A\times A'}$ of rigidified line bundles.

The proof of this more general result requires the notion of cubical structures on line bundles on $A \times_K S$ which is related to the theorem of the cube in Theorem 7.1.6. Any rigidified line bundle on $A \times_K S$ canonically carries such a cubical structure. One uses this to show that the pull-back $(p, \mathrm{id})^* L$ on $T \times_K S$ is trivial as a cubical line bundle locally over S cf. Proposition 6.2.10. This is a stronger version of Lemma 2.7.4.

Furthermore, the M-linearization on $(\mathrm{id}, p)^* L$ on $T \times_K S$ induces morphisms of cubical line bundles so that they are given in the manner (r, λ), where $r(m) : S \to \mathbb{G}_{m,K}$ is a holomorphic map for each $m \in M$. Our weaker result could be derived from Proposition 1.3.4. Using the technique of cubical structures, the universal property mentioned above follows in the same way as was explained in the proof of Theorem 2.7.7; cf. Corollary 6.2.6. A more general situation will be treated in Sect. 6.3.

Remark 2.7.9. Without using the stronger version in Remark 2.7.8 of Theorem 2.7.7, the following can be proved. Every line bundle L gives rise to a holomorphic morphism

$$\varphi_L : A \longrightarrow A', \ a \longmapsto \varphi_L(x) := \tau_a^* L \otimes L^{-1}.$$

If (r, λ) is the M-linearization corresponding to a line bundle L, then the lifting $\Phi_L : T \longrightarrow T'$ of φ_L is given by

$$\begin{array}{ccccccc} T & \xrightarrow{\ \Phi_L\ } & T' & \qquad & t & \longmapsto & t \circ \lambda \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \downarrow & & \downarrow \\ A & \xrightarrow{\ \varphi_L\ } & A' & & a & \longmapsto & \varphi_L(a) = \tau_a^* L \otimes L^{-1}, \end{array}$$

where the point $t \in T$ is viewed as a group homomorphism $t : M' \to \mathbb{G}_{m,K}$. Then the composition $t \circ \lambda : M \to \mathbb{G}_{m,K}$ is a group homomorphism as well, and hence it can be viewed as a point of T'; cf. Proposition 2.7.1(a). In particular, the morphism Φ_L corresponds to the homomorphism $\lambda : M \to M'$ of their character groups in the sense of Proposition 2.7.1(b). Furthermore, φ_L is surjective if and only if λ is injective.

Proof. A point $t \in T$ is mapped to the M-linearization $t' : M \to \mathbb{G}_{m,K}$ which is given by

$$m \longmapsto \frac{r(m) \cdot \langle tz, \lambda(m)\rangle}{r(m) \cdot \langle (z, \lambda(m)\rangle} = \big\langle t, \lambda(m)\big\rangle = t \circ \lambda(m).$$

If $\lambda : M \to M'$ is injective, then the index $[M' : \lambda(M)] < \infty$ and hence Φ_L is surjective, because $K[M] \to K[M']$ is finite. If φ_L is surjective, so is Φ_L, and hence the index $[M' : \lambda(M)] < \infty$ is finite. Thus, λ is injective. □

Definition 2.7.10. Let $M \subset T$ be a lattice of full rank. A *polarization* of the couple (T, M) is a linear map $\lambda : M \to M'$ such that the bilinear form

$$M \times M \longrightarrow K^{\times}, \quad (m_1, m_2) \longmapsto \big\langle m_2, \lambda(m_1)\big\rangle,$$

is symmetric and positive definite; i.e., $|\langle m, \lambda(m)\rangle| < 1$ for all $m \in M - \{0\}$.

Remark 2.7.11. Let $M \subset T$ be a lattice.

(a) By Proposition 2.7.1(b) every morphism $\lambda : M \to M'$ gives rise to a morphism

$$\varphi : T \longrightarrow T', \; t \longmapsto \big[\varphi(t) : M \longrightarrow \mathbb{G}_{m,K}, m \longmapsto t\big(\lambda(m)\big) = \big\langle t, \lambda(m)\big\rangle\big].$$

If $\lambda : M \to M'$ is symmetric, i.e., if λ satisfies the rule

$$\big\langle m_2, \lambda(m_1)\big\rangle = \big\langle m_1, \lambda(m_2)\big\rangle \quad \text{for all } m_1, m_2 \in M,$$

then $\varphi(M) \subset M'$, and hence φ induces a morphism $\overline{\varphi} : T/M \to T'/M'$.

(b) For every linearization (r, λ) of the trivial line bundle $T \times \mathbb{A}^1_K$, the bilinear form on M given by λ as in (a) is symmetric, i.e., the rule in (a) is satisfied.

(c) If $\lambda : M \to M'$ is a symmetric group homomorphism, then there exists an M-linearization (r, λ) on the trivial line bundle on T which gives rise to a line bundle on T/M.

Proof. (a) To prove the inclusion $\varphi(M) \subset M'$, consider an element $m_1 \in M$. Then

$$\big\langle m_2, \varphi(m_1)\big\rangle = \big\langle m_1, \lambda(m_2)\big\rangle = \big\langle m_2, \lambda(m_1)\big\rangle \quad \text{for all } m_2 \in M.$$

Thus, we see that $\varphi(m_1) = \lambda(m_1) \in M'$.

(b) The symmetry follows from the relation

$$\big\langle m_1, \lambda(m_2)\big\rangle = \frac{r(m_1 + m_2)}{r(m_1) \cdot r(m_2)} = \big\langle m_2, \lambda(m_1)\big\rangle$$

which was explained in Proposition 2.7.5.

(c) Let $(e_1, \ldots, e_r)$ be a basis of M. We choose values $c_1, \ldots, c_r \in K^{\times}$ and put $r(e_i) = c_i$ for $i = 1, \ldots, g$. Then we define $r : M \to \mathbb{G}_{m,K}$ inductively via the formula

$$r(m_1 + m_2) := r(m_1) \cdot r(m_2) \cdot \big\langle m_1, \lambda(m_2)\big\rangle.$$

Due to the symmetry, the definition of $r(m)$ is independent of the representation of $m \in M$ as a linear combination in the chosen basis of M. □

Theorem 2.7.12. *In the situation of Proposition 2.7.5 let (r, λ) be a pair corresponding to an invertible sheaf $\mathcal{L}$ on A. Then the following conditions are equivalent*:

(a) *$\mathcal{L}$ is ample in the sense of Definition 1.7.3.*
(b) *The morphism λ gives rise to a polarization of (T, M).*
(c) *λ is injective and there is a non-trivial global section of $\mathcal{L}$.*

In particular, $\dim \Gamma(A, \mathcal{L}) = \#(M'/\lambda(M))$.

Proof. By Remark 1.7.2 we can regard the sections of $\mathcal{L}$ as sections of the line bundle L, whose corresponding pair in the sense of Proposition 2.7.5 is given by $(r^{-1}, -\lambda)$.

In order to show the equivalence of (b) and (c), consider a global section f of $\Gamma(A, \mathcal{L})$. Switching back to the line bundle L, we view f as a section of L over A. Pulling back the situation to T, the line bundle p^*L becomes trivial, and hence the pull-back of f becomes a holomorphic function on $f : T \to \mathbb{A}^1_K$ which is invariant under the M-action given by $(r^{-1}, -\lambda)$ of M. Now consider the Laurent series

$$f(z) = \sum_{m' \in M'} a_{m'} \langle z, m' \rangle$$

of f. The invariance of f under the M-action; i.e., the commutativity of the diagram

$$\begin{array}{ccccccc}
T \times \mathbb{A}^1 & \xrightarrow{c_m} & T \times \mathbb{A}^1 & \qquad & (z, \ell) & \longmapsto & (m \cdot z, r(m)^{-1} \langle -\lambda(m), z \rangle \cdot \ell) \\
{\scriptstyle f} \uparrow \downarrow & & \downarrow \uparrow {\scriptstyle f} & & \downarrow & & \downarrow \\
T & \xrightarrow{m} & T & & z & \longmapsto & m \cdot z,
\end{array}$$

yields

$$\begin{aligned}
r(m)^{-1} \langle z, -\lambda(m) \rangle \cdot f(z) &= \sum_{m' \in M'} r(m)^{-1} \cdot a_{m'} \cdot \langle z, m' - \lambda(m) \rangle \\
= f(m \cdot z) \qquad &= \sum_{m' \in M'} a_{m'} \cdot \langle m \cdot z, m' \rangle
\end{aligned}$$

for all $m \in M$. Thus, by checking coefficients, it follows that

$$r(m)^{-1} \cdot a_{m'} = \langle m, m' - \lambda(m) \rangle \cdot a_{m' - \lambda(m)} = \langle m, m' \rangle \cdot \langle m, -\lambda(m) \rangle \cdot a_{m' - \lambda(m)},$$

and equivalently

$$a_{m' - \lambda(m)} = \frac{\langle m, \lambda(m) \rangle}{r(m) \cdot \langle m, m' \rangle} \cdot a_{m'}.$$

Since $r(i \cdot m) = r(m)^i \cdot \langle m, \lambda(m)\rangle^{i(i-1)/2}$ for $i \in \mathbb{N}$, iteration by m yields

$$a_{m'-\lambda(i\cdot m)} = \frac{\langle m, \lambda(m)\rangle^{i(i+1)/2}}{r(m)^i \cdot \langle m, m'\rangle^i} \cdot a_{m'}. \tag{$*$}$$

(c) → (b): As seen in Proposition 2.7.5, the map $\lambda : M \to M'$ fulfills the symmetry condition $\langle m_2, \varphi(m_1)\rangle = \langle m_1, \lambda(m_2) = \langle m_2, \lambda(m_1)\rangle$ for all $m_1, m_2 \in M$. Let f be a global section of $\mathcal{L}$ which is not 0. This means that the Fourier expansion of $f = \sum a_{m'}\langle z, m'\rangle$ has a coefficient $a_{m'} \neq 0$. Since λ is injective, we have that $\lambda(m) \neq 0$ for every $m \in M - \{0\}$. By the formula ($*$) we see that all the coefficients $(a_{m'-\lambda(i\cdot m)}; i \in \mathbb{Z})$ are nonzero. Since the Fourier series converges on T, it follows that for all positive $c \in \mathbb{R}$ and $m \in M - \{0\}$ the coefficients satisfy $|a_{m'-\lambda(i\cdot m)}|/c^{|i|} \to 0$ for $|i| \to \infty$. Now formula ($*$) implies that this happens only if $|\langle m, \lambda(m)\rangle| < 1$ for all $m \in M - \{0\}$. Thus, $\lambda : M \to M'$ is a polarization.

(b) → (c): If the inequality $|\langle m, \lambda(m)\rangle| < 1$ is satisfied for all $m \in M$ with $m \neq 0$, we see that $\lambda : M \to M'$ is injective. Moreover, one can construct a non-trivial Fourier series by prescribing coefficients on a set of representatives of $M'/\lambda(M)$ by the formula ($*$). Then these Fourier series give rise to non-trivial global sections of $\mathcal{L}$.

In particular, one obtains the formula $\dim \Gamma(A, \mathcal{L}) = \#(M'/\lambda(M))$.

(a) → (c): If $\mathcal{L}$ is ample, then A is an abelian variety. By [74, p. 60] the map $\varphi_{\mathcal{L}}$ is surjective. Then it follows from Remark 2.7.9 that λ is injective. Moreover, $\mathcal{L}$ admits a non-vanishing global section; cf. [74, Cor. on p. 159].

(c) → (a): Note that the morphism $\varphi_{\mathcal{L}}$ is surjective due to Remark 2.7.9, since λ is injective by assumption. Then the ampleness of $\mathcal{L}$ follows from Lemma 7.1.9 below which is a general basic fact on abeloid varieties. □

Corollary 2.7.13. *Let $A := T/M$ be rigid analytic torus as above. The following conditions are equivalent*:

(a) *A is an abelian variety.*
(b) *There exists a polarization of (T, M).*
(c) *The transcendence degree of the field of meromorphic functions on A is equal to the dimension of A.*

Proof. (a) → (b): If A is an abelian variety, there exists an ample invertible sheaf on A; cf. Proposition 7.1.10. Then the assertion follows from Theorem 2.7.12.

(b) → (a): We may assume that K is algebraically closed. Then this follows from Theorem 2.7.12 as well. Indeed, if the polarization λ is given, one defines an ample invertible sheaf $\mathcal{L}$ on A by constructing an M-linearization (r, λ) on $\mathbb{A}^1_T$; cf. Remark 2.7.11(c). Then the assertion follows from Proposition 7.1.10.

(a) ↔ (c): It follows from Proposition 7.1.8(b) which is a general basic fact on abeloid varieties as well. □

2.8 Jacobian Variety of a Mumford Curve

In this section we will present an analytic construction of the Jacobian variety of a Mumford curve. For its precise definition in the general context see Definition 5.1.2.

Let $\Gamma \subset \mathrm{PGL}(2, K)$ be a Schottky group of rank $g \geq 1$. We denote by

$$p_X : \Omega := \Omega_\Gamma \longrightarrow X := X_\Gamma := \Gamma \backslash \Omega$$

the quotient map from the set of ordinary points of Γ to the associated Mumford curve; cf. Theorem 2.3.1. Note that X has genus g due to Theorem 2.3.1(b). Moreover, we fix a K-rational point $z_0 \in \Omega$.

First we will construct a rigid analytic torus $J = T/M$ which parameterizes Γ-linearizations on the trivial line bundle $\Omega \times \mathbb{A}^1_K$. In a second step we will produce a canonical map $J \to \mathrm{Jac}\, X$ from J to the Jacobian $\mathrm{Jac}\, X$ of the algebraic curve X. Finally, it will turn out that this map is an isomorphism. In the following let

$$H := \Gamma/[\Gamma, \Gamma]$$

be the maximal abelian quotient of Γ. There is a canonical isomorphism

$$H := \Gamma/[\Gamma, \Gamma] \xrightarrow{\sim} H_1(X, \mathbb{Z}) := H_1(S_X, \mathbb{Z}), \quad \overline{\alpha} \longmapsto \overline{c}_\alpha,$$

from Γ to the first homology group; cf. Notation 2.4.16. See Remark 2.4.18 for the identification of $H_1(X, \mathbb{Z}) = H_1(S_X, \mathbb{Z})$. Hereby Γ is viewed as the deck transformation group of $p_X : \Omega \to X$ and of their skeletons $p_S : S_\Omega \to S_X$ which kills the homology group of 1-chains. Now we consider the affine torus

$$T := \mathrm{Spec}\, K[H].$$

Its closed points are group homomorphisms $t : H \to \overline{K}^\times$ from H to the multiplicative group of an algebraic closure $\overline{K}$ of K; cf. Proposition 2.7.1(a). Then the group H is identified with the character group

$$h' : H \xrightarrow{\sim} M' := \mathrm{Hom}(T, \mathbb{G}_{m,K}), \; c \longmapsto \big[c : t \longmapsto t(c)\big],$$

of T in a canonical way; cf. Proposition 2.7.1(b).

There is a second map h of Γ onto a lattice of T. For this, recall Drinfeld's pairing

$$c : \mathcal{D}_0^f \times \Gamma \longrightarrow \mathbb{G}_{m,K}, \; (d, \alpha) \longmapsto c(d)(\alpha) = \prod_{\gamma \in \Gamma} \frac{\theta(d; \gamma z_0)}{\theta(d; \gamma \alpha^{-1} z_0)},$$

where $\mathcal{D}_0^f$ is the set of finite divisors on Ω of degree 0, and where $c(d)$ is the automorphy factor of $\Theta(d; z)$; cf. Proposition 2.5.5. By Proposition 2.5.8 we have the morphism

$$h : H \longrightarrow T, \; \overline{\alpha} \longmapsto \big[c(\alpha z_1 - z_1) : H \to K^\times\big].$$

The map h sends a transformation $\alpha \in \Gamma$ to the automorphy factor $c(\alpha z_1 - z_1)$ of $\Theta(\alpha z_1 - z_1; z)$; this map is independent of the choice of z_1; cf. Proposition 2.5.8. It depends only on the class $\overline{\alpha}$ of α in H. The group homomorphism $c(\alpha z_1 - z_1)$ sends $\beta \in \Gamma$ to the automorphy factor $c(\alpha z_1 - z_1)(\beta) \in K^\times$ evaluated at β. Note that this depends only on the class of β in H. Moreover, we know by Theorem 2.6.2 that the mapping is injective and that its image

$$M := h(H) \subset T$$

is a lattice in T of full rank. Thus, $h : H \to M$ is bijective.

Definition 2.8.1. The isomorphism

$$\lambda := h' \circ h^{-1} : M \xrightarrow{\sim} M'$$

is called the *canonical polarization*.

We next introduce the dual affine torus

$$T' := \operatorname{Spec} K[M].$$

Its closed points t' are the group homomorphisms $t' : M \to \overline{K}^\times$. We have a canonical map

$$M' \longrightarrow T', \quad m' \longmapsto \left[m'\big|_M : M \to K^\times \right],$$

which maps a character m' to the restriction $m'|_M$ onto the lattice M. So we consider M' as a subset of the dual torus T' in a canonical way.

We want to describe the Jacobian in terms of Γ-linearizations on line bundles as proposed by Proposition 1.7.14. This can be done by means of automorphic functions. In the following we consider the rigid analytic torus

$$J := T/M$$

which, as it will turn out in Theorem 2.8.7, is the analytic Jacobian variety of X. By the chosen K-rational point $z_0 \in \Omega$ we have a commutative diagram

$$\begin{array}{ccccccc} \Omega & \xrightarrow{\widehat{j}} & T & \qquad & z & \longmapsto & c(z - z_0) \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_J} & & \downarrow & & \downarrow \\ X & \xrightarrow{j} & J & & \overline{z} & \longmapsto & \overline{c(z - z_0)}. \end{array}$$

The map $\widehat{j}$ sends a point z to the automorphy factor of $\Theta(z - z_0; \_)$. Because of $c(\gamma z_0 - z_0) \in M$ we have the asserted factorization. Let $\mathcal{L} := \mathcal{O}_X(\overline{z} - \overline{z}_0)$ be the invertible sheaf associated to the divisor $\overline{z} - \overline{z}_0$ and let $L = V(\mathcal{L})$ be its associated line bundle on X. Then the invertible sheaf of its sections is $\mathcal{S}(L) = \mathcal{L}^{-1} = \mathcal{O}_X(\overline{z}_0 - \overline{z})$; cf. Remark 1.7.2. Thus, $\Theta(z - z_0; \_)$ is a non-vanishing section of $p_X^* L$. Therefore, the Γ-linearization of $p_X^* L$ is given by $c(z - z_0)$. Thus, we see that $\overline{c(z - z_0)}$ is the isomorphism class of the line bundle $V(\mathcal{O}_X(\overline{z} - \overline{z}_0))$; cf. Remark 1.7.2.

Thus, we obtain the following lemma.

Lemma 2.8.2. *In the above situation we have*:

(a) $c(z - z_0)$ *is the* Γ*-linearization associated to* $V(\mathcal{O}_X(\overline{z} - \overline{z}_0))$.
(b) *If* $z = \beta z_0$ *belongs to the orbit* Γz_0 *of* z_0, *then* $c(z - z_0)(\alpha) = \langle \beta, \alpha \rangle$ *for* $\alpha \in \Gamma$, *and hence* $c(z - z_0)$ *belongs to* M.

Later on, we need further computations for $\widehat{j} : \Omega \to T$.

Lemma 2.8.3. *Let* $z_0, z \in \Omega$ *be* K*-rational points and* $\alpha \in \Gamma$. *Then*

(a) $c(z - z_0)(\alpha) = \Theta(\alpha z_0 - z_0; z)$.
(b.1) $h'(\alpha) \circ \widehat{j}(z) = \Theta(\alpha z_0 - z_0; z)$, *where* $h'(\alpha)$ *is viewed as a character.*
(b.2) $h(\alpha) \cdot \widehat{j}(z) = \widehat{j}(\alpha z)$, *where* $h(\alpha)$ *is viewed as a point of* T.
(c) *If* $\alpha, \beta \in \Gamma$, *then the evaluation of* $h'(\alpha)$ *at* $h(\beta)$ *is given by*

$$h'(\alpha)\big(h(\beta)\big) = \langle \beta, \alpha \rangle,$$

where the right-hand side is the symmetric bilinear form of Proposition 2.5.8.

Proof. (a) At first, we assume $\Gamma z_0 \neq \Gamma z$. Then (a) follows from Proposition 2.5.5 and the projective invariance of the cross-ratio. Indeed, we have that

$$\begin{aligned} c(z - z_0)(\alpha) &= \prod_{\gamma \in \Gamma} \frac{\theta(z - z_0; \gamma z_0)}{\theta(z - z_0; \gamma \alpha^{-1} z_0)} \\ &= \prod_{\gamma \in \Gamma} \frac{\gamma z_0 - z}{\gamma z_0 - z_0} \frac{\gamma \alpha^{-1} z_0 - z_0}{\gamma \alpha^{-1} z_0 - z} \\ &= \prod_{\gamma \in \Gamma} \frac{\gamma z - z_0}{\gamma z - \alpha^{-1} z_0} \frac{\gamma z_0 - \alpha^{-1} z_0}{\gamma z_0 - z_0} \\ &= \prod_{\gamma \in \Gamma} \frac{\theta(z_0 - \alpha^{-1} z_0; \gamma z)}{\theta(z_0 - \alpha^{-1} z_0; \gamma z_0)} \\ &= \prod_{\gamma \in \Gamma} \frac{\theta(\alpha z_0 - z_0; \gamma z)}{\theta(\alpha z_0 - z_0; \gamma z_0)} \\ &= \Theta(\alpha z_0 - z_0; z). \end{aligned}$$

If $z = \beta z_0$, then $c(\beta z_0 - z_0)(\alpha) = \langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = c(\alpha z_0 - z_0)(\beta)$ by Proposition 2.5.8. The latter coincides with $\Theta(\alpha z_0 - z_0; \beta z_0)$, because $c(\alpha z_0 - z_0)$ is the automorphy factor of $\Theta(\alpha z_0 - z_0; \_)$ and $\Theta(\alpha z_0 - z_0; z_0) = 1$.

(b.1) $h'(\alpha) \circ \widehat{j}(z)$ is the evaluation of the homomorphism $\widehat{j}(z) : H \to \mathbb{G}_{m,K}$ at α, which is here considered as an element of H. Then it follows from (a) that

$$h'(\alpha) \circ \widehat{j}(z) = c(z - z_0)(\alpha) = \Theta(\alpha z_0 - z_0; z).$$

(b.2) $h(\alpha) \cdot \widehat{j}(z)$ is the automorphy factor of $\Theta(\alpha z_0 - z_0; \_) \cdot \Theta(z - z_0, \_)$. Since $c(\alpha z_0 - z_0)(\beta)$ is independent of z_0 due to Proposition 2.5.8, it follows that

$$c(\alpha z_0 - z_0)(\beta) \cdot c(z - z_0)(\beta) = c(\alpha z - z)(\beta) \cdot c(z - z_0)(\beta)$$

for every $\beta \in \Gamma$. The right-hand side is the automorphy factor of the product

$$\Theta(\alpha z - z; \_) \cdot \Theta(z - z_0, \_) = \Theta(\alpha z - z + z - z_0; \_) = \Theta(\alpha z - z_0; \_),$$

which equals $c(\alpha z - z_0)$. Thus, we see $h(\alpha) \cdot \widehat{j}(z) = \widehat{j}(\alpha z)$.

(c) We know that $h'(\alpha)(h(\beta))$ is the evaluation of the automorphy factor of $h'(\alpha)$ at $h(\beta)$. This coincides with the evaluation of the automorphy factor of $\Theta(\alpha z_0 - z_0; \_)$ at β due to (b). Therefore,

$$h'(\alpha)\big(h(\beta)\big) = \frac{\Theta(\alpha z_0 - z_0, \beta z)}{\Theta(\alpha z_0 - z_0, z)} = \langle \beta, \alpha \rangle,$$

cf. Remark 2.5.7 and Proposition 2.5.8. □

The assertion (c) of Lemma 2.8.3 tells us that Drinfeld's polarization is the canonical one; namely the evaluation of characters at the points of the lattice.

Theorem 2.8.4. *Let Γ be Schottky group of rank $g \geq 1$ and H its maximal abelian quotient. Let Ω be the set of ordinary points of Γ and $p_X : \Omega \to X$ the rigid analytic quotient by Γ. Then H can canonically be identified with $H_1(X, \mathbb{Z})$; cf. Remark 2.4.18. Let $z_0 \in \Omega$ be a K-rational point. Then with the above notations we have*:

(a) *$J := T/M$ is an abelian variety.*
(b) *The map $\lambda := h' \circ h^{-1} : M \to M'$ is a principal polarization and induces an isomorphism $\varphi : J \xrightarrow{\sim} J'$ from J to its dual $J' = T'/M'$.*
(c) *The canonical map $j : X \to J, z \mapsto [\overline{c(z - z_0)}]$, admits the lifting $\widehat{j}$, which maps a point $z \in \Omega$ to the automorphy factor $c(z - z_0)$ of the function $\Theta(z - z_0; \_)$. Thus, we have the following commutative diagram*

$$\begin{array}{ccccccc} \Omega & \xrightarrow{\ \widehat{j}\ } & T & \qquad & z & \longmapsto & c(z - z_0) \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_J} & & \downarrow & & \downarrow \\ X & \xrightarrow{\ j\ } & J & & \overline{z} & \longmapsto & [\overline{c(z - z_0)}]. \end{array}$$

Proof. (a) The claim follows from (b) and Corollary 2.7.13.

(b) Viewing M and M' as the character groups of T' and of T, respectively, the isomorphism λ gives rise to an isomorphism $\widehat{\varphi} : T \to T'$. Thus, it remains to verify that $\widehat{\varphi}(M) \subset M'$. This follows from the symmetry

$$\begin{aligned}\big\langle h(\alpha), \lambda\big(h(\beta)\big)\big\rangle &= \big\langle h(\alpha), h'(\beta)\big\rangle = \langle \alpha, \beta\rangle \\ &= \langle \beta, \alpha\rangle = \big\langle h(\beta), h'(\alpha)\big\rangle = \big\langle h(\beta), \lambda\big(h(\alpha)\big)\big\rangle,\end{aligned}$$

by Proposition 2.5.8 and Lemma 2.8.3(c). Thus, $\widehat{\varphi}$ induces a morphism $\varphi : J \to J'$. The bilinear form

$$M \times M \longrightarrow K^\times, \quad (m_1, m_2) \longmapsto \lambda(m_1)(m_2),$$

coincides with the symmetric bilinear form studied in Proposition 2.5.8 and is positive definite due to Theorem 2.6.2.

(c) Due to Lemma 2.8.3(a) the maps $\widehat{j}$ and j are holomorphic. □

Next we want to compare J with the Jacobian $\operatorname{Jac}(X)$ of the algebraic curve X^{alg}. We remind the reader that our rigid analytic variety J classifies a certain class of Γ-linearizations on the trivial line bundles on Ω, but this equivalence relation is not the usual one.

To be precise, we identified two Γ-linearizations c_1, c_2 if and only if $c_1 \otimes c_2^{-1}$ is induced by $u(\alpha z)/u(z)$, where u is an invertible holomorphic function on Ω of the special type $u(z) = \Theta(d; z)$, where $\sum_{\gamma\in\Gamma} \gamma d = 0$; cf. Proposition 2.5.5(b). On the other hand, to classify isomorphism classes of line bundles on X, we have to allow any invertible holomorphic function u on Ω.

In the following we will see that our equivalence relation is the correct one. In other words, we will show that all rational functions on X are of type $\Theta(d; z)$ which are Γ-invariant.

In order to clarify this point, let us have a closer look at the K'-valued points of J, where K'/K is a finite field extension. Consider the group functor $\mathcal{D}$ which associates to K' the group

$$\mathcal{D}\big(K'\big) := \big\{d; \text{finite } K'\text{-divisor on } \Omega\big\}.$$

We will also consider its subgroup functors

$$\begin{aligned}\mathcal{D}^0\big(K'\big) &:= \big\{d; \text{finite } K'\text{-divisor of degree 0 on } \Omega\big\},\\ \mathcal{D}^{00}\big(K'\big) &:= \big\{d \in \mathcal{D}^0\big(K'\big); c(d) \in M\big\};\end{aligned}$$

cf. Proposition 2.5.5. Then we have the quotient

$$\mathcal{D}^0\big(K'\big)/\mathcal{D}^{00}\big(K'\big) = J\big(K'\big),$$

which can be identified K'-valued points of J.

Each line bundle L on X admits a global meromorphic section by Lemma 1.7.5. Thus, the sheaf of sections of L is isomorphic to $\mathcal{O}_X(d)$ for some divisor d of X. Therefore, the classification of isomorphism classes of line bundles on X is equivalent to the classification of isomorphism classes of divisors up to linear equivalence. It follows from Proposition 2.5.5(a) that for each line bundle L on X of degree 0 its pull-back p^*L admits a trivializing section $\Theta(d; z)$ and a Γ-linearization on p^*L which is given by an automorphy factor $c(d)$. Now two line bundles L_1 and L_2 of degree 0 on X with induced Γ-linearizations c_1 and c_2, respectively, are isomorphic if and only if there exists an invertible holomorphic function u on Ω such that

$$\frac{c_1(\alpha)}{c_2(\alpha)} = \frac{u(\alpha z)}{u(z)} \quad \text{for all } z \in \Omega.$$

The equivalence modulo $\mathcal{D}^{00}$ is a stronger condition, because it requires that there exists a finite divisor d_0 with $\sum_{\gamma \in \Gamma} \gamma d_0 = 0$ such that

$$\frac{c_1(\alpha)}{c_2(\alpha)} = \frac{\Theta(d_0; \alpha z)}{\Theta(d_0; z)} = c(d_0)(\alpha).$$

Thus, we obtain an isomorphism of functors

$$j : \mathcal{D}^0/\mathcal{D}^{00} \xrightarrow{\sim} J, \; d \bmod \mathcal{D}^{00} \longmapsto c(d) \bmod M,$$

on the category of finite field extensions of K.

Furthermore, we obtain a canonical surjective map

$$\varrho : J \longrightarrow \mathrm{Jac}(X), \; c(d) \bmod M \longmapsto \left[\mathbb{A}^1_\Omega / c(d)\right],$$

by sending the class of $c(d)$ to the isomorphism class of the line bundle $\mathbb{A}^1_\Omega / c(d)$. Indeed, on the trivial line bundle $(T \times \Omega) \times \mathbb{A}^1_K$ we have the Γ-linearization, which sends a point (t, z, ℓ) of $(T \times \Omega) \times \mathbb{A}^1_K$ to the point $(t, \gamma z, t(\gamma) \cdot \ell)$ for $\gamma \in \Gamma$. Thus, we obtain a line bundle on $T \times X$. By GAGA in Theorem 1.6.11 this line bundle is algebraic over every affinoid subdomain of T, and hence it induces a canonical morphism $T \to \mathrm{Jac}(X)$ by the universal property of $\mathrm{Jac}(X)$; cf. Theorem 5.1.1. Obviously, this map factors through $\varrho : J \to \mathrm{Jac}(X)$ as a holomorphic map, because the points of lattice M are sent to the unit element.

Lemma 2.8.5. *The canonical map* $\varrho : J \to \mathrm{Jac}(X)$ *is an isomorphism.*

Proof. Let $g = g(X)$ be the genus of X. Then both J and $\mathrm{Jac}(X)$ have dimension g. The map ϱ is surjective, because every line bundle on X of degree 0 is induced by a Γ-linearization $c(d)$ on the trivial line bundle with $d \in \mathcal{D}^0(K')$ for some finite field extension K' of K. Since J and $\mathrm{Jac}(X)$ are group varieties of the same dimension and the map ϱ is a surjective group homomorphism, the fibers of ϱ have dimension 0. Now consider the induced canonical maps

$$\begin{array}{ccccc} \Omega^{(g)} & \xrightarrow{\widehat{j}^{(g)}} & J & \xrightarrow{\varrho} & \mathrm{Jac}(X) \\ d = z_1 + \cdots + z_g & \longmapsto & c(d - g z_0) \bmod M & \longmapsto & \sum_{i=1}^{g} \big[p_X(z_i) - p_X(z_0) \big] \end{array}$$

from the g-fold symmetric product $\Omega^{(g)}$ to J. The map $\widehat{j}^{(g)}$ factors through $j^{(g)} : X^{(g)} \to J$, since $c(\gamma z - z)$ belongs to M for all $z \in \Omega$ and all $\gamma \in \Gamma$. The fibers of $\varrho \circ j^{(g)}$ are isomorphic to the linear system $\mathbb{P}(H^0(X, \mathcal{O}_X(D)))$ over the isomorphism class $[D - g \cdot p_X(z_0)]$. In particular, they are connected; cf. [68, §5]. Therefore, the fibers of ϱ are connected. Since the fibers have dimension 0, they consist of exactly one reduced point. Thus, we see that ϱ is an isomorphism. □

Lemma 2.8.5 implies the following result.

Corollary 2.8.6. *In the above situation we have*:

(a) *Any rational function on X is* (*up to a multiplicative scalar*) *induced by a Γ-invariant Weierstraß product $\Theta(d; z)$, which is associated to a finite divisor of degree* 0 *on Ω.*
(b) *Every automorphic function on Ω without zeros and poles is induced by a Weierstraß product $\Theta(d; z)$ for a finite divisor d with $\sum_{\gamma \in \Gamma} \gamma d = 0$.*

Proof. (a) If m is a rational function on X with divisor $d := \mathrm{div}(m)$, then $p_X^* m$ has the divisor $p^* d$ and $p^* d$ can be solved by an automorphic function $\Theta(d; z)$. Since d is a principal divisor, the automorphy factor $c(d)$ of $\Theta(d; z)$ belongs to M, because the morphism $\varrho : J \to \mathrm{Jac}(X)$ is an isomorphism by Lemma 2.8.5. Thus there exists a transformation $\gamma \in \Gamma$ such that $c(d)$ is the automorphic function of $\Theta(\gamma z_0 - z_0; \_)$. This automorphy form is an invertible holomorphic function on Ω. Thus, we see that $\Theta(d; z)/\Theta(\gamma z_0 - z_0; z)$ is a meromorphic function on Ω, which is Γ-invariant, and hence it gives rise to a meromorphic function f on X. Since f has the same divisor as m, the quotient m/f is an invertible holomorphic function on X, and hence constant. This settles our assertion.

(b) An automorphic function u on Ω without zeros and poles gives rise to a Γ-linearization of the trivial line bundle on Ω. By Lemma 2.8.5 its automorphy factor is the automorphy factor of some $\Theta(\gamma z_0 - z_0; \_)$. Then $u(z)/\Theta(\gamma z_0 - z_0; z)$ is Γ-invariant. Thus, the quotient is induced by an invertible holomorphic function on X and hence constant. □

Finally, let us sum up the results of this section.

Theorem 2.8.7. *In the situation of Theorem* 2.8.4 *we have*:

(a) *The Jacobian variety $J = \mathrm{Jac}(X)$ of X has a Raynaud representation*

$$\mathrm{Jac}(X) = T/M$$

as rigid analytic torus with $T = \operatorname{Spec} K[H]$ *and* $M = h(H)$*, where* $h : H \to T$ *maps* $\overline{\alpha} \in H$ *to the automorphy factor of* $\Theta(\alpha z_0 - z_0; \_)$.

(b) *The universal line bundle on* $X \times J$ *is given by the* $(\Gamma \times M)$*-action*

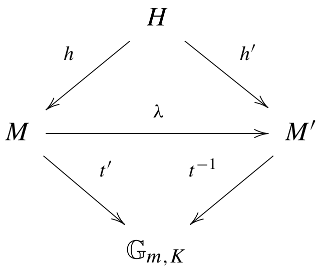
$$C(z,t)(\alpha, m) \cdot \ell = \langle \widehat{j}(z), -\lambda(m) \rangle \cdot \langle t, -h'(\alpha) \rangle \cdot \langle h(\alpha), -\lambda(m) \rangle \cdot \ell$$

on the trivial line bundle $(\Omega \times T) \times \mathbb{A}^1_K$*. Here* $(\alpha, m) \in \Gamma \times M$ *is a pair and* $((z,t), \ell) \in (\Omega \times T) \times \mathbb{A}^1_K$ *a point on* $(\Omega \times T) \times \mathbb{A}^1_K$*. On the right-hand side we have that* $\lambda := h' \circ h^{-1} : M \to M'$ *as in Definition 2.8.1.*

(c) *Let* $P_{J \times J'}$ *be the Poincaré bundle on* $J \times J'$*. The lifting* $\widehat{\varphi}' : T' \to T$ *of the autoduality map* $\varphi' : J' \to J, x' \mapsto [j^* P_{J \times x'}]$*, is induced by a group homomorphism* $\lambda' : M' \to M$*.*

Moreover, λ' *equals* $-\lambda^{-1} = -h \circ h'^{-1}$*; cf. Corollary 2.9.16.*

For the notion of the Poincaré bundle see Theorem 5.1.4 and for the notion of the universal bundle on $X \times J$ see Theorem 5.1.1. The relationship $\widehat{\varphi}'(t) = t'$ concerning the autoduality can be expressed by the commutative diagram



Proof of Theorem 2.8.7. (a) follows from Theorem 2.8.4 by Lemma 2.8.5.

(b) The Poincaré bundle $P_{J \times J'}$ on $J \times J'$ is given by the $(M \times M')$-linearization

$$C(t, t')(m, m') = m'(t) \cdot m(t') \cdot \langle m, m' \rangle$$

on the trivial line bundle over $T \times T'$; cf. Theorem 2.7.7. The pull-back of the Poincaré bundle $P_{J \times J'}$ under the mapping

$$(j, \mathrm{id}_{J'}) : X \times J' \longrightarrow J \times J'$$

induces the autoduality map $\varphi' : J' \to J$; cf. Theorem 5.1.6 below. Then the pull-back of the linearization is given by

$$\begin{aligned}(\widehat{j}, \mathrm{id}_{T'})^* C(z, t')(\alpha, m') &= m'(\widehat{j}(z)) \cdot h(\alpha)(t') \cdot \langle h(\alpha), m' \rangle \\ &= \langle \widehat{j}(z), m' \rangle \cdot \langle h(\alpha), t' \rangle \cdot \langle h(\alpha), m' \rangle.\end{aligned}$$

This presents a line bundle on $X \times J'$. It remains to identify the image $t \in T$ of the point $t' \in T'$.

Of course, the image t equals $t := \widehat{\varphi}'(t')$, where $\widehat{\varphi}' : T' \to T$ is the lifting of the autoduality map φ'. This map corresponds to a group homomorphism $\lambda' : M' \to M$ of the character groups by Proposition 2.7.1(b). Then we see by Remark 2.7.11 that $\widehat{\varphi}'$ sends $t' \in T'$, which is a group homomorphism $t' : M \to \mathbb{G}_{m,K}$, to the point $t := t' \circ \lambda' : M' \to \mathbb{G}_{m,K}$. We will see in the proof of (c) below that $\lambda' = -\lambda^{-1}$. Thus, we obtain $\langle h(\alpha), t' \rangle = \langle t, -h'(\alpha) \rangle$. Likewise we see that the element $m' \in M'$ corresponds to the element $-\lambda(m)$.

(c) It is a general fact on Jacobians in Theorem 5.1.6(d) that the autoduality map φ' is the inverse of the morphism

$$-\varphi_\Theta = \varphi_{-\Theta} : J \longrightarrow J', \quad a \longmapsto \tau_a^* \mathcal{O}_J(-\Theta) \otimes \mathcal{O}_J(\Theta) = \mathcal{O}_J(\Theta - \tau_{-a}\Theta),$$

where Θ is the usual theta divisor; i.e., the image $\Theta := j^{(g-1)}(X^{(g-1)})$. The morphism φ_Θ corresponds to a homomorphism $\lambda_\Theta : M \to M'$. Thus, we see that λ' equals $-\lambda_\Theta^{-1}$. Therefore, it remains to show that $\lambda = \lambda_\Theta$. This will follow from Riemann's Vanishing Theorem 2.9.13 below. □

Theorem 2.8.8. *In the situation of Theorem 2.8.4, the vector space $\Gamma(X, \Omega^1_{X/K})$ of global differential forms on X is generated by the differentials*

$$\operatorname{dlog}\big(\Theta(\alpha_i z_0 - z_0; z)\big) \quad \textit{for } i = 1, \ldots, g,$$

where $\alpha_1, \ldots, \alpha_g$ is a basis of Γ.

Proof. The logarithmic differentials $\operatorname{dlog}\chi$ of the characters χ of T are M-invariant and generate $\Gamma(J, \Omega^1_{J/K})$. Indeed, if $(\xi_1, \ldots, \xi_g)$ is a system of coordinate functions on T, then $(d\xi_1/\xi_1, \ldots, d\xi_g/\xi_g)$ is a basis of $\Gamma(J, \Omega^1_{J/K})$. The pull-back under the canonical map

$$j^* : \Gamma\big(J, \Omega^1_{J/K}\big) \xrightarrow{\sim} \Gamma\big(X, \Omega^1_{X/K}\big)$$

is known to be bijective; cf. [68, 9.5]. Since all the characters are of the form $h'(\alpha)(z)$ for $\alpha \in \Gamma$, it follows Lemma 2.8.3 that

$$j^* \operatorname{dlog}\big(h'(\alpha)\big) = \operatorname{dlog}\big(\Theta(\alpha z_0 - z_0; \_)\big) \quad \text{for } \alpha \in \Gamma.$$

Since these differential forms depend additively on α, a basis of $\Gamma(X, \Omega^1_{X/K})$ is given by the pull-backs of a basis $\alpha_1, \ldots, \alpha_g$ of Γ; cf. Corollary 2.5.14. □

2.9 Riemann's Vanishing Theorem

We continue with the notations of Sect. 2.8. Assume, in addition, that K is algebraically closed. In Proposition 2.7.1 we introduced the canonical bilinear form

$$\langle \_, \_ \rangle : T \times M' \longrightarrow \mathbb{G}_{m,K}, \ \big(t, m'\big) \longmapsto m'(t) = t\big(m'\big),$$

by evaluating characters $m' \in M'$ at points $t \in T$ or, equivalently, by evaluating the group homomorphism $t : M' \to \mathbb{G}_{m,K}$ at $m' \in M'$. Furthermore, we will use the isomorphism

$$\lambda := h' \circ h^{-1} : M \xrightarrow{\sim} M',$$

which was defined in Definition 2.8.1. Writing $m_1 = h(\alpha)$ and $m_2 = h(\beta)$ for elements $\alpha, \beta \in H$, we have the symmetric bilinear form

$$M \times M \longrightarrow K^\times, \quad (m_1, m_2) \longmapsto \big\langle m_1, \lambda(m_2)\big\rangle = \langle \alpha, \beta \rangle,$$

which positive definite; cf. Theorem 2.8.4.

Lemma 2.9.1. *There is a symmetric bilinear form $b : M \times M \to K^\times$ satisfying $b(m_1, m_2)^2 = \langle m_1, \lambda(m_2)\rangle$ for all $m_1, m_2 \in M$.*

Proof. Let $(e_1, \ldots, e_r)$ be a basis of M and consider a representation of the symmetric quadratic form $\langle \_, \lambda(\_)\rangle$ by the matrix $(\ell_{i,j})$ with entries $\ell_{i,j} := \langle e_i, \lambda(e_j)\rangle$. If $m, n \in \mathbb{Z}^r$, then the bilinear form is given by

$$\Big\langle \sum_i m_i e_i, \lambda\Big(\sum_j n_j e_j\Big)\Big\rangle = \prod_{i,j=1}^r \ell_{i,j}^{m_i \cdot n_j}.$$

Next take a square root $b_{i,j} \in K^\times$ of $\ell_{i,j}$ for $i \le j$ and put $b_{i,j} := b_{j,i}$ for $i > j$. Define the quadratic form b via the symmetric matrix $(b_{i,j})$. Then we obtain

$$b\Big(\sum_i m_i e_i, \sum_j n_j e_j\Big)^2 = \prod_{i,j=1}^r b_{i,j}^{2 \cdot m_i \cdot n_j} = \prod_{i,j=1}^r \ell_{i,j}^{m_i \cdot n_j}.$$

This proves the assertion of the lemma. □

For the remainder of this section we fix a symmetric bilinear form b as in Lemma 2.9.1 and introduce the theta function.

Definition 2.9.2. The formal Laurent series

$$\vartheta(t) := \sum_{m \in M} b(m, m) \cdot \big\langle t, \lambda(m)\big\rangle \quad \text{for } t \in T$$

is called the *theta function* attached to the Mumford curve X.

Lemma 2.9.3. *The theta function is a holomorphic function*

$$\vartheta : T \longrightarrow \mathbb{A}^1_K$$

on T. It does not vanish identically and satisfies the functional equation

$$\vartheta(t) = b(m, m) \cdot \big\langle t, \lambda(m)\big\rangle \cdot \vartheta(m \cdot t)$$

for all $m \in M$ and $t \in T$.

Proof. The series converges on the whole of T, because by Theorem 2.6.2 we have $|b(m,m)| \le c^{2\cdot|m|}$ for some constant $c < 1$, where $|m|$ is a suitable norm on the additive group M.

The functional equation can be verified in the following way. Let $m \in M$. If we enumerate the terms of the sum by an index $n \in M$ on both sides, we have the following formulas. Since $m + M = M$, the left-hand side yields

$$\begin{aligned}\vartheta(t) &= \sum_{n\in\mathbb{Z}^g} b(m+n, m+n)\cdot\big\langle t, \lambda(m+n)\big\rangle \\ &= \sum_{n\in\mathbb{Z}^g} b(m,m)\cdot b(m,n)^2\cdot b(n,n)\cdot\big\langle t,\lambda(m+n)\big\rangle.\end{aligned}$$

For the right-hand side we have that

$$\begin{aligned}&b(m,m)\cdot\big\langle t,\lambda(m)\big\rangle\cdot\vartheta(m\cdot t)\\ &\quad= \sum_{n\in\mathbb{Z}^g} b(m,m)\cdot\big\langle t,\lambda(m)\big\rangle\cdot b(n,n)\cdot\big\langle mt,\lambda(n)\big\rangle\\ &\quad= \sum_{n\in\mathbb{Z}^g} b(m,m)\cdot b(n,n)\cdot\big\langle m,\lambda(n)\big\rangle\cdot\big\langle t,\lambda(m+n)\big\rangle.\end{aligned}$$

Since $b(m,n)^2 = \langle n,\lambda(m)\rangle = \langle m,\lambda(n)\rangle$, both sides of the formula coincide. □

Definition 2.9.4. The divisor

$$\widehat{\Theta} := \operatorname{div}\vartheta$$

is an effective divisor on T which is invariant under the action of the lattice M by left translations on T, as follows from Lemma 2.9.3. It is called the *theta divisor* associated to the canonical polarization λ. For $s \in T$ let

$$\widehat{\Theta}_s := \tau_s\widehat{\Theta}$$

be the translate of the divisor $\widehat{\Theta}$ by s, which is the divisor of the function

$$\vartheta_s : T \longrightarrow \mathbb{A}^1_K,\ t \longmapsto \vartheta\big(s^{-1}\cdot t\big).$$

Lemma 2.9.5. *Consider the holomorphic function*

$$\zeta : \Omega \longrightarrow \mathbb{A}^1_K,\ z \longmapsto \zeta(z) := \vartheta\big(\widehat{j}(z)\big),$$

where $\widehat{j} : \Omega \to T$ *is as defined in Lemma* 2.8.3(b). *For* $\alpha \in \Gamma$ *put* $m = h(\alpha) \in M$. *Then we have*

$$\zeta(\alpha z) = \big[b(m,m)\cdot\Theta(\alpha z_0 - z_0; z)\big]^{-1}\cdot\zeta(z).$$

In particular, a point $z \in \Omega$ *is a zero of* ζ *of order* k *if and only if* αz *is a zero of* ζ *of order* k *for all* $\alpha \in \Gamma$.

Proof. One knows from Lemma 2.8.3(b.2) that $\widehat{j}(\alpha z) = m \cdot \widehat{j}(z)$. Thus, using Lemma 2.9.3 we obtain

$$\begin{aligned}\zeta(\alpha z) &= \vartheta\big(\widehat{j}(\alpha z)\big) = \vartheta\big(m \cdot \widehat{j}(z)\big) \\ &= \big[b(m,m) \cdot \big\langle \widehat{j}(z), \lambda(m)\big\rangle\big]^{-1} \cdot \vartheta\big(\widehat{j}(z)\big) \\ &= \big[b(m,m) \cdot \Theta(\alpha z_0 - z_0; z)\big]^{-1} \cdot \zeta(z).\end{aligned}$$

Indeed, by Lemma 2.8.3(b.1) we know that $\langle \widehat{j}(z), \lambda(m)\rangle = \Theta(\alpha z_0 - z_0; z)$. □

With the same calculation as in the proof of Lemma 2.9.5 one shows

Lemma 2.9.6. *Fix $s \in T$ and consider the function*

$$\zeta_s(z) := \vartheta\big(s^{-1} \cdot \widehat{j}(z)\big)$$

on Ω. For $\alpha \in \Gamma$ put $m = h(\alpha) \in M$. Then ζ_s satisfies the rule

$$\zeta_s(\alpha z) = \big[b(m,m) \cdot \big\langle s^{-1}, \lambda(m)\big\rangle \cdot \Theta(\alpha z_0 - z_0; z)\big]^{-1} \cdot \zeta_s(z).$$

Thus, for each $s \in T$ the function ζ_s gives rise to a well defined effective divisor $\operatorname{div} \zeta_s$ on X unless ζ_s vanishes identically on Ω.

Proposition 2.9.7. *Let $s \in T$ with $\widehat{j}(\Omega) \not\subset \widehat{\Theta}_s$. Now consider the function*

$$\psi_s(z) := \frac{\zeta_s(z)}{\zeta(z)}$$

on Ω. Then $\psi_s(z)$ is an automorphic function with automorphy factor

$$s : \Gamma \longrightarrow K^\times,\ \alpha \longmapsto \big\langle s, \lambda\big(h(\alpha)\big)\big\rangle.$$

Moreover, let $\kappa \in T$ with $\widehat{j}(\Omega) \not\subset \widehat{\Theta}_\kappa$ and $\widehat{j}(\Omega) \not\subset \widehat{\Theta}_{\kappa \cdot s}$. Put

$$\psi_{s,\kappa}(z) := \frac{\zeta_{s \cdot \kappa}(z)}{\zeta_\kappa(z)}.$$

Then $\psi_{s,\kappa}$ and ψ_s have the same automorphy factor which is $\langle s, \lambda(\_)\rangle$.

Proof. Note that $\zeta(\Omega) \not\subset \widehat{\Theta}$ because $\widehat{j}(z_0) \notin \widehat{\Theta}$. In fact, the absolute value of $\zeta(z_0)$ is $|\zeta(z_0)| = 1$ because of the ultrametric inequality, as $b(0,0) = 1$ and $|b(m,m)| < 1$ for $m \neq 0$. □

Remark 2.9.8. Consider the map

$$\widehat{\varphi} : T \longrightarrow T',\ s \longmapsto s' := c(\psi_s) = s \circ \lambda$$

which associates to s the automorphy factor $c(\psi_s)$. The latter is given by

$$s' : M \longrightarrow \mathbb{G}_{m,K}, \ \alpha \longmapsto \langle s, \lambda(m) \rangle = s \circ \lambda(m).$$

Here we regard $c(\psi_s)$ as an M-linearization of the trivial line bundle $T \times \mathbb{A}^1_K$ on T. Let $L(c(\psi_s))$ be the associated line bundle on J; its sheaf of sections is $\mathcal{S}(L(c(\psi_s))) = \mathcal{O}_J(\Theta - \tau_s\Theta)$, where Θ denotes the image $p_J(\widehat{\Theta})$. Thus, the associated invertible sheaf $\mathcal{L}(c(\psi_s))$ is given by

$$\mathcal{L}\big(c(\psi_s)\big) = \mathcal{O}_J(\tau_s\Theta - \Theta);$$

cf. Proposition 1.7.14. Since $\psi_{s,\kappa}$ and ψ_s have the same automorphy factor, we have for every $\kappa \in T$ that the divisor classes

$$[\tau_s\widehat{\Theta} - \widehat{\Theta}] = [\tau_s\widehat{\Theta}_\kappa - \widehat{\Theta}_\kappa] = \tau_\kappa[\tau_s\widehat{\Theta} - \widehat{\Theta}],$$

coincide. This reflects the fact that the line bundle associated to $[\tau_s\widehat{\Theta} - \widehat{\Theta}]$ is translation invariant.

Lemma 2.9.9. *Let $s \in T$ with $\widehat{j}(\Omega) \not\subset \widehat{\Theta}_s$. Then*

$$\operatorname{div} \zeta_s := \Gamma z_1(s) + \cdots + \Gamma z_g(s)$$

is an effective divisor of degree g.

Proof. See [35, p. 199]. This computation is similar to that of the proof of Theorem 2.3.1. Let $\alpha_1, \ldots, \alpha_g$ be a separating basis of Γ; put $\alpha_{g+i} = \alpha_i^{-1}$. Consider the fundamental domain

$$E := \mathbb{P}^1_K - \bigcup_{i=1}^{2g} W_i^-,$$

where $W_i^{\pm} := W_{\alpha_i}^{\pm}$. By varying the separating morphism $\rho : \Gamma \to \overline{K}^\times$, we may assume that ζ_s has no zeros on $A_i := W_i^+ - W_i^-$, for $i = 1, \ldots, 2g$.

In order to compute the number of zeros of ζ_s we have to compute the order of ζ_s on the annuli A_i with respect to the coordinate functions $z - a_i$, where $a_i \in W_i^-$. From the transformation formula in Lemma 2.9.6 we obtain that

$$\zeta_s(\alpha z) = \big[b(m,m) \cdot \langle s^{-1}, \lambda(m) \rangle \cdot \Theta(\alpha z_0 - z_0; z)\big]^{-1} \cdot \zeta_s(z),$$

which implies that

$$\operatorname{ord}_{A_i} \zeta_s + \operatorname{ord}_{A_{i+g}} \zeta_s = \operatorname{ord}_{A_i} \Theta(\alpha_i z_0 - z_0; z)$$

because of $\alpha_i(\mathbb{P}^1_K - W_i^-) = W_{i+g}^+$. We have

$$\Theta(\alpha_i z_0 - z_0; z) = \frac{z - \alpha_i z_0}{z - \alpha_i^{-1} z_0} \left[\frac{z_0 - \alpha_i^{-1} z_0}{z_0 - \alpha_i z_0} \prod_{\gamma \in \Gamma, \gamma \neq 1, \alpha_i} \frac{\gamma z - \alpha_i z_0}{\gamma z - z_0} \frac{\gamma z_0 - z_0}{\gamma z_0 - \alpha_i z_0} \right].$$

On $(W^+_{\alpha_i} - W^-_{\alpha_i})$ the factor in the brackets has order 0 on A_i and the other factor has order 1. Note that z_0 is chosen in E in such a way that it does not lie on any A_i. In Proposition 2.4.10 we provided a formula for the degree of the divisor of a meromorphic function on a subdomain of the projective line. This formula yields that

$$\deg \operatorname{div} \zeta_s = \sum_{i=1}^{2g} \operatorname{ord}_{A_i} \zeta_s = \sum_{i=1}^{g} \operatorname{ord}_{A_i} \Theta(\alpha z_0 - z_0; z) = g.$$

The sign in the formula has changed, because we use here a coordinate function which is oriented in the opposite direction; cf. Proposition 2.4.8. Thus, the number of zeros is equal to g. □

Definition 2.9.10. A point $t \in T$ and $a \in J$, respectively, is said to be *in general position* if the fiber of the map $\widehat{j}^{(g)} : \Omega^{(g)} \to T$ and $j^{(g)} : X^{(g)} \to J$, respectively, above t and a, respectively, has dimension 0.

Obviously, $t \in T$ is in general position if and only if $p_J(t) \in J$ is so. If $p_J(t) = [D - g \cdot z_0]$ represents the isomorphism class of the divisor $[D - g \cdot z_0]$, where D is effective of degree g, then $p_J(t)$ is in general position if and only if $\dim H^0(X, \mathcal{O}_X(D)) = 1$. Indeed, the fiber above t is given by the linear system $\mathbb{P}(H^0(X, \mathcal{O}_X(D)))$, as was mentioned in the proof of Lemma 2.8.5.

It is well known from the theory of algebraic curves that the set of points in J which are in general position is open and dense in J. An even stronger assertion is valid; cf. [15, 9.3/4].

Lemma 2.9.11. *Let $W_{g-1} \subset \operatorname{Jac}(X)$ be the image of the canonical morphism $j^{(g-1)} : X^{(g-1)} \to \operatorname{Jac}(X)$. The subset $U \subset \operatorname{Jac}(X)$ of all points $\delta \in \operatorname{Jac}(X)$ which are in general position is open and dense in $\operatorname{Jac}(X)$ and intersects W_{g-1} in a dense set.*

Lemma 2.9.12. *Let $\Gamma z_1, \dots, \Gamma z_g \subset \Omega$ be the zeros of ζ and put*

$$\widehat{\kappa} := \widehat{j}(z_1) \cdot \ldots \cdot \widehat{j}(z_g) \in T.$$

If $\widehat{j}(\Omega) \not\subset \widehat{\Theta}_s$, we have that

$$\widehat{j}\big(z_1(s)\big) \cdot \ldots \cdot \widehat{j}\big(z_g(s)\big) \cdot \widehat{\kappa}^{-1} \equiv s \bmod M,$$

where $\Gamma z_1(s), \dots, \Gamma z_g(s)$ are the zeros of ζ_s for $s \in T$.

Proof. On the one hand, it follows from Proposition 2.9.7 that the automorphy factor of ζ_s/ζ is s. On the other hand, by Lemma 2.9.9 the functions ζ_s and ζ, respectively, have exactly g zeros $z_1(s), \dots, z_g(s)$ and $z_1, \dots, z_g$, respectively. From Corollary 2.8.6 we obtain that

$$\zeta_s(z)/\zeta(z) = \Theta\big(z_1(s) - z_1; z\big) \cdot \ldots \cdot \Theta\big(z_g(s) - z_g; z\big) \cdot \Theta(d; z),$$

for a finite divisor d with $\sum_{\gamma\in\Gamma}\gamma d = 0$. Since

$$\Theta\big(z_i(s) - z_i; z\big) = \Theta\big(z_i(s) - z_0; z\big)\cdot\Theta(z_i - z_0; z)^{-1}\quad\text{for } i = 1,\dots,g$$

and since $\Theta(d; z)$ has no zeros and poles, the automorphy factor of ζ_s/ζ is given by

$$\widehat{j}\big(z_1(s)\big)\cdot\ldots\cdot\widehat{j}\big(z_g(s)\big)\cdot\widehat{j}(z_1)^{-1}\cdot\ldots\cdot\widehat{j}(z_g)^{-1}\equiv s \bmod M,$$

which coincides with $s \bmod M$. Therefore,

$$\widehat{\kappa} := \widehat{j}(z_1)\cdot\ldots\cdot\widehat{j}(z_g)\equiv\widehat{j}\big(z_1(s)\big)\cdot\ldots\cdot\widehat{j}\big(z_g(s)\big)\cdot s^{-1} \bmod M$$

is independent of s. □

Theorem 2.9.13 (Riemann's vanishing theorem). *Let $\widehat{\kappa}\in T$ be the constant of Lemma 2.9.12. Let $\widehat{W}_n\subset T$ be the image of $\Omega^{(n)}$ under the canonical map*

$$\widehat{j}^{(n)}:\Omega^{(n)}\times\{z_0\}\longrightarrow T,\ d\longmapsto c(d - n\cdot z_0),$$

for $n = 1,\dots,g$. Then we have that $\widehat{\Theta}_{\widehat{\kappa}} = \widehat{W}_{g-1}$.

Proof. We follow the complex analytic proof [38, p. 338].

First, we show that $\widehat{W}_{g-1}\subset\widehat{\Theta}_{\widehat{\kappa}}$. There is a dense open subset $U\subset\widehat{W}_{g-1}$ such that every element $\delta\in U$ is in general position; cf. Lemma 2.9.11. Furthermore, we can assume that all $\delta\in U$ satisfy $\widehat{j}(\Omega)\not\subset\widehat{\Theta}_s$ for $s := \delta\cdot\widehat{\kappa}^{-1}$, because we know that $\widehat{j}(\Omega)\cdot\widehat{W}_{g-1} = T$. If $\delta\in U$, then we have that

$$\widehat{j}\big(z_1(s)\big)\cdot\ldots\cdot\widehat{j}\big(z_g(s)\big) = s\widehat{\kappa} = \delta \bmod M,$$

where $z_1(s),\dots,z_g(s)$ are the zeros of ζ_s; cf. Lemma 2.9.12. Since $\delta\in\widehat{W}_{g-1}$, we can assume $\delta = \widehat{j}(p_1)\cdot\widehat{j}(p_2)\cdot\ldots\cdot\widehat{j}(p_g)$ with $p_1 = z_0$. Since δ is in general position, we know

$$z_1(s)+\cdots+z_g(s)\equiv p_1+\cdots+p_g \text{ in } X^{(g)}.$$

Therefore, we may assume that $z_1(s) = p_1 = z_0$ and hence that $\zeta_s(p_1) = 0$.

The crucial point now is $\vartheta(t) = \vartheta(t^{-1})$ as seen from Definition 2.9.2. Then it follows that

$$\begin{aligned}\vartheta_{\widehat{\kappa}}(\delta) = \vartheta\big(\delta\cdot\widehat{\kappa}^{-1}\big) &= \vartheta\big(\widehat{j}(p_2)\cdot\ldots\cdot\widehat{j}(p_g)\cdot\widehat{\kappa}^{-1}\big)\\ &= \vartheta\big(s\cdot\widehat{j}(p_1)^{-1}\big)\\ &= \vartheta\big(\widehat{j}(p_1)\cdot s^{-1}\big) = \zeta_s(p_1) = 0,\end{aligned}$$

because $\widehat{j}(p_1) = 1$ and $\zeta_s(p_1) = 0$. Thus, we see that $\vartheta_{\widehat{\kappa}}(\delta) = 0$ for all $\delta\in U$. Then we obtain that $U\subset\widehat{\Theta}_{\widehat{\kappa}}$, and hence $\widehat{W}_{g-1}\subset\widehat{\Theta}_{\widehat{\kappa}}$, because the topological closure of U equals $\widehat{\Theta}_{\widehat{\kappa}}$.

Next we show that $\widehat{\Theta}_{\widehat{\kappa}} \subset \widehat{W}_{g-1}$. By the first part we can write

$$\widehat{\Theta}_{\widehat{\kappa}} = n \cdot \widehat{W}_{g-1} + \widehat{\Theta}'$$

with effective divisors, where $n \geq 1$ is an integer and where $\widehat{\Theta}'$ is an effective divisor on T. We will first show $n = 1$ and then $\widehat{\Theta}' = \emptyset$. From Lemma 2.9.9 we know that

$$\deg \zeta_{\widehat{\kappa}} = g. \tag{2.1}$$

On the other hand, we know that

$$\deg\bigl(\widehat{j}(\Omega) \cap \widehat{W}_{g-1}\bigr) \geq g.$$

Indeed, let

$$s = \widehat{j}(p_1) \cdot \ldots \cdot \widehat{j}(p_g) \in T$$

be a generic point such that $[-1](\widehat{j}(\Omega)) \not\subset \widehat{W}_{g-1} \cdot s^{-1}$, where $[-1]$ denotes the inverse mapping on T. Then $[-1](\widehat{j}(\Omega))$ and $\widehat{W}_{g-1} \cdot s^{-1}$ meet in isolated points. We have that

$$\widehat{j}(p_\mu)^{-1} = \Bigl(\prod_{\nu \neq \mu} \widehat{j}(p_\nu)\Bigr) \cdot s^{-1} \in \widehat{W}_{g-1} \cdot s^{-1} \tag{2.2}$$

and thus $\#([-1](j(X)) \cap W_{g-1}) \geq g$, where now and in the following we denote by W_{g-1} and Θ' the image of $\widehat{W}_{g-1}$ and of $\widehat{\Theta}'$ in J, respectively. For the intersection number we obtain that

$$\bigl([-1]\bigl(j(X)\bigr) \cdot W_{g-1}\bigr) = \bigl(j(X) \cdot [-1](W_{g-1})\bigr) = \bigl(j(X) \cdot W_{g-1}\bigr),$$

because a translate of W_{g-1} is symmetric; cf. Lemma 2.9.14 below.

Then it follows from (2.1) and (2.2) that $n = 1$ and that $(j(X) \cdot \Theta') = 0$. In fact, one can always find an $s \in T$ such that $\widehat{j}(\Omega) \not\subset \widehat{\Theta}' \cdot s^{-1}$. Then we have

$$g = n \cdot \bigl(j(X) \cdot W_{g-1}\bigr) + \bigl(j(X) \cdot \Theta'\bigr).$$

Since $(j(X) \cdot \Theta') \geq 0$ and $(j(X) \cdot W_{g-1}) \geq g$, we see that $n = 1$ and that $(j(X) \cdot \Theta') = 0$ for the intersection number.

It remains to show that $\Theta' = \emptyset$. For $s \in J$ put $\Theta'_s := \tau_s \Theta'$. Since $(j(X) \cdot \Theta') = 0$, we also have that $(j(X) \cdot \Theta'_s) = 0$. Therefore, we obtain that

$$j(X) \cap \Theta'_s \neq \emptyset \quad \Longrightarrow \quad j(X) \subset \Theta'_s. \tag{$*$}$$

From this one deduces for every $s \in J_K$

$$\Theta'_s \cap W_2 \neq \emptyset \quad \Longrightarrow \quad W_2 \subset \Theta'_s.$$

Indeed, there are the following implications:

$$\begin{aligned}
\Theta'_s \ni j(p_1)\cdot j(p_2) &\implies \Theta'_{s\cdot j(p_1)^{-1}} \ni j(p_2)\\
&\implies \Theta'_{s\cdot j(p_1)^{-1}} \ni j(p'_2) \quad \text{for all } p'_2 \in X \text{ by } (*)\\
&\implies \Theta'_s \ni j(p_1)\cdot j(p'_2) \quad \text{for all } p'_2 \in X.
\end{aligned}$$

Then we also have that $\Theta'_s \ni j(p'_1)\cdot j(p'_2)$ for all $p'_1, p'_2 \in X$, and hence $W_2 \subset \Theta'_s$. Repeating this argument yields the implication:

$$\Theta'_s \cap W_n \neq \emptyset \quad \implies \quad \Theta'_s \supset W_n$$

for all $n \in \mathbb{N}$. But this is impossible, because Θ'_s is a divisor. Therefore, $\Theta' \cap W_g = \emptyset$, and hence $\Theta' = \emptyset$. □

In the sequel of the proof of Theorem 2.9.13 we made use of the following fact which is a general statement about the theta divisor of a curve over an algebraically closed field. For the convenience of the reader we will add the proof.

Lemma 2.9.14. *Let X be a connected smooth projective curve of genus g with $g \geq 1$ over an algebraically closed field k. Let $x_0 \in X$ be a k-rational point and let $W_{g-1} \subset J := \operatorname{Jac} X$ be the canonical theta divisor with respect to x_0. Then a suitable translate W of W_{g-1} is symmetric*; *i.e.*, $[-1]_J(W) = W$.

Proof. Let $\mathcal{L}$ be an invertible sheaf on X of degree $g-1$. Then we have that $\mathcal{L}$ is isomorphic to $\mathcal{O}_X(D)$ with an effective divisor D if and only if $\Gamma(X,\mathcal{L}) \neq 0$. Furthermore, by the Riemann-Roch Theorem and Serre's duality Corollary 1.8.2 we have $\Gamma(X,\mathcal{L}) \neq 0$ if and only if $\Gamma(X, \Omega^1_{X/K} \otimes \mathcal{L}^{-1}) \neq 0$. Since the degree of $\Omega^1_{X/K}$ is $2\cdot(g-1)$, we can write $\Omega = \mathcal{N}^{\otimes 2}$ for some invertible sheaf $\mathcal{N}$ of degree $g-1$. Now put $\mathcal{K} := \mathcal{N}((1-g)x_0)$, which has degree 0. If $\mathcal{L}$ is an invertible sheaf of degree 0, then we have the equivalence

$$\begin{aligned}
[\mathcal{L}\otimes\mathcal{K}] \in W_{g-1} &\iff \Gamma(X, \mathcal{L}\otimes\mathcal{N}) \neq 0\\
&\iff \Gamma\big(X, \mathcal{L}^{-1}\otimes\mathcal{N}\big) \neq 0 \iff \big[\mathcal{L}^{-1}\otimes\mathcal{K}\big] \in W_{g-1}.
\end{aligned}$$

Now let $W \subset J$ be the translate of W_{g-1} by the inverse of $\mathcal{K}$. Then we have that the divisor W is symmetric. □

Remark 2.9.15. Let D be a divisor on a rigid analytic torus $A = T/M$ and then let $\mathcal{L} := \mathcal{O}_A(D)$ be the associated invertible sheaf. Then there is a canonical morphism of rigid analytic tori

$$\varphi_{\mathcal{L}} : A \longrightarrow A',\ a \longmapsto \tau_a^*\mathcal{L}\otimes\mathcal{L}^{-1} = \mathcal{O}_A(\tau_{-a}D - D),$$

from A to its dual A'. If the M-linearization data (d,δ) are associated to $\mathcal{L}$, then $\varphi_{\mathcal{L}}$ is induced by the homomorphism $\delta : M \to M'$ of their character groups. As a map of points, the lifting $\widehat{\varphi}_{\mathcal{L}}$ of $\varphi_{\mathcal{L}}$ is given by

$$\widehat{\varphi}_{\mathcal{L}} : T \longrightarrow T',\ t \longmapsto \big[t' : m \longmapsto t\big(\delta(m)\big)\big].$$

Proof. The assertion follows from Remark 2.7.9. □

Riemann's vanishing theorem tells us that $\widehat{W}_{g-1}$ equals a translate $\widehat{\Theta}_{\widehat{\kappa}}$ of the theta divisor $\widehat{\Theta}$. Thus, their M-linearization data coincide at their group homomorphism.

Corollary 2.9.16. *In the situation of Theorem 2.9.13, consider the M-linearization data (c, λ_Θ) associated to $\mathcal{L} := \mathcal{O}_J(W_{g-1})$ and the data (d, λ) associated to $\mathcal{H} := \mathcal{O}_J(p_J(\Theta_{\widehat{\kappa}}))$. Then the group homomorphisms λ and λ_Θ coincide; i.e., the canonical polarization coincides with the theta polarization.*

In particular, the autoduality map φ' is associated to the inverse $-\lambda^{-1}$.

Proof. The invertible sheaf associated to the theta divisor has a trivialization given by $1/\vartheta(t)$. Due to Lemma 2.9.3 the M-linearization is given by

$$b(m,m) \cdot \big\langle t, \lambda(m)\big\rangle \cdot \frac{1}{\vartheta(t)} = \frac{1}{\vartheta(m \cdot t)}.$$

The homomorphism λ is not altered by a translation. Thus, by Theorem 2.9.13 we obtain that $\lambda_\Theta = \lambda$ and hence $\varphi_{\mathcal{L}} = \varphi_{\mathcal{H}}$.

Now it is a general fact on Jacobians of algebraic curves that the morphism $-\varphi_{\mathcal{L}} : J \to J'$ is the inverse of the autoduality map; cf. Theorem 5.1.6(d) below. So, the autoduality map is induced by $-\lambda^{-1} : M' \to M$. □

Remark 2.9.17. If one knows that the canonical polarization λ is equal to the theta polarization λ_Θ, then it is easy to see that there exists a point $\widehat{\kappa} \in T$ such that $\widehat{W}_{g-1} = \widehat{\Theta}_{\widehat{\kappa}}$. Thus, Riemann's vanishing theorem is a consequence of equality $\lambda = \lambda_\Theta$.

Proof. Put $\mathcal{L} := \mathcal{O}_J(W_{g-1})$ and let (c, λ_Θ) be its M-linearization data as in Corollary 2.9.16. The theta divisor $\widehat{\Theta}$ of Definition 2.9.4 gives rise to an effective divisor $p_J(\widehat{\Theta})$ on J with M-linearization data (d, λ) by Lemma 2.9.3.

If the homomorphisms λ and λ_Θ coincide, the invertible sheaf $\mathcal{O}_J(W_{g-1} - p_J(\widehat{\Theta}))$ is translation invariant. Its M-linearization is associated to the group homomorphism $\widehat{\kappa}' := d^{-1} \cdot c : M \to K^\times$ which is a point $\widehat{\kappa}' \in T'$. Then $\mathcal{O}_J(p_J(\widehat{\Theta}_{\widehat{\kappa}}) - W_{g-1})$ is associated to the M-linearization data $d \cdot c^{-1} \cdot \widehat{\kappa}' = 1$ as follows from Lemma 2.9.3, where $\widehat{\kappa} := \widehat{\kappa}' \circ \lambda^{-1}$. Thus, the divisor $\widehat{W}_{g-1} - \widehat{\Theta}_{\widehat{\kappa}}$ can be solved by a meromorphic function $\widehat{f}$ on T satisfying $\widehat{f}(m \cdot t) = \widehat{f}(t)$ for all $m \in M$ and $t \in T$. Then $\widehat{f}$ is induced by a meromorphic function f on J, and $p_J(\widehat{\Theta}_{\widehat{\kappa}}) - W_{g-1}$ is the principal divisor $\mathrm{div}(f)$. Thus, it remains to verify that f is constant.

If $s \in J$, then consider the morphism

$$j_s : X \longrightarrow J,\ z \longmapsto s \cdot j(z)^{-1}.$$

It follows from Lemma 2.9.11 that there is an open dense subset $U \subset J$ such that $j_s^{-1}(W_{g-1})$ consists of exactly g distinct points for $s \in U$. Now consider such a point $s = j(p_1) \cdot \ldots \cdot j(p_g) \in U$. Then we have that

$$j_s(p_\mu) = s \cdot j(p_\mu)^{-1} = j(p_1) \cdot \ldots \cdot j(p_{\mu-1}) \cdot j(p_{\mu+1}) \cdot \ldots \cdot j(p_g) \in W_{g-1}.$$

The inverse image $D_s := j_s^{-1}(W_{g-1})$ agrees with the set $\{p_1, \ldots, p_g\}$ and constitutes an effective divisor of degree g so that $j_s^* \mathcal{O}_J(W_{g-1}) = \mathcal{O}_X(D_s)$ has degree g and $H^1(X, \mathcal{O}_X(D_s))$ vanishes for $s \in U$. Moreover, we may assume that $j_s(p_1) = j(p_2) \cdot \ldots \cdot j(p_g) \in W_{g-1}$ is in general position and so

$$\dim \Gamma\big(X, \mathcal{O}_X(D_s)\big) = 1.$$

Now put $E_s = j_s^{-1}(p_J(\widehat{\Theta}_{\widehat{\kappa}}))$. Then the section $j_s^* f$ is a global generator of $\mathcal{O}_X(D_s - E_s)$. Up to a nonzero scalar the sections $j_s^* f$ and 1 of $\Gamma(X, \mathcal{O}_X(D_s))$ coincide. This shows that $E_s = D_s$ for all $s \in U$. In particular, we have that $s = j_s(p_1) \in W_{g-1}$ and $p_1 \in D_s$.

Since $D_s = E_s$, the point p_1 belongs to E_s. Thus, we see that the point $s' := j(p_2) \cdot \ldots \cdot j(p_g)$ belongs to $p_J(\widehat{\Theta}_{\widehat{\kappa}})$ for general points $s' \in W_{g-1}$. So we have that $W_{g-1} \subset p_J(\widehat{\Theta}_{\widehat{\kappa}})$, and hence that the divisor $p_J(\widehat{\Theta}_{\widehat{\kappa}}) - W_{g-1} = \operatorname{div}(f)$ is effective. Thus, f is constant. □

Chapter 3
Formal and Rigid Geometry

In 1974 Raynaud proposed a program [80], where he introduced groundbreaking ideas to rigid geometry by interpreting a rigid analytic space as the generic fiber of a formal schemes over $\mathrm{Spf}\,R$. Here $\mathrm{Spf}\,R$ is always the formal spectrum of a complete valuation ring R of height 1, where its topology is given by an ideal (π) for some element $\pi \in R$ with $0 < |\pi| < 1$. Due to results on flat modules [82] his approach also works in the non-Noetherian case of formal schemes of topologically finite presentation over $\mathrm{Spf}(R)$.

In Sect. 3.1 we start with a mild attempt to understand formal schemes by considering formal analytic structures on rigid analytic spaces; these consist of extra data on a given space. This allows us to define a reduction of a rigid analytic space without using the abstract method of formal schemes. For the first time Bosch introduced such spaces in [8].

In Sect. 3.2 admissible formal R-schemes and formal blowing-ups are defined. In a canonical way the generic fiber of an admissible formal R-scheme is a formal analytic space.

In Sect. 3.3 we will discuss the important result in Theorem 3.3.3 of Raynaud about the relationship between formal schemes and rigid analytic spaces. We omit the proof of this theorem. It heavily relies on the flattening technique [82]; all details were worked out by Mehlmann in [67] and by Bosch and the author in [14, Part II]. Moreover, by using the flattening technique many properties of rigid analytic morphisms can be transferred to suitable formal R-models. In particular, the notions of properness in rigid and formal geometry correspond to each other; cf. Theorem 3.3.12.

Already at the level of Sect. 3.1 a major problem shows up; namely, if the structural rings of a formal analytic space are of topologically finite type over R. When the base field is algebraically closed, this question was answered by Grauert-Remmert [36]. One can also approach the problem from an opposite direction; namely, how to arrange an R-model of an affinoid algebra which is of topologically finite type over R with reduced special fiber. This is a deep problem which was settled by Epp if R is a discrete valuation ring and by Bosch, Raynaud and the

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author in the relative case. The latter approach is quite natural as it also works over a general admissible formal scheme in Theorem 3.4.8.

In Sect. 3.4 we explain the major steps of this approach. In particular, it is a first step to provide a semi-stable R-model of a curve in Theorem 4.4.3 and of a curve fibration; cf. Theorem 7.5.2.

In the last Sect. 3.6 we provide new methods about approximations which are only used in Chap. 7. This part is deeply related to the significance of properness of rigid analytic spaces and to Elkik's method on approximation of solutions of equations in restricted power series.

In the whole chapter let K, R, k, π be the standard notations as defined in the Glossary of Notations.

3.1 Canonical Reduction of Affinoid Domains

In this section we will introduce a special concept of formal schemes which is quite canonical in rigid geometry. We hope that it can relieve the reader from the hesitation to work with formal schemes, because these schemes come equipped with a geometric interpretation.

3.1.1 Functors $A_K \rightsquigarrow \mathring{A}_K$ and $A_K \rightsquigarrow \widetilde{A}_K$

The building blocks of rigid geometry over a non-Archimedean field K are the affinoid spaces $X_K = \operatorname{Sp} A_K$, which are defined as the set of the maximal ideals of A_K. We put here an subindex "K", since we will associate a formal scheme X over $\operatorname{Spf} R$ to X_K. If A_K is a reduced affinoid K-algebra, then we consider the R-algebra

$$A := \mathring{A}_K := \{ f \in A_K ; |f|_{X_K} \le 1 \},$$

where the sup-norm is defined by

$$|f|_{X_K} := \max\{ |f(x)| ; x \in X_K \}.$$

Note that $\mathring{A}_K$, as a subset of A_K, consists of all elements which are *power bounded*. If $A_K = T_n$, then the sup-norm coincides with the Gauss norm, which is defined by the maximum of the absolute values of the coefficients, and hence

$$\mathring{T}_n = R\langle \xi_1, \ldots, \xi_n \rangle := \left\{ \sum_{n \in \mathbb{N}^n} a_\nu \xi_\nu \in R[[\xi]]; \lim_{\nu \to \infty} a_\nu = 0 \right\}.$$

Since every ideal T_n is closed, the epimorphism

$$\alpha : T_n \longrightarrow A_K$$

induces the residue norm on A_K, which we denote by $|.|_\alpha$. This gives rise to the R-algebra

$$A_\alpha := \alpha\big(R\langle\xi_1,\dots,\xi_n\rangle\big),$$

which is of topologically finite type (tft) over R, actually of topologically finite presentation. Obviously, we have that $|f|_{\sup} \le |f|_\alpha$ for all $f \in A_K$. One can show that the sup-norm is in fact a norm on A_K if A_K is reduced; cf. [10, 6.2.4/1]. It was a question of Tate [92] if one can write $\mathring{A}_K$ also as a suitable A_α. We will settle this in Sect. 3.4. In Definition 1.4.4 we introduced the canonical reduction of A_K, which is defined by

$$\widetilde{A}_K := \mathring{A}_K/\check{A}_K,$$

where

$$\check{A}_K := \big\{f \in A_K;\, |f|_{\sup} < 1\big\}.$$

$\widetilde{A}_K$ is a reduced k-algebra of finite type. There is a canonical map

$$\rho : X_K := \mathrm{Sp}(A_K) \longrightarrow \widetilde{X}_K := \mathrm{MaxSpec}(\widetilde{A}_K),\ x \longmapsto x \cap \mathring{A}_K \bmod \check{A}_K.$$

The *reduction map* ρ is surjective and continuous with respect to the Grothendieck topology on the domain and the Zariski topology on the codomain. $\widetilde{X}_K$ is called the *reduction* of X_K.

For the functors $A_K \rightsquigarrow \mathring{A}_K$ and $A_K \rightsquigarrow \widetilde{A}_K$ we have the following finiteness conditions, which can be shown without too much effort; cf. [10, 6.3.5/1].

Proposition 3.1.1. *Let $\varphi : A_K \to B_K$ be a morphism of affinoid algebras. Then the following conditions are equivalent*:

(a) *φ is finite.*
(b) *$\mathring{\varphi}$ is integral.*
(c) *$\widetilde{\varphi}$ is finite.*

Finiteness conditions for $\mathring{\varphi}$ will be dealt with in Sect. 3.1.3.

Remark 3.1.2. If the value group of K is divisible, then it is obvious that $\mathring{A}_K$ has a reduced special fiber. However, even if the value set of $|A_K|_{X_K}$ coincides with $|K|$, the R-algebra $\mathring{A}_K$ does not need to be of topologically finite type; cf. [10, §6.4.1].

For later use we mention the following result which can easily be verified; cf. [10, 7.3.4/3].

Lemma 3.1.3. *Let $\varphi : A_K \to B_K$ be a homomorphism of affinoid algebras and let $(f_1,\dots,f_n)$ be an affinoid generating system of B_K over A_K. If $|\_|$ is a norm defining the topology of B_K, then there exists an $\varepsilon > 0$ such that the following holds*:

If $(g_1,\dots,g_n)$ is a system of elements in B_K with $|g_i - f_i| < \varepsilon$ for all $i = 1,\dots,n$, then $(g_1,\dots,g_n)$ is an affinoid generating system of B_K over A_K as well.

For R-algebras of power bounded functions it is even simpler to prove.

Lemma 3.1.4. *Let A_K be a reduced affinoid algebra such that its R-subalgebra $\mathring{A}_K = R\langle f_1, \dots, f_n\rangle$ is of topologically finite type over R. If $g_1, \dots, g_n$ are elements of A_K with $|f_i - g_i|_{X_K} < 1$ for all $i = 1, \dots, n$, then $g_1, \dots, g_n$ generate $\mathring{A}_K$ over R as well.*

3.1.2 Formal Analytic Spaces

In this section we keep the notations of Sect. 3.1.1. An important consequence of the maximum principle is the following; cf. [10, 7.2.6/3].

Proposition 3.1.5. *Let $f \in A_K$ be an element with $|f|_{X_K} = 1$ and let*

$$X_K(1/f) := \{x \in X_K; \, |f(x)| = 1\} \subset X_K := \operatorname{Sp} A_K$$

be the affinoid subdomain of X_K. Then $A_K\langle 1/f\rangle = \mathcal{O}_{X_K}(X_K(1/f))$ and the canonical open embedding $\varphi : X_K(1/f) \to X_K$ induces an isomorphism

$$\widetilde{\varphi} : \operatorname{Spec} \widetilde{A_K\langle 1/f\rangle} = \widetilde{X_K(1/f)} \xrightarrow{\sim} (\widetilde{X}_K)_{\tilde{f}} \subset \widetilde{X}_K$$

of the reduction of $X_K(1/f)$ to the Zariski open subset $(\widetilde{X}_K)_{\tilde{f}}$ of $\widetilde{X}_K$.

If K is algebraically closed, then $\mathring{A}_K$ is of topologically finite type over R, actually of topologically finite presentation over R; i.e.,

$$\mathring{A}_K = R\langle \xi_1, \dots, \xi_n\rangle/\mathfrak{a},$$

where $\mathfrak{a} \subset R\langle \xi_1, \dots, \xi_n\rangle$ is finitely generated; see Theorem 3.1.17 further down. Moreover, if $\widetilde{U} \subset \widetilde{X}_K$ is Zariski open and affine, then $U_K := \rho^{-1}(\widetilde{U}) \subset X_K$ is an affinoid subdomain associated to the algebra $B_K := \mathcal{O}_{X_K}(U_K)$ with reduction $\widetilde{B}_K = \mathcal{O}_{\widetilde{X}_K}(\widetilde{U})$, where $\rho : X_K \to \widetilde{X}_K$ is the canonical reduction map. Thus, one can look at the formal R-scheme

$$X := \operatorname{Spf} A \longrightarrow \operatorname{Spf} R$$

with $A := \mathring{A}_K$. The topology of definition is given by the ideals $A\pi^n$ for $n \in \mathbb{N}$ and some element $\pi \in R$ with $0 < |\pi| < 1$. The geometric topology on the generic fiber $X_K = X \otimes_R K$ is induced by the pull-back under the reduction map $\rho : X_K \to \widetilde{X}_K$ from the Zariski topology of the special fiber

$$\widetilde{X} := \widetilde{X}_K = X \otimes_R k.$$

The structure sheaf $\mathcal{O}_X$ is the functor which associates the R-algebra $\mathring{B}_K$ to an open affine subset $\widetilde{U}$, where $B_K := \mathcal{O}_{X_K}(\rho^{-1}(\widetilde{U}))$. Note that $\mathring{B}_K$ is in general not of topologically finite type over R. But, if X_K is reduced and K algebraically closed, $\mathring{B}_K$ is of topologically finite type over R due to Theorem 3.1.17.

Definition 3.1.6.

(a) Let A_K be an affinoid algebra. An affinoid subdomain $V = \mathrm{Sp}(B_K)$ of $X_K := \mathrm{Sp}(A_K)$ is called *formal analytic open* if V is the inverse image of a Zariski open subset $\widetilde{V} \subset \widetilde{X}_K := \mathrm{MaxSpec}(\widetilde{A}_K)$ under the reduction map.
(b) An admissible covering $\mathfrak{U} = \{U_i; i \in I\}$ of a rigid analytic space X_K by affinoid open subvarieties U_i is called *formal analytic* if each intersection $U_i \cap U_j$ is formal analytic open in U_i for all $i, j \in I$.
(c) A *formal analytic space* X is a couple (X_K, ρ) consisting of a rigid analytic space X_K and a map $\rho : X_K \to \widetilde{X}$ from X_K to a reduced k-scheme $\widetilde{X}$ of locally finite type such that there exists an open affine covering $\mathfrak{U}$ of $\widetilde{X}$ with the following properties:

    (i) The covering $\{\rho^{-1}(\widetilde{U}); \widetilde{U} \in \mathfrak{U}\}$ is admissible and formal analytic.
    (ii) For every $\widetilde{U} \in \mathfrak{U}$ the inverse image $U := \rho^{-1}(\widetilde{U}) \subset X_K$ is an open affinoid subdomain of X_K and $\rho|_U : U \to \widetilde{U}$ is the canonical reduction.

    The map ρ is called *reduction map* and $\widetilde{X}$ a *reduction* of X.
(d) Let $\rho : X \to \widetilde{X}$ be a formal analytic space. For $\tilde{x} \in \widetilde{X}$ the subset

$$X_+(\tilde{x}) := \left\{x \in X_K; \rho(x) = \tilde{x}\right\}$$

is called the *formal fiber of* $\tilde{x}$ *with respect to the reduction* ρ. Here $X_+(\tilde{x})$ is viewed as a rigid analytic space.
(e) A *morphism of formal analytic space* is a morphism of the underlying rigid analytic spaces which respects the formal structures.

The data of a formal analytic space can be given by a rigid analytic space equipped with a formal analytic covering.

Example 3.1.7. There are the following standard examples.

(a) Let $X = \mathbb{P}^n_K$ be the n-dimensional projective space with homogeneous coordinate functions $\xi_0, \ldots, \xi_n$. Set

$$U_j := \left\{x \in \mathbb{P}^n_K; \left|\xi_i(x)\right| \le \left|\xi_j(x)\right| \text{ for } i = 0, \ldots, n\right\}.$$

Then $\{U_0, \ldots, U_n\}$ is a formal analytic covering. The reduction with respect to this covering is the projective space $\mathbb{P}^n_k$ over the residue field k.
(b) Let X be a connected smooth projective curve and let f be a rational function on X which is not constant. Then the covering consisting of the subsets

$$U_0 := \left\{x \in X; \left|f(x)\right| \le 1\right\} \quad \text{and} \quad U_\infty := \left\{x \in X; \left|f(x)\right| \ge 1\right\}$$

is a formal analytic covering of X.

(c) Let $X = \operatorname{Sp} A_K$ be an affinoid space and $f_0, \dots, f_n \in A_K$ without common zeros and $\mathfrak{U} := \{X(f_0/f_j, \dots, f_r/f_j); j = 0, \dots, n\}$ the associated rational covering; cf. Definition 1.3.1. Then $\mathfrak{U}$ is a formal analytic covering of X.

Definition 3.1.8. Let X be a smooth rigid analytic curve. An admissible open subset $U \subset X$ is called a *closed* and/or *open disc* if U is isomorphic to the closed and/or open unit disc $\mathbb{D}^{\pm}$. Similarly, U is called an *annulus of height* $c \in |K^\times|$ if it is isomorphic to an annulus $A(c, 1)^{\pm}$; cf. Definition 1.3.3.

Lemma 3.1.9. *In the situation of Definition 3.1.8 let $B \subset X$ be a closed disc and X projective. If $b \in B$ and $a \in X - B$, then there exists a meromorphic function ζ on X with $\zeta(b) = 0$ and a unique pole in a such that $\zeta|_B$ is a coordinate function on B.*

Proof. Due to the theorem of Riemann-Roch there exists a meromorphic function f on X with $f(b) = 0$, $\operatorname{Pol}(f) = \{a\}$ and $|f|_B = 1$. Then $f : X \to \mathbb{P}^1_K$ is finite and $f|_B : B \to \mathbb{D}$ is finite as well. Thus, B is a connected component of $f^{-1}(\mathbb{D})$. Since the set of algebraic regular functions on $X - \{a\}$ is dense in $\mathcal{O}_X(B)$, one can approximate a coordinate function on B by a meromorphic function on X which has a pole only in a. Since a good approximation ξ is again a coordinate function, the function $\zeta := \xi - \xi(b)$ satisfies the claim. □

Proposition 3.1.10. *Let X be the analytification of a smooth projective curve which is geometrically connected. Let $b_1, \dots, b_n \in X$ be K-rational points and let $B_i \subset X$ be closed discs with $b_i \in B_i$ for $i = 1, \dots, n$. Assume that $B_1, \dots, B_n$ are pairwise disjoint. Let B_i^- be the formal fiber of $\tilde{b}_i$ with respect to the canonical reduction $\operatorname{Sp} B_i \to \operatorname{Spec} \widetilde{B}_i$ for $i = 1, \dots, n$. Put*

$$B_0 := X - \left(B_1^- \cup \dots \cup B_n^-\right).$$

Then $\mathfrak{B} := \{B_0, B_1, \dots, B_n\}$ is a formal analytic covering of X. If $X_{\mathfrak{B}}$ denotes the associated formal analytic space, then its reduction $\widetilde{X}_{\mathfrak{B}}$ is isomorphic to a configuration of n projective rational curves, which meet in a single point.

Proof. Since X is connected, there exists a point $b \in X$ which is not contained in any B_i for $i = 1, \dots, n$. Let $b_i \in B_i^-$ for $i = 1, \dots, n$. By the theorem of Riemann-Roch we see that there exists a rational function f_i on X with $f_i(b) = 0$ such that f_i has only a pole in b_i. Furthermore, we can adjust f_i such that $|f_i|_{B_i - B_i^-} = 1$ for $i = 1, \dots, n$.

Let ξ_i be a coordinate function on B_i with $\xi_i(b_i) = 0$. If $\varepsilon_i < 1$ and close to 1, then one has by Proposition 1.3.4 that

$$\left|f_i(x)\right| = \left|\xi_i(x)\right|^{n_i + r_i} \quad \text{for all } x \in B_i \text{ with } 1 > \left|\xi_i(x)\right| \geq \varepsilon_i,$$

where n_i is the number of zeros of f_i on B_i^- and r_i is the order of f_i at b_i. Since the degree of the divisor of f_i is 0 and f_i has at least one zero outside B_i^-, we have $n_i + r_i < 0$. Since the map $f_i : X \to \mathbb{P}^1_K$ is finite and $f_i^{-1}(\infty) = \{b_i\}$, we see $\{x \in X; |f_i(x)| > 1\}$ is contained in B_i^-, and hence $|f_i|_{X-B_i^-} = 1$. Then $X - B_i^-$ is a connected component of $D_i := \{x \in X; |f_i(x)| \le 1\}$, because the morphism $f_i : X - B_i^- \to \mathbb{D}_K$ to the unit disc is finite. Thus, $X - B_i^-$ is a formal open subdomain of D_i and $X - B_i$ is a disjoint union of finitely many formal fibers with respect to the canonical reduction of D_i, because the map $\tilde{f}_i$ is finite. Then

$$\left(X - B_i^-\right) \cap B_i = \left(X - B_i^-\right) - (X - B_i) = B_i - B_i^-$$

is the complement of finitely many formal fibers of $X - B_i^-$ with respect to its canonical reduction. Thus, we see that $(X - B_i^-) \cap B_i$ is formal open in $(X - B_i^-)$. Obviously, $(X - B_i^-) \cap B_i$ is formal open in B_i. This shows that $(X - B_i^-, B_i)$ is a formal covering of X whose reduction is a rational curve with a unique singular point. The covering $\mathfrak{B}$ is the common refinement of the coverings $\{B_i, (X - B_i^-)\}$ for $i = 1, \ldots, n$. □

Example 3.1.11. Let $B_1, \ldots, B_n$ be pairwise disjoint closed rational discs in the projective line and let $B_\nu^- \subset B_\nu$ be a formal fiber with respect to the canonical reduction of B_ν for $\nu = 1, \ldots, n$. Consider the affinoid subdomain

$$X := \mathbb{P}^1_K - \left(B_1^- \dot{\cup} \cdots \dot{\cup} B_n^-\right).$$

Let ζ be a coordinate function on $\mathbb{P}^1_K$ with a pole in X. Let $b_\nu \in B_\nu^-$ be a K-rational point and $\gamma_\nu \in K^\times$ such that

$$\zeta_\nu := \frac{\gamma_\nu}{\zeta - \zeta(b_\nu)}$$

has sup-norm 1 on X for $\nu = 1, \ldots, n$. Then the coordinate ring $\widetilde{A}$ of the canonical reduction $\widetilde{X}$ of X is given by

$$\widetilde{A} = k[\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n]/(\tilde{\zeta}_\mu \tilde{\zeta}_\nu;\ 1 \le \mu < \nu \le n).$$

For $n = 1$ the domain $D := \mathbb{P}^1_K - B_1^-$ is a closed disc with coordinate ζ_1.

For $n = 2$ the domain $A := \mathbb{P}^1_K - (B_1^- \cup B_2^-)$ is a closed rational annulus with coordinate $\xi := (\zeta - \zeta(b_2))/(\zeta - \zeta(b_1))$. The height of A is equal to

$$\left|\frac{\gamma_1 \cdot \gamma_2}{(\zeta(b_1) - \zeta(b_2))^2}\right|$$

or its inverse. It is less than 1, because B_1 and B_2 are disjoint.

Proof. The assertion about the $\widetilde{A}$ follows from Proposition 2.4.8, because we have $|\zeta_\mu \cdot \zeta_\nu|_X < 1$ if $\mu \ne \nu$, as the discs are pairwise disjoint.

In the special case $n = 1$ the function ζ_1 is a coordinate on $\mathbb{P}^1_K$ with a pole in b_1 outside of $D := \mathbb{P}^1_K - B_1^-$ and a zero inside D.

In the case $n = 2$ the function ξ is a coordinate on $\mathbb{P}^1_K$ with a pole b_1 in B_1^- and a zero b_2 in B_2^-, both are outside of A. For computing the height one has to determine the sup-norm of ξ on $A_i := B_i - B_i^-$ for $i = 1, 2$. Since $\zeta - \zeta(b_2)$ has no zero or pole in B_1, it follows

$$|\xi|_{A_2} = |\xi|_{B_2} = \left|\frac{\gamma_2}{\zeta(b_2) - \zeta(b_1)}\right|.$$

Similarly, one obtains

$$|\xi|_{A_1} = \frac{1}{|1/\xi|_{B_1}} = \left|\frac{\zeta(b_2) - \zeta(b_1)}{\gamma_1}\right|.$$

The height is the quotient of both or of its inverse. The height is less than 1, since the absolute function $|\xi|$ cannot be constant on A because of the position of its zero and its pole. □

For later use we provide the following proposition.

Proposition 3.1.12. *Let X be a formal analytic space whose reduction $\widetilde{X}$ is irreducible. If $\mathfrak{U} := \{U_1, \ldots, U_N\}$ is an admissible covering of X, then there exists an index $i \in \{1, \ldots, N\}$ such that U_i contains a non-empty formal open subset V of X.*

Proof. We may assume that X is an affinoid space. Due to Theorem 1.3.7 the covering $\mathfrak{U}$ admits a refinement by a rational covering given by functions $f_0, \ldots, f_n$. Each function has a sup-norm $c_i = |f_i|_X$ with $c_i \in \sqrt{|K|^\times}$. We may assume $c_i \in |K^\times|$, because we can replace $f_0, \ldots, f_n$ by suitable powers $f_0^m, \ldots, f_n^m$, and hence that $c_0 = 1 \geq c_i$ for $i = 1, \ldots, n$. Since the reduction $\widetilde{X}$ is irreducible, the sup-norm is multiplicative by Remark 1.4.6. Therefore, $X_0 := X(f_1/f_0, \ldots, f_n/f_0)$ is a subdomain of X which contains the non-empty formal open part $X(1/f_0)$. □

Lemma 3.1.13. *Let $\overline{K}$ be a complete algebraic closure of K and let X_K be a quasi-compact rigid analytic space. Consider a formal analytic structure on $X_K \otimes_K \overline{K}$ given by a finite admissible covering $\mathfrak{V} = \{V_1, \ldots, V_r\}$ with affinoid open subspaces V_i of $X_K \otimes_K \overline{K}$. Then there exists a finite separable field extension K'/K such that $\mathfrak{V}$ is defined over K'.*

Proof. First assume that X_K is affinoid. For each $V \in \mathfrak{V}$ there exists rational covering $\mathfrak{U} := \{U_1, \ldots, U_n\}$ of $X_K \otimes_K \overline{K}$ such that $V \cap U_j$ is a rational subdomain of U_j; cf. Theorem 1.3.7. The functions, which define U_j and $V \cap U_j$ as a subset of X_K and U_j, respectively, can be approximated by linear combinations of functions in $\mathcal{O}_{X_K}(X_K)$ with coefficients in $\overline{K}$, and hence with coefficients in the algebraic closure K_{alg}. Since there are only finitely many coefficients involved, they are contained in a finite extension K'/K. If the characteristic of K is 0, the extension K'/K is separable.

If the characteristic is $p > 0$, we rise the functions to suitable p^e-power, then the coefficients become separable and we succeed in that case also. Thus, there exists a finite separable field extension K'/K such that each V_i is defined over K'; i.e., there exists an open subvariety $W_i \subset X_K \otimes_K K'$ with $W_i \otimes_{K'} \overline{K} = V_i$ for $i = 1, \ldots, r$.

It remains to see that each W_i is affinoid. Since V_i is affinoid, there exist functions $f_1, \ldots, f_n$ which constitute a generating system of the $\overline{K}$-algebra of V_i. Since by Lemma 3.1.3 good approximations of $f_1, \ldots, f_n$ constitute a generating system as well, we may assume that they are defined over a finite field extension K''/K'.

If the characteristic of K is $p > 0$, one can raise $f_1, \ldots, f_n$ to p^e-powers such that they are defined on $W_i \otimes_{K'} K''$, where K''/K' is separable and finite. Now they are no longer a generating system over K'', but they give rise to a finite morphism of $W_i \otimes_{K'} K''$ to an affinoid space. Then $W_i \otimes_{K'} K''$ is affinoid as well.

In the general case one starts with a finite covering $\{X_K^1, \ldots, X_K^N\}$ by affinoid subvarieties. Due to the above reasoning, every $V \in \mathfrak{V}$ is defined over a finite separable extension K'/K. To verify that each $V \in \mathfrak{V}$ is affinoid, one can copy the above given proof. □

3.1.3 Finiteness Theorem of Grauert-Remmert-Gruson

The notion of a formal analytic structure can be carried out over every non-Archimedean field K. The disadvantage is that the formal R-algebras $\mathring{A}_K$ are in general not of topologically finite type over R, whereas the reduction $\widetilde{A}_K$ is of finite type over k and reduced. Thus, it would suffice to produce a model with reduced special fiber; i.e., to find an R-algebra of topologically finite type $A \subset \mathring{A}_K$ with $A \otimes_R K = A_K$ such that $A \otimes_R k$ is reduced. We will see in Proposition 3.4.1 that such a model coincides with $\mathring{A}_K$.

This problem was stated by Tate in his Harvard Notes [92] as a finiteness conjecture for affinoid morphisms; cf. Theorem 3.1.17. Originally, this result was achieved by Grauert-Remmert [36] for algebraically closed base fields; the general case was shown by Gruson [43] using a different proof. A crucial definition is the following notion of a stable field.

Definition 3.1.14. Let K be a field with a non-Archimedean valuation, not necessarily complete.

(a) Let L/K be an algebraic field extension. The *spectral norm* of an element $x \in L$ is defined by

$$|x|_{\mathrm{sp}} := \max_{1 \le \nu \le n} \sqrt[\nu]{|c_\nu|},$$

where $T^n + c_1 T^{n-1} + \cdots + c_n \in K[T]$ is the minimal polynomial of x.

(b) The field K is called *stable* if every finite field extension L of K is a K-*Cartesian* vector space with respect to its spectral norm $|.|_{\mathrm{sp}}$; i.e., there exists a

K-basis $(v_1, \ldots, v_n)$ of L such that

$$|c_1 v_1 + \cdots + c_n v_n|_{\mathrm{sp}} = \max\{|c_1| \cdot |v_1|_{\mathrm{sp}}, \ldots, |c_n| \cdot |v_n|_{\mathrm{sp}}\};$$

cf. [10, 3.6.1/1 and 2.4.1/1].

For example, every non-Archimedean field with a discrete valuation is stable; cf. [10, 3.6.2/1]. Obviously, every algebraically closed field is stable. The main theorem in this approach is the following result; cf. [10, 5.3.2/1].

Theorem 3.1.15 (Transitivity of stability). *The field of fractions $Q(T_n)$ is stable if the ground field K is stable.*

This leads to the so-called *Gradgleichung*.

Theorem 3.1.16 (Gradgleichung). *Let K be a non-Archimedean field and assume that K is stable. Let $\varphi : A_K \hookrightarrow B_K$ be finite monomorphism of affinoid algebras of pure dimension d, where the reduction $\widetilde{A}_K$ is an integral domain and where B_K is reduced. If $|K| = |B_K|_{\sup}$, then we have*

$$[B_K : A_K] = [\widetilde{B}_K : \widetilde{A}_K].$$

Proof. If $A_K = T_d$ is a Tate algebra, the assertion follows from [10, 3.6.2/8] as mentioned above. Indeed, due to Proposition 3.1.1 the extension $\widetilde{T}_d \to \widetilde{B}_K$ is finite. Furthermore, the field of fractions $Q(T_d)$ is stable. Due to Theorem 3.1.17 for every $f \in T_d$ with $|f| = 1$ holds

$$\big[B_K\langle 1/f\rangle : T_d\langle 1/f\rangle\big] = [B_K : T_d] \quad \text{and} \quad \big[\widetilde{B}_K[1/\tilde{f}] : T_d[1/\tilde{f}]\big] = [\widetilde{B}_K : \widetilde{T}_d].$$

Since there exists an $f \in T_d$ with $|f| = 1$ such that $\widetilde{B}_K[1/\tilde{f}]$ splits into a product of domains, we may assume that the reduction of $B_K\langle 1/f\rangle$ is irreducible. Then the assertion follows from [10, 3.6.2/8]. In the general case, by Corollary 1.2.6 there exists a Noether normalization $T_d \hookrightarrow A_K$. So we have the equalities $[B_K : T_d] = [\widetilde{B}_K : \widetilde{T}_d]$ and $[A_K : T_d] = [\widetilde{A}_K : \widetilde{T}_d]$. Thus, the assertion follows from the degree formula for finite extensions. □

Actually, the argumentation in the proof of Theorem 3.1.15 goes the opposite direction. The preliminary of the proof of Theorem 3.1.15 is the criterion that a finite field extension L/K is K-Cartesian and fulfills $|L|_{\mathrm{sp}} = |K|$ if and only if the degrees of the field extension $[L : K]$ and of the residue extension $[\widetilde{L} : \widetilde{K}]$ coincide; [10, 3.6.2/8]. Therefore, the main burden in the proof is to establish this equality. In fact, it is well-known for discretely valued fields. In [10, 5.3.3/4] it is directly shown for stable fields with divisible value group by a clever consideration. In [10, §5.3.4] it is descended to a general stable field.

For the following result see [10, §6.4.1]; a proof will be discussed in Sect. 3.4.

Theorem 3.1.17 (Finiteness theorem). *Let K be a non-Archimedean field and A_K be a reduced affinoid K-algebra.*

(a) *Assume that K is stable and that A_K satisfies $|K| = |A_K|_{\sup}$. If $\varphi : B_K \to A_K$ is a finite morphism of affinoid algebras, then the morphism $\mathring{\varphi} : \mathring{B}_K \to \mathring{A}_K$ is finite.*

In particular, $\mathring{A}_K$ is an R-algebra of topologically finite type.

(b) *If K is stable and $|K^\times|$ is divisible (e.g. K is algebraically closed), then the R-algebra $\mathring{A}_K$ is of topologically finite type over R.*

(c) *If K is discretely valued, then every finite morphism $\varphi : B_K \to A_K$ of affinoid algebras induces a finite morphism $\mathring{\varphi} : \mathring{B}_K \to \mathring{A}_K$.*

3.2 Admissible Formal Schemes

In this section we will give a survey on the main tools of formal geometry and thereafter in Sect. 3.3 we will discuss its relation to rigid analytic spaces. Instead of starting with affinoid algebras, Raynaud started with R-algebras of type

$$A := R\langle \xi_1, \dots, \xi_n \rangle / \mathfrak{a}$$

which do not have R-torsion and which are of topologically finite presentation. The latter means that $\mathfrak{a}$ is finitely generated. Note that an R-module M has *no R-torsion* if and only if M is *flat over R*. We call R-algebras as above *admissible*.

In the theory of formal schemes one fixes an ideal of definition. In our case it is given by πA, where $0 \neq \pi \in \mathfrak{m}_R$ is any element in the maximal ideal $\mathfrak{m}_R$ of R. Thus, these R-algebras are equipped with the ideal topology given by $(A\pi^n; n \in \mathbb{N})$ and they are complete and separated with respect to this topology. Moreover, we have that

$$A = \varprojlim_{n \in \mathbb{N}} A_n \quad \text{with } A_n := A/A\pi^{n+1}.$$

The topology does not depend on the choice of π, because any other $\pi' \in R$ with $0 < |\pi'| < 1$ induces the same topology.

Since we consider arbitrary valuation rings R of rank 1, the following result is important in the study of such R-algebras; cf. [82, 3.4.6].

Theorem 3.2.1 (Gruson-Raynaud). *Let A be an R-algebra of topologically finite type and M a finitely generated A-module which is flat over R. Then M is an A-module of finite presentation. In particular, if $A^n \to M$ is a surjective homomorphism, then its kernel is finitely generated.*

Corollary 3.2.2. *Let $A = R\langle \xi_1, \dots, \xi_n \rangle / \mathfrak{a}$ be of topologically finite type.*

(a) *If A is a flat R-algebra, it is of topologically finite presentation.*
(b) *Every R-algebra A of topologically finite presentation is a coherent.*

Moreover, we need to know the procedure of complete localizations to obtain formal affine schemes; cf. [39, I, §10]. Let A be an R-algebra of topologically finite presentation and $f \in A$. Then set

$$A\langle 1/f\rangle := \varprojlim_{n\in\mathbb{N}} A_f/A_f\pi^{n+1} = \varprojlim_{n\in\mathbb{N}} \left(A/A\pi^{n+1}\right)_f,$$

which is an R-algebra of topologically finite presentation as well.

If elements $f_1, \dots, f_n$ of A generate the unit ideal, then the sequence

$$A \to \prod_{i=1}^{n} A\langle 1/f_i\rangle \rightrightarrows \prod_{i,j=1}^{n} A\langle 1/f_i f_j\rangle \tag{$*$}$$

is exact. In the usual way one can define an *affine formal scheme*

$$X := \operatorname{Spf} A$$

associated to an R-algebra A of topologically finite presentation. Its underlying topological space is $X_0 := \operatorname{Spec} A \otimes_R (R/R\pi)$, which coincides with $X \otimes_R k$. The infinitesimal levels $X_n := X \otimes_R (R/R\pi^{n+1})$ of formal schemes are indicated by a subindex "n". The structure sheaf on open sets U of $\operatorname{Spf} A$ is given by

$$\mathcal{O}_X(U) := \ker\left(\prod_{i=1}^{r} A\langle 1/f_i\rangle \rightrightarrows \prod_{i,j=1}^{r} A\langle 1/f_i f_j\rangle\right),$$

where $f_1, \dots, f_r \in A$ induce a covering of U; i.e.,

$$U = X_0(1/f_1) \cup \cdots \cup X_0(1/f_r).$$

Since the sequence ($*$) is exact, $\mathcal{O}_X$ is in fact a sheaf. Furthermore, we define the stalk at a point $x \in \operatorname{Spf} A$ by the local ring

$$\mathcal{O}_{X,x} = \varinjlim_{x\in U} \mathcal{O}_X(U).$$

Since admissible R-algebras are coherent, every *finitely generated* A-module M gives rise to a coherent $\mathcal{O}_X$-module $\mathcal{F}^M$ in the usual way. The notion of coherence is local on X. In fact, if $\mathcal{F}$ is an $\mathcal{O}_X$-module which is of type $\mathcal{F}|_{U_i} \cong \mathcal{F}^{M_i}$ for an affine open covering $\{U_1, \dots, U_n\}$ of $X = \operatorname{Sp} A$ with finitely generated $\mathcal{O}_X(U_i)$-modules M_i, then $M := \mathcal{F}(X)$ is a finitely generated A-module and the canonical morphism $\mathcal{F}^M \xrightarrow{\sim} \mathcal{F}$ is an isomorphism.

Definition 3.2.3. A *formal R-scheme* X (in this book) is a locally ringed space $(X, \mathcal{O}_X)$ such that each point $x \in X$ admits an open neighborhood U, where $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine formal scheme $\operatorname{Spf} A$, where A is an R-algebra of topologically finite presentation and its topology is defined by the ideal $A\pi$.

X is called an *admissible formal R-scheme* if, in addition, its structure sheaf $\mathcal{O}_X$ does not have R-torsion; i.e., $\mathcal{O}_X$ is R-flat.

Example 3.2.4. A typical case is the completion of an R-scheme of finite presentation along its the special fiber.

(a) Let $X := \mathbb{A}^n_R$ be the affine R-space of dimension n. The π-adic completion of $R[\xi_1, \dots, \xi_n]$ is the restricted power series ring $R\langle\xi_1, \dots, \xi_n\rangle$. Thus, we have that $X_0 = \mathbb{A}^n_{R_0}$ is the n-dimensional affine R_0-space and $\mathrm{Spf}(R\langle\xi_1, \dots, \xi_n\rangle)$ is the *formal n-dimensional polydisc*.
(b) Let $X = \mathbb{G}_{m,R} = \mathrm{Spec}\, R[\xi, 1/\xi]$ be the 1-dimensional affine R-torus, its π-adic completion is $\mathrm{Spf}\, R\langle\xi, 1/\xi\rangle$, the so-called *formal torus*. Its underlying topological space is the affine torus $X_0 = \mathbb{G}_{m,R_0}$ over R_0.
(c) Let $X = \mathbb{P}^n_R$ be the projective n-space over R. If we introduce homogeneous coordinates $\xi_0, \dots, \xi_n$ and equip X with the associated canonical covering $\{U_0, \dots, U_n\}$ where $U_\nu := \{x \in X; \xi_\nu(x) \neq 0\}$, then the π-adic completion is the formal scheme with underlying topological space $X_0 = \mathbb{P}^n_{R_0}$ and coordinates $\xi_{0,0}, \dots, \xi_{n,0}$, where $\xi_{j,0} = \xi_j \bmod \pi$. Their associated R-algebras are $\mathcal{O}_X(U_\nu) = R\langle\xi_0/\xi_\nu, \dots, \xi_n/\xi_\nu\rangle$ for $\nu = 0, \dots, n$.

Note that in cases (a) and (b) not all K-valued points of X specialize on the formal completion, whereas in the case (c) all K-valued points specialize due to the valuative criterion of properness.

Remark 3.2.5. Assume that K is algebraically closed. Then a formal analytic space X in the sense of Definition 3.1.6 with reduction $\rho : X \to \widetilde{X}$ induces an admissible formal R-scheme with underlying topological space $\widetilde{X}$ and algebras

$$\mathcal{O}_X(U) = \mathring{\mathcal{O}}_X\big(\rho^{-1}(U)\big)$$

for open affine subsets U of $\widetilde{X}$. Note that the topology of X is induced by the Zariski topology of $\widetilde{X}$ via ρ.

Since K is algebraically closed, $\mathcal{O}_X(U)$ is topologically of finite type over R due to Theorem 3.1.17. However, in general, it suffices to ask that there exists a open formal analytic covering $\{U_i = \mathrm{Sp}\, A_i; i \in I\}$ of X such that the R-algebras $\mathring{A}_i$ are of topologically finite type; cf. Corollary 3.2.2.

The crucial idea of Raynaud is the use of admissible formal blowing-ups.

Definition 3.2.6. Let A be an admissible R-algebra and let $X = \mathrm{Spf}(A)$ be the associated affine formal scheme.

(a) An ideal $I = (f_0, \dots, f_r) \subset A$ is called *open* if $\pi^n \in I$ for an $n \in \mathbb{N}$.
(b) An *admissible formal blowing-up* $X' \to X$ of a finitely generated open ideal I on $\mathrm{Spf}(A)$ is the completion of the algebraic blowing-up $X'_{\mathrm{alg}} \to X$ of I on $\mathrm{Spec}(A)$ with respect to its special fiber $X'_{\mathrm{alg}} \otimes_R R/\pi R$.

It is clear that the generic fiber of the algebraic blowing-up is not changed, since $I \otimes K = A \otimes_R K$ is invertible. A system of generators $(f_0, \dots, f_r)$ of the ideal I

gives rise to a canonical covering of X'_{alg} and hence of X'. Indeed, X'_{alg} can be regarded as the closed subscheme

$$X'_{\mathrm{alg}} := V\big(\big((\xi_i f_j - \xi_j f_i; i, j = 0, \ldots, r) : \pi^{\mathbb{N}}\big)\big) \subset \mathbb{P}^n_A,$$

where the homogeneous ideal consists of all elements which are multiplied by a big power π^N into the first ideal. The latter process is necessary in order the make the locally principal ideal $I\mathcal{O}_{X_{\mathrm{alg}}}$ invertible. Let X'_i be the open subscheme of X', where $\xi_{i,0} := \xi_i \bmod \pi$ does not vanish, then

$$\mathcal{O}_{X'}\big(X'_i\big) = A\langle f_0/f_i, \ldots, f_r/f_i\rangle/\big(\pi^{\mathbb{N}}\text{-torsion}\big).$$

Of course, one has to show that every non-zero divisor on X'_{alg} remains a non-zero divisor on X'. The latter is true, since the extension $\mathcal{O}_{X'_{\mathrm{alg}}} \to \mathcal{O}_{X'}$ is flat. This is well known if R is Noetherian; for the general case see Corollary 3.5.9 below. One can also introduce the formal blowing-up by the following formal scheme

$$X' = \varinjlim_{n\in\mathbb{N}} \operatorname{Proj}\Big(\bigoplus_{m\in\mathbb{N}} I^m \otimes_R R/\pi^n R\Big).$$

This procedure can be generalized to blowing-ups of coherent sheaves of open ideals $\mathcal{I} \subset \mathcal{O}_X$ on an admissible formal scheme

$$X_{\mathcal{I}} := \varinjlim_{n\in\mathbb{N}} \operatorname{Proj}\Big(\bigoplus_{m\in\mathbb{N}} \mathcal{I}^m \otimes_R R/\pi^n R\Big) \longrightarrow X.$$

The admissible formal blowing-up has the following properties, which can be proved easily; cf. [9, §8.2].

Proposition 3.2.7. *Let X be an admissible formal scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a coherent sheaf of open ideals. Then we have the following.*

(a) *$X_{\mathcal{I}}$ is an admissible formal scheme over* Spf R.
(b) *The ideal $\mathcal{I}\mathcal{O}_{X_{\mathcal{I}}}$ is invertible.*
(c) *If $\varphi : Y \to X$ is a morphism of admissible formal schemes such that $\mathcal{I}\mathcal{O}_Y$ is invertible, then φ factorizes through $X_{\mathcal{I}} \to X$ in a unique way.*

In particular, the admissible formal blowing-up is uniquely determined.

(d) *If K'/K is a finite field extension with associated extension R'/R of their valuation rings, then every K'-valued point of X extends uniquely to an R'-valued point of $X_{\mathcal{I}}$.*
(e) *Admissible formal blowing-ups commute with flat base change.*
(f) *If X is quasi-compact, then the composition of admissible blowing-ups $X' \to X$ and $X'' \to X'$ is an admissible formal blowing-up $X'' \to X$ of a suitable open ideal on X.*

(g) *Assume that* X *is quasi-compact. If* $U \subset X$ *is an open subscheme and if* $\mathcal{J} \subset \mathcal{O}_X|_U$ *is a coherent sheaf of open ideals, then* $\mathcal{J}$ *is the restriction of a coherent sheaf* $\mathcal{I} \subset \mathcal{O}_X$ *of open ideals.*

In particular, every admissible formal blowing-up $U_{\mathcal{J}} \to U$ *extends to an admissible formal blowing-up* $X_{\mathcal{I}} \to X$.

3.3 Generic Fiber of Admissible Formal Schemes

The interpretation of rigid spaces as generic fibers of admissible formal schemes was the innovative idea of Raynaud. His new insight made the theorems of Kiehl on coherent modules [50] and [51] into corollaries of well-known facts in algebraic geometry [61, §1]. Moreover, it opens an access to the work of Grothendieck [39]. Many interesting geometric results could be shown by this technique; cf. [14, II, §5].

If $A = R\langle \xi_1, \dots, \xi_n \rangle / \mathfrak{a}$ is an admissible formal R-algebra, then we put

$$A_{\text{rig}} := A \otimes_R K = K\langle \xi_1, \dots, \xi_n \rangle / \mathfrak{a} \cdot K.$$

A_{rig} is an affinoid K-algebra. Its maximal spectrum

$$X_{\text{rig}} := \text{Sp}(A_{\text{rig}})$$

is called *the generic fiber of the formal affine scheme* $X := \text{Spf}(A)$. Obviously this association applies to morphisms, and hence it gives rise to a functor. Therefore, one can generalize this procedure for every *admissible formal scheme over* $\text{Spf}(R)$.

Definition 3.3.1. Let X be an admissible formal $\text{Spf}\, R$-scheme, then the rigid analytic space X_{rig} associated to X is called the *generic fiber of* X.

If X_K is a rigid analytic space which is the generic fiber of an admissible formal $\text{Spf}\, R$-scheme X, then X is called an *R-model* of X_K. Similar definitions apply to morphisms and coherent sheaves.

An important observation of this functor is the following fact which easily follows from Proposition 3.2.7.

Proposition 3.3.2. *Let* $X = \text{Spf}\, A$ *be an admissible affine formal* R*-scheme. Consider an open ideal* $I = (f_0, \dots, f_n) \subset A$ *and let* $\varphi : X' \to X$ *be the admissible formal blowing-up of* I *on* X*. Then we have the following*:

(i) $\varphi_{\text{rig}} : X'_{\text{rig}} \to X_{\text{rig}}$ *is an isomorphism.*
(ii) *Put* $X'_i := \{x \in X';\, f_i \mathcal{O}_{X'} = I\mathcal{O}_{X'}\}$*. Then* φ_{rig} *maps* $(X'_i)_{\text{rig}}$ *onto the rational subdomain* $X_{\text{rig}}(f_0/f_i, \dots, f_n/f_i)$ *for* $i = 0, \dots, n$; *cf. Definition* 1.3.1.

The main point in the approach of Raynaud is to determine the essential image of his functor $X \rightsquigarrow X_{\text{rig}}$. For its proof see [14, Part II].

Theorem 3.3.3 (Existence of integral models). *The functor*

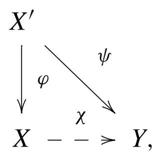
$$\text{rig} : (\text{Admissible Formal } R\text{-Schemes}) \longrightarrow (\text{Rigid Analytic } K\text{-Spaces})$$

gives rise to an equivalence of categories between

(1) *the category of quasi-compact, quasi-separated admissible formal R-schemes, localized by admissible formal blowing-ups, and*
(2) *the category of rigid analytic K-spaces which are quasi-compact and quasi-separated.*

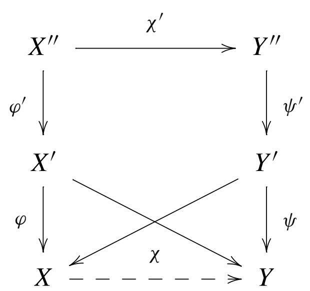
An admissible formal scheme X or a rigid analytic space X_K is called *quasi-compact* if X or X_K, respectively, admits a finite (admissible) covering by affine or affinoid open subspaces $\{U_i; i \in I\}$, respectively. It is called *quasi-separated* if the intersection of two affine or affinoid subspaces, respectively, is quasi-compact.

There is a general procedure of localization by a family of morphisms in a category. This simply means that the morphisms of the localizing family are regarded as isomorphisms. Thus, a morphism $\chi : X \dashrightarrow Y$ in the localized category is given by a diagram



where φ belongs to the localizing family and ψ is a true morphism.

In our case, we localize the category of admissible formal R-scheme by the family of admissible formal blowing-ups. Thus, a morphism $X \dashrightarrow Y$ in this localized category is a diagram as above, where φ is an admissible formal blowing-up and ψ is a true morphism such that $\chi_{\text{rig}} \circ \varphi_{\text{rig}} = \psi_{\text{rig}}$. Therefore, χ is an isomorphism if and only if there exists a commutative diagram



where all vertical arrows, $\varphi \circ \varphi'$, and $\psi \circ \psi'$ are blowing-ups and χ' is an isomorphism. For technical reasons it is useful to have the more precise statement [14, II, 5.5–7].

Theorem 3.3.4. *Let X_K be a rigid analytic space which is quasi-compact and quasi-separated. Let $\mathfrak{U}_K$ be a finite covering of X_K by open affinoid subvarieties of X_K. Then there exists an admissible formal R-model X of X_K such that $\mathfrak{U}_K$ is induced by an open covering $\mathfrak{U}$ of X.*

Let X and Y be admissible formal R-schemes and assume that X and Y are quasi-compact. Let $f_K : X_{\mathrm{rig}} \to Y_{\mathrm{rig}}$ be a rigid analytic morphism. Then there exists an admissible blowing-up $X' \to X$ such that f_K is induced by a formal morphism $f' : X' \to Y$.

Remark 3.3.5. Let X_K be a rigid-analytic space which is quasi-compact and quasi-separated.

(a) If X is an admissible formal R-scheme with generic fiber X_K, then the generic fiber of every open subscheme of X is admissible. Likewise every open covering of X induces an admissible covering of X_K.
(b) Let X_K be a rigid analytic space and $V_K \subset X_K$ an affinoid subdomain. If $V_+(v)$ is a formal fiber of a point $v \in V_K$, then the covering $\{V_K, X_K - V_+(v)\}$ is an admissible covering of X_K.

Proof. (a) This follows from Theorem 3.3.3.

(b) By Theorem 3.3.4 we have an admissible formal R-model X of X_K such that V_K is the generic fiber of an open formal subscheme V' of X. Due to Proposition 3.2.7 we may assume that V' is presented as an admissible formal blowing-up $p : V' \to V$. Let $v_0 \in V_0$ be the reduction of v. Since $A_0 = p^{-1}(v_0)$ is a closed subscheme of X_0, we see that $X_0 - A_0$ is an open subscheme of X, and hence that the covering $\{V_K, X_K - X_+(v)\}$ is induced by $\{V'_0, X - A_0\}$. Thus, by (a) the covering $\{V_K, X_K - V_+(v)\}$ is admissible. □

Sometimes we will consider morphisms $f : Y \to X$ of admissible formal schemes whose generic fiber $f_{\mathrm{rig}} : Y_{\mathrm{rig}} \to X_{\mathrm{rig}}$ has a certain property. Therefore we introduce the following notions; for details see [14, Part III, §3].

Definition 3.3.6. Let $f : X \to Y$ be a morphism of admissible formal schemes over Spf R.

(a) f is called a *rig-étale cover* if f is surjective and f_{rig} is étale.
(b) f is called *rig-flat* if f_{rig} is flat.
(c) f is called a *rig-isomorphism* if f_{rig} is an isomorphism.
(d) f is called *rig-quasi-finite* if f_{rig} is quasi-finite.
(e) f is called *rig-finite* if f_{rig} is finite.

The main tool for proving Theorem 3.3.3 was provided by the paper [82] of Gruson and Raynaud. In the following we will state some of its consequences in order to explain the relationship between formal and rigid analytic geometry. Proofs and details can be found in [14, II, 4.1].

Theorem 3.3.7 (Flattening). *Let $f : X \to T$ be a morphism of quasi-compact admissible formal R-schemes. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Let $n \in \mathbb{N}$ such that $\mathcal{M}$ is rig-flat over T in dimension $\geq n$.*

Then there exists an admissible formal blowing-up $T' \to T$ such that the strict transform $\overline{\mathcal{M}}'$ of $\mathcal{M}$ on $X' := X \times_T T'$ is T'-flat in dimension $\geq n$; i.e., there exists an open subscheme U' of X' with

$$\dim\bigl((X_0' - U_0')/T_0'\bigr) \leq n - 1$$

such that

$$\overline{\mathcal{M}}'\big|_{U'} = \mathcal{M}'\big|_{U'}/(\pi\text{-torsion})$$

is T'-flat, where $\mathcal{M}'$ is the pull-back of $\mathcal{M}$ to X'.

A coherent $\mathcal{O}_X$-module $\mathcal{M}$ is called *rig-flat over T in dimension $\geq n$* if there exists a closed subscheme Y of X with $\dim(Y/T)_{\text{rig}} \leq (n-1)$ such that $\mathcal{M}|_{(X-Y)}$ is rig-flat over T. For example, if $\dim(X/T)_{\text{rig}} \leq (n-1)$, then $X \to T$ is rig-flat in dimension $\geq n$.

The flattening technique can be applied to morphisms and is a means to improve properties of R-models; cf. [14, II, §5].

Corollary 3.3.8. *Let $f : X \to Y$ be a morphism of quasi-compact admissible formal R-schemes as above.*

(a) *If $\dim(X/Y)_{\text{rig}} \leq n$, then there exists an admissible formal blowing-up $Y' \to Y$ such that the induced map $f' : X' \to Y'$ of the strict transform has relative dimension $\leq n$.*
(b) *If f_{rig} is quasi-finite, one can choose $Y' \to Y$ such that $f' : X' \to Y'$ is quasi-finite as well.*
(c) *If f_{rig} is an open (resp. closed) immersion, then one can choose $Y' \to Y$ such that $f' : X' \to Y'$ is an open (resp. closed) immersion.*
(d) *If f_{rig} is flat, then the image of f_{rig} is a finite union of open affinoid subvarieties of Y_{rig}.*

The assertion in Corollary 3.3.8(d) can be generalized to a remarkable statement on the image of a rigid analytic morphism which will be used in Remark 7.2.4.

Proposition 3.3.9. *Let $u_K : X_K \to Y_K$ be a morphism of quasi-compact, separated rigid analytic spaces. Assume that X_K admits a smooth formal R-model X with geometrically connected special fiber. Then there exist a dense open formal subscheme U of X and an admissible formal R-scheme V, whose generic fiber V_K is an admissible open subvariety of Y_K with the following properties:*

(i) *The map $v_K := u_K|_{U_K} : U_K \longrightarrow Y_K$ factorizes through the generic fiber of a closed subscheme V' of V.*
(ii) *V' is a smooth formal R-scheme.*
(iii) *The map v_K extends to a faithfully flat morphism $v : U \to V'$.*

Proof. Since the special fiber of X is irreducible, we may assume that X_K and Y_K are affinoid spaces due to Proposition 3.1.12. Then there exists an affine formal R-model Y of Y_K such that u_K is induced by a morphism $u : X \to Y$.

Consider the product $X_K \times X_K$. Inside the product there is the fibered product $Z_K := X_K \times_{Y_K} X_K$ as a closed subvariety which can be regarded as a graph of an equivalence relation. Denote by $Z \subset X \times X$ the schematic closure of Z_K. Since X is smooth over R, there exists a dense open subscheme $U \subset X$ such that each projection $Z \to X$ is faithfully flat over U, because the map $Z_k \to X_k$ of the special fibers is generically flat. Thus, we may replace X by U and assume that both projections $Z \to X$ are faithfully flat. Now consider the projections

$$p_{i,j} : X \times X \times X \longrightarrow X \times X$$

for $1 \le i < j \le 3$ from the triple to the two-fold product. Since Z_K is the graph of an equivalence relation and the schematic closure commutes with flat base change, we have

$$p_{1,2}^* Z \cap p_{2,3}^* Z = p_{1,3}^* Z \cap p_{2,3}^* Z,$$

meaning that Z is a graph of an equivalence relation on X.

Next we assert that, after a further shrinking of X, the formal $fppf$-quotient X/Z exists as a formal affine scheme over $\operatorname{Spf} R$. Indeed, Z induces an equivalence relation Z_n on $X_n := X \otimes_R R_n$ for every $n \in \mathbb{N}$, where $R_n := R/R\pi^{n+1}$. Due to [4, Theorem 7.1] there exists the $fppf$-quotient $Q_n := X_n/Z_n$ as an algebraic space in the sense of sheaves. Moreover, Q_n is flat over R_n. In [4] the theorem is only stated in the case, where one considers schemes of finite type over a Noetherian base. However, by the usual techniques [41, Exp. V, §9] it can be generalized to the case of finite presentation over a general base. Unfortunately, there is no proof in [4], but it is implicitly given in [49]. Indeed, if Z is a flat equivalence relation on X, then by [54, Lemma 3.3] one reduces the problem to a flat quasi-finite equivalence relation. Then, after an étale localization, one arrives at a finite flat equivalence relation on an affine scheme by [54, Proposition 4.2]. The latter is effective by [41, Exp. V, Theorem 4.1(iv)]. So the quotient is representable by an algebraic space.

By an étale localization one can only expect an algebraic space. Due to [55, I, Proposition 5.19] there exists a dense open subset $U_0 \subset Q_0$ which is an affine scheme; alternatively, one can use [41, Exp. V, Theorem 8.1] to have a generic quotient X_0/Z_0 as an affine scheme. After a suitable shrinking of X, we may assume that Q_0 is smooth over R_0 and affine. Then the open subscheme $U_n \subset Q_n$ with $U_n \otimes R/R\pi = U_0$ is an affine scheme as well by [55, III, Theorem 3.3]. Moreover, it is smooth over R_n. Indeed, since the special fiber $Q_k := Q \otimes_R k$ of Q_n is dominated by the geometrically reduced fiber X_k, there is an open dense subscheme of Q_k which is smooth. Since Q_n is flat over R_n, the schemes Q_n are smooth over R_n for all $n \in \mathbb{N}$. Then the limit $Q := \varinjlim Q_n$ is the $fppf$-quotient $Q := X/Z$ in the category of admissible formal schemes and the morphism $X \to Q$ is faithfully flat. After the shrinkings we made, Q is an affine smooth formal R-scheme.

The map $u_K : X_K \to Y_K$ can be factorized through a monomorphism $i_K : Q_K \to Y_K$. Since such a map is quasi-finite, it follows from Corollary 3.3.8

that there is an admissible formal blowing-up $Y' \to Y$ such that the strict transform $Q' \to Y'$ is quasi-finite. Then there exists a dense open part $U \subset Q$ and an open subscheme $V \subset Y'$ such that $U \to V$ is finite. Since $Q_K \to Y_K$ is a monomorphism, the restriction $v : U \to V$ is a closed immersion. The image V' of v is isomorphic to Q, and hence a smooth formal R-scheme. □

Let us briefly explain how flattening techniques are used as an important tool in the proof of the existence of R-models in Theorem 3.3.3.

Any affinoid space $X_K = \operatorname{Sp} A_K$ has an admissible formal R-model $X = \operatorname{Spf} A$. Indeed, if $A_K = K\langle\underline{\xi}\rangle/\mathfrak{a}_K$ for a system $\underline{\xi} := (\xi_1, \ldots, \xi_n)$ of variables, then set $\mathfrak{a} := \mathfrak{a}_K \cap R\langle\underline{\xi}\rangle$. Now $A := R\langle\underline{\xi}\rangle/\mathfrak{a}$ is an admissible R-model due to Corollary 3.2.2.

The construction of an admissible formal R-model for a quasi-compact quasi-separated rigid space is local with respect to the Grothendieck topology. Thus, it suffices to consider an open immersion φ_K of an affinoid subdomain $U_K := \operatorname{Sp} B_K$ of $X_K := \operatorname{Sp} A_K$ and to explain how to glue the associated R-models. If $(f_1, \ldots, f_n)$ is an affinoid generating system of A_K, then each $g_j := \varphi_K^*(f_j)$ has sup-norm $|g_j|_{U_K} \leq 1$. Thus, we may assume by Proposition 3.1.1 that $(g_1, \ldots, g_n)$ belong to B, where B is an R-model of B_K as above. Then the morphism φ_K induces a morphism $\varphi^* : Y := \operatorname{Spf} B \to X := \operatorname{Spf} A$ of the above mentioned R-models. However, it does not need to be an open immersion, but $\varphi_{\text{rig}} : Y_{\text{rig}} \to X_{\text{rig}}$ is flat. Now the flattening technique provides an admissible formal blowing-up $X' \to X$ such that its strict transform $\varphi' : Y' \to X'$ is flat. Then it is not difficult to see that $\varphi' : Y' \to X'$ is an open immersion. The same reasoning can be used to show the existence of an R-model for a rigid analytic map $\varphi_K : Y_{\text{rig}} \to X_{\text{rig}}$ between R-models of affinoid spaces. Indeed, one considers the graph $\Gamma_K \subset Y \times_R X$ and looks at the projection $p_2 : \Gamma_K \to X_{\text{rig}}$. Then one can argue as above.

One can also show a similar result for coherent sheaves; cf. [61, 2.2].

Proposition 3.3.10. *Let X_K be quasi-compact quasi-separated rigid analytic space. If $\mathcal{F}_K$ is a coherent $\mathcal{O}_{X_K}$-module, then there exists an admissible formal R-model X of X_K and a coherent formal $\mathcal{O}_X$-module $\mathcal{F}$ on X without R-torsion such that $\mathcal{F}_{\text{rig}} := \mathcal{F} \otimes_R K$ is isomorphic to $\mathcal{F}_K$.*

One can use Proposition 3.3.10 to deduce Kiehl's Theorem 1.6.2 from Grothendieck's formal existence theorem [39, III, §5] in the case of a Noetherian base ring R; cf. [61, 2.3]. The case of a general R is proved in [1, 2.11.10 and 4.7.36].

Definition 3.3.11. A morphism $\varphi : Y \to X$ of admissible formal schemes is called *proper* if $\varphi_0 : Y_0 \to X_0$ is proper.

Note that every morphism $\varphi : Y \to X$ of admissible formal schemes gives rise to morphisms $\varphi_n := \varphi \otimes R/R\pi^{n+1}$ for all $n \in \mathbb{N}$. It is well known that φ_n is separated or proper if and only if φ_0 is. A deeper result is the relationship between properness in the formal and in the rigid analytic case; cf. [61, 3.1] for a Noetherian base ring R and [94] for the general case.

Theorem 3.3.12. *Let $\varphi : Y \to X$ be a morphism of admissible formal schemes. Then the following conditions are equivalent*:

(a) *$\varphi : Y \to X$ is proper.*
(b) *$\varphi_{\mathrm{rig}} : Y_{\mathrm{rig}} \to X_{\mathrm{rig}}$ is proper in the sense of Definition 1.6.3.*

Similarly as before, one can deduce the Finiteness Theorem 1.6.4 of Kiehl from Grothendieck's finiteness theorem [39, III, 4.1.5] for a Noetherian base ring R; cf. [61, 2.7]. For the general case see [1, 2.11.5 and 4.8.22].

Remark 3.3.13. The implication "(b) → (a)" in Theorem 3.3.12 is easy to verify. Indeed, we may assume that $X = \operatorname{Spf} A$ is affine. Then consider two open coverings $(U_K^1, \ldots, U_K^n)$ and $(V_K^1, \ldots, V_K^n)$ of Y_{rig} by admissible open affinoid subvarieties such that $U_K^j \Subset_{X_{\mathrm{rig}}} V_K^j$ for $j = 1, \ldots, n$; cf. Definition 1.6.3.

If $U_K = \operatorname{Sp} B_K \Subset_{X_K} V_K = \operatorname{Sp} C_K$ with $X_K := \operatorname{Sp} A_K$, then for suitable affine models $\varphi : U := \operatorname{Spf} B \to V := \operatorname{Spf} C$ the schematic image $\overline{\varphi_0(U_0)} \subset V_0$ is finite over X_0. In fact, due to Definition 1.6.3 we may assume that C_K can be represented in the form $A_K\langle \xi_1, \ldots, \xi_N \rangle / \mathfrak{c}$ such that $|\xi_i|_{U_K} < 1$. The latter implies that $\varphi_0 : U_0 \to V_0$ factorizes through the zero section Z_0 of $\operatorname{Spec} C_0 \to \operatorname{Spec} A_0$. Therefore, the schematic closure $\overline{\varphi_0(U_0)}$ in V_0 is finite over X_0.

By Theorem 3.3.4 there exists an R-model $Y' \to Y$, which can be chosen as an admissible formal blowing-up, such that all the subvarieties U_K^j and V_K^j are induced by open subschemes U^j and V^j of Y', respectively. Moreover, each V^j is an admissible formal blowing-up of some $\operatorname{Spf} C^j$ as considered above. By the above reasoning we obtain that the schematic closure of U_0^j in Y_0' is proper over X_0 for $j = 1, \ldots, n$. Thus, $Y_0' \to X_0$ is proper, and hence $Y \to X$ is proper as well.

The opposite implication is much harder to prove, since it requires the construction of suitable affinoid coverings; cf. [61, §3] for the Noetherian case. The proof loc. cit. works also in the general case, after Abbes has generalized parts of Grothendieck's theory about formal geometry to schemes of topologically finite presentation [1].

The implication "(a) → (b)" will not be used in this book, because one can proceed as Abbes by defining a proper rigid morphism $\varphi_K : Y_K \to X_K$ by requiring an R-model $\varphi : Y \to X$ which is proper in the formal sense; cf. [1, 7.2.9]. Then the Finiteness Theorem 1.6.4 remains true, cf. [1, 7.3.12].

The implication "(a) → (b)" is easy in the case of dimension 1, since a proper formal curve is algebraic due to Grothendieck existence theorem [39, III, §5] and hence proper as rigid analytic variety in Remark 1.6.10.

3.4 Reduced Fiber Theorem

In Sect. 3.3 we have seen that every quasi-compact quasi-separated rigid analytic space X_K admits an admissible formal R-model X. In general, the special fiber

$X \otimes_R k$ of X will not be reduced; one even cannot expect that without extending the base ring R.

The existence of R-models with reduced special fiber is the objective of this section. It is closely related to the Finiteness Theorem 3.1.17 and, moreover, it provides the first major step towards the stable reduction theorem for algebraic curves. In this section we will look at the problem from different points of view. The first two subsections will report on the literature by explaining the main ideas of their approaches while the third one, which is the natural one, will be studied in more details.

3.4.1 Analytic Method of Grauert-Remmert-Gruson

Given a geometrically reduced affinoid algebra A_K, we are interested in finding a R-model A of A_K such that the special fiber $A \otimes_R k$ is geometrically reduced. That means to find a representation

$$\alpha : K\langle \xi_1, \ldots, \xi_n \rangle \longrightarrow A_K$$

such that

$$A_\alpha := \alpha\big(R\langle \xi_1, \ldots, \xi_n \rangle\big)$$

has geometrically reduced special fiber. The surjective map α induces a Banach norm $|\_|_\alpha$ on A_K. The next proposition explains the meaning of the reduced special fiber.

Proposition 3.4.1. *In the above situation assume that A_K is reduced. Then the following assertions are equivalent*:

(a) *$A_\alpha \otimes_R k$ is reduced.*
(b) *$|f|_\alpha = |f|_{\sup}$ for all $f \in A_K$.*

Obviously, the condition (b) *implies $A_\alpha = \mathring{A}_K$.*

Proof. (a) → (b): It is always true that $|f|_{\sup} \le |f|_\alpha$ for all $f \in A_K$. Consider now an $f \in A_K$ with $f \neq 0$. There exists a $c \in K^\times$ such that $|c| = |f|_\alpha$. Thus, we have $g := f/c \in A_\alpha$ and $0 \neq \overline{g} \in A_\alpha \otimes_R k$. If $|g|_{\sup} < 1$, then $g^N \to 0$ for $N \to \infty$, because $|\_|_{\sup}$ is norm on A_K by [10, 6.2.1/4(iii)]. Since $A_\alpha \subset A_K$ is open, the residue class $\overline{g}$ is nilpotent in $A_\alpha \otimes_R k$. Since $A_\alpha \otimes_R k$ is reduced, we see that $\overline{g}$ is 0. This contradicts the assumption. Thus, we see that $|f|_{\sup} = |c| = |f|_\alpha$.

(b) → (a): Since $|f|_\alpha = |f|_{\sup}$ for all $f \in A_K$ and $|A_K|_\alpha = |K|$, so $|A_K|_{\sup} = |K|$ and, hence $\mathring{A}_K \otimes_R k = \tilde{A}_K$ is reduced. Thus, we see that $A_\alpha \otimes_R k$ is reduced. □

The Finiteness Theorem 3.1.17 implies the Reduced Fiber Theorem 3.4.2. Actually, the Reduced Fiber Theorem is equivalent to the Finiteness Theorem.

Theorem 3.4.2 (Reduced fiber theorem). *Let A_K be a geometrically reduced affinoid algebra. Then there exists an étale field extension K'/K such that the power bounded subalgebra $\mathring{A}_{K'}$ of $A_{K'} := A_K \otimes_K K'$ is of topologically finite type over the valuation ring R' of K' and has a geometrically reduced special fiber $\mathring{A}_{K'} \otimes_{R'} k'$.*

Moreover, if $A \subset \mathring{A}_K$ is an R-subalgebra of topologically finite type such that $A \otimes_R K = A_K$, then $A \otimes_R R' \to \mathring{A}_{K'}$ is finite.

Proof. First, assume that K is separably closed. Then K is stable due to [10, 3.6.2/2]. Since A_K is geometrically reduced, we have $|K| = |A_K|$ due to the maximum principle in Theorem 1.4.2. Then, it follows from Theorem 3.1.17 that $\mathring{A}_K$ is an R-algebra of topologically finite type with geometrically reduced reduction.

In the general case, let $\overline{K}$ be the completion of a separable closure of K and let K_{sep} be the algebraic separable closure of K in $\overline{K}$. As we discussed above, there exists an epimorphism

$$\overline{\alpha} : \overline{K}\langle \xi_1, \ldots, \xi_n \rangle \longrightarrow A_K \widehat{\otimes}_K \overline{K}$$

such that the associated residue norm is the sup-norm of $A_K \widehat{\otimes}_K \overline{K}$. Writing $\overline{R}$ for the valuation ring of $\overline{K}$, we see that $\overline{A} = \overline{\alpha}(\overline{R}\langle \xi_1, \ldots, \xi_n \rangle)$ is an $\overline{R}$-model of $A_K \widehat{\otimes}_K \overline{K}$ with reduced special fiber. Due to Lemma 3.1.4 we can replace $\overline{\alpha}(\xi_i)$ by approximations belonging to the image of $A_K \otimes_K K_{\mathrm{sep}}$. Moreover, we may assume that there is a finite étale extension K' of K such that the epimorphism $\overline{\alpha}$ is obtained from an epimorphism

$$\alpha : K'\langle \xi_1, \ldots, \xi_n \rangle \longrightarrow A_K \widehat{\otimes}_K K'$$

by tensoring with $\overline{K}$ over K'. Let R' be the valuation ring of K' and set $A' := \alpha(R'\langle \xi_1, \ldots, \xi_n \rangle)$. Since $\overline{A} = A' \widehat{\otimes}_{R'} \overline{R}$, we see that the special fiber of A' is geometrically reduced. Due to Proposition 3.4.1 the residue norm $|\_|_\alpha$ coincides with the sup-norm on $A_{K'} := A_K \otimes_K K'$ and $\mathring{A}_{K'}$ coincides with A'.

The additional assertion follows from Proposition 3.1.1. □

Corollary 3.4.3. *Let X_K be a quasi-compact geometrically reduced rigid analytic space. Let $\overline{K}$ be a complete algebraic closure of K.*

Consider a formal analytic structure on $X_K \otimes_K \overline{K}$ given by a finite admissible covering $\mathfrak{V} = \{V_1, \ldots, V_r\}$ by affinoid subspaces V_i of $X_K \otimes_K \overline{K}$. Then there exists a finite separable field extension K'/K such that $\mathfrak{V}$ is defined over K' and $\mathring{\mathcal{O}}_{X'_K}(V)$ is an R'-algebra of topologically finite type for every formal open affine subspace $V \subset X'_K := X_K \otimes_K K'$.

In particular, for any quasi-compact formal analytic space X_K there exists a suitable étale base field extension K'/K such that $X_K \otimes_K K'$ is the generic fiber of an admissible formal R'-scheme.

Proof. As seen in Lemma 3.1.13 there exists an étale field extension K'/K such that $\mathfrak{V}$ is defined over K'. So we may assume $K = K'$. Due to Theorem 3.4.2, there exists an étale field extension K'/K such that each $\mathring{\mathcal{O}}_{X_K}(V_i \otimes_K K')$ is an R'-algebra of topologically finite type with geometrically reduced reduction for $i = 1, \ldots, r$. Then it follows from Proposition 3.1.5 that, for every open affine subscheme $\tilde{V} \subset \tilde{X}$, the pre-image $V := \rho^{-1}(\tilde{V})$ is affinoid and its subalgebra of power bounded functions $\mathring{A}_K$ is topologically of finite type over R' with geometrically reduced reduction. □

3.4.2 Elementary Method of Epp

H. Epp found an elementary approach to prove the reduced fiber theorem for discrete valuation rings. However elementary does not mean simple. Epp's approach is limited to discrete valuation rings R; it does not work for arbitrary valuation rings of height one. Therefore, we assume in this subsection that R is a complete discrete valuation ring. In the following we present the statement of Epp and explain the main ideas of his paper [25].

An extension of discrete valuation rings $R \subset S$ which is not necessarily finite is called *dominating* if the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective. A dominating extension of discrete valuation rings is called *weakly unramified* if the uniformizer π of R is a uniformizer of S as well; this means the morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ has reduced fibers.

Epp's paper is written in the style of commutative algebra. One fixes a universal domain Ω which contains all discrete valuation rings under consideration. The join T_1T_2 of two discrete valuation rings T_1 and T_2 contained in Ω is the normalization of the ring theoretic join of T_1 and T_2 in $Q(\Omega)$; the latter is the smallest subring of the field of fractions $Q(\Omega)$ which contains T_1 and T_2. For the following statement see [25, 1.9].

Theorem 3.4.4 (Epp). *Let $R \subset S_0 \subset S$ be a dominating extension of (complete) discrete valuation rings such that $S_0 \subset S$ is finite and $R \subset S_0$ is weakly unramified. If the characteristic of k is $p > 0$, assume that*

$$\ell^{p^\infty} := \bigcap_{i=1}^{\infty} \ell^{p^i} \qquad (*)$$

is separable algebraic over k, where $k := k_R$, $\ell := k_{S_0}$ are the residue fields of R resp. S_0. Then there exists a finite extension $R \subset R'$ of discrete valuation rings such that the normalization $\overline{S'}$ of the ring theoretic join of SR' in its field of fractions is weakly unramified over R'.

Since the valuation rings under consideration in Theorem 3.4.4 are complete and SR' is a finite extension of S, the ring $\overline{S'}$ is a discrete valuation ring.

Remark 3.4.5. The condition $(*)$ is always satisfied in geometric situations, because such residue fields k_S are finitely generated over k; for example, if S is the local ring of a (formal) R-model of an algebraic or rigid analytic space over the field of fractions $Q(R)$.

What is the application of this result?

Example 3.4.6. Let $X := \mathrm{Spf}(A) \to \mathrm{Spf}(R)$ be a formal affine curve with smooth generic fiber and assume X to be normal. More generally, one can consider a normal topologically finite type algebra A over R with geometrically reduced generic fiber. Let $x \in X$ be a generic point of the special fiber; this corresponds to a prime ideal $\mathfrak{p} \subset A$ of height 1.

Thus, $R \subset S := A_{\mathfrak{p}}$ is a dominating extension of discrete valuation rings. Moreover, let $X \to \mathbb{D}^d := \mathrm{Spf}\, R\langle\xi\rangle$ be a Noether normalization. Then, in a similar way as above, the generic point of the special fiber of $\mathbb{D}^d$ over R gives rise to a discrete valuation ring $S_0 := R\langle\xi\rangle_{\mathfrak{q}}$ which is weakly unramified over R. Thus, $R \subset S_0 \subset S$ is an extension as considered by Epp. Then Epp's result yields a R'-model of $X_K \otimes_K K'$, whose special fiber is generically geometrically reduced.

This example can be used to show the Reduced Fiber Theorem 3.4.2 without too much effort; only the additional tool of Lemma 3.4.17 is needed, which is not so difficult to verify.

Epp's proof is split into the equal and the unequal characteristic case. The method of Epp consists in explicit calculations of equations.

Note first that, in the *case of residue characteristic* 0 or the case, where the multiplicity N of the special fiber of S over S_0 is prime to the residue characteristic, the statement directly follows from the famous lemma of Abhyankar [40, Exp. X, 3.6]; in fact, after the base change by the extension $R \subset R' := R[\sqrt[N]{\pi}]$ the assertion is fulfilled for a suitable N.

Let us now consider the *unequal characteristic case*: Using Kummer theory and the lemma of Abhyankar, which works also in Epp's situation, one can reduce to the case, where the degree $[S : S_0] = p^r$ is a power of p and, moreover, that $S_0 \subset S$ is Galois and p-cyclic; and finally to the case

$$S = S_0[\zeta]/\big(\zeta^p - u\big),$$

where $u = 1 + m \in S_0$ with $m \in \mathfrak{m}_{S_0}$ is a principal unit. Then there is a statement which precisely characterizes, when S/S_0 is weakly unramified. There is the following lemma of Epp [25, 1.4].

Lemma 3.4.7. *In the above situation assume that a primitive p-th root of unity is contained in R. Let $e \in \mathbb{N}$ be the absolute ramification index*; *i.e.*, $\pi^e \sim p$ *up to a unit, and $e_1 := [e/(p-1)]$. Assume $\zeta^p = 1 + a\pi^{rp}$ with $rp > 0$ and a unit $a \in S_0^\times$. Then $S_0 \subset S$ is weakly unramified if one of the following conditions is satisfied*

(1) $0 < rp < e + e_1$ *and* $\overline{a} \in \ell - \ell^p$,
(2) $rp \geq e + e_1$.

Then Epp uses Hensel's expansion of principal units

$$u = \big(1+a_1\pi^1\big)\cdot\ldots\cdot\big(1+a_{e+e_1-1}\pi^{e+e_1-1}\big)\cdot(1+u_\infty) = (1+u_0)\cdot(1+u_1)\cdot(1+u_\infty),$$

where $a_i \in S_0$ are liftings of $\overline{a}_i \in \ell$ and $u_\infty \in R\pi^{e+e_1}$. The term u_0 collects all factors, where $\overline{a}_i$ belongs to $\ell - \ell^p$, and u_1 collects all terms with $\overline{a}_i \in \ell^p - k$. The expansion can be stopped after $e+e_1$, because one can extract the p-th root from the higher terms. Using Lemma 3.4.7, one can settle the term $(1+u_0)$ after extending R by $R' := R[\sqrt{\pi}]$. The term $(1+u_1)$ can be dealt with in a similar way by reducing the exponent $j(i)$ in $a_i \in \ell^{p^{j(i)}}$. The term $(1+u_\infty) \in R$ will be canceled after the base change to $R' := R[\sqrt[p]{1+u_\infty}]$. The only extensions of R used here are of type $t^{p^r} = \pi$ and $t^p = u_\infty$.

Let us now consider the *equal characteristic case* $\operatorname{char}(R) = p \neq 0$. In this case, one has $R = k[[\pi]]$ and $S_0 = \ell[[\pi]]$. One can easily reduce to the case $\ell^{p^\infty} = k$. Since one can assume that S/S_0 is Galois and p-cyclic, then by Artin-Schreier theory one has $S = S_0[[\zeta]]$, where ζ satisfies an equation of type

$$\zeta^p - \zeta = a_{-N}\pi^{-N} + \cdots + a_{-1}\pi^{-1} + a_0 \quad \text{with } a_i \in \ell,\ N \in \mathbb{N}.$$

If $N = 0$, the extension $S_0 \subset S$ yields an unramified extension of R.

If $N \neq 0$ and all $a_\nu \in k$, the extension $S_0 \subset S$ is associated to an equation of type $\zeta^p - \zeta = d + a_0$ with d contained in the field of fractions of R and a_0 in R. If $a_0 \in k$, the equation stems from a finite separable extension of R and we are done. If $a_0 \notin k$, one first performs the extension of R given by the equation $z^p - z = d$. Then the defining equation becomes the form $z^p - z = a_0$ which is unramified.

Now assume that there is an $\nu < 0$ with $a_\nu \in \ell - k$. Then, after an extension $R' = R[t]$ of the base ring with $t = \pi^{p^r}$ for a suitable $r \in \mathbb{N}$, one can transform the situation in such a way that $a_{-N} \in \ell - \ell^p$. Thus, one can rewrite the defining equation in the form

$$\zeta^p - \zeta = t^{-N}(a_{-N} + dt) \quad \text{with } d \in S_0,$$

where the integer N is divisible by p, so that $N = pq$. If one puts $\eta := t \cdot \zeta$, one obtains

$$\eta^p - \eta \cdot t^{(p-1)q} = a_{-N} + dt \quad \text{with } d \in S_0.$$

Modulo t, the equation becomes $\eta^p = a_{-N}$. Since $a_{-N} \notin \ell^p$, the equation is irreducible over ℓ, and so the extension $S_0 \subset S$ is weakly unramified. □

3.4.3 The Natural Approach

In this section we will give a further approach to show the Reduced Fiber Theorem. We call this approach *natural*, since it follows a naive idea explained after Remark 3.4.19. This also works in the relative situation, but it is rather technical; cf. [14, Part IV, 2.1]. The case of relative dimension 1 will be used for proving the uniformization of abeloid varieties in Sect. 7.6. Therefore we mainly focus on this case. It should be remarked that even in the approach of Grauert-Remmert the proof is done by reduction to the case of curves. In the following let R be a complete valuation ring of height 1 and set $S := \operatorname{Spf} R$.

Theorem 3.4.8 (Relative reduced fiber theorem). *Let T be an admissible formal R-scheme and $X \to T$ a quasi-compact morphism of admissible formal R-schemes* *such that X/T is flat and $X_{\mathrm{rig}}/T_{\mathrm{rig}}$ has reduced geometric fibers, equidimensional of dimension d. Then there exists a commutative diagram of admissible formal T-schemes*

$$\begin{array}{ccccc} X & \longleftarrow & X' & \longleftarrow & Y' \\ \downarrow & & \downarrow & \swarrow & \\ T & \longleftarrow & T' & & \end{array}$$

such that

(i) *$T' \to T$ is a rig-étale cover,*
(ii) *$X' := X \times_T T'$,*
(iii) *$Y' \to X'$ is a finite rig-isomorphism,*
(iv) *$Y' \to T'$ is flat and has reduced geometric fibers.*

A morphism $X \to T$ has *equidimensional fibers of dimension d* if every fiber is non-empty and of pure dimension d. For $T = \mathrm{Spf}\, R$ this theorem settles the absolute case of Theorem 3.4.2 and covers the Finiteness Theorem 3.1.17 for algebraically closed fields by Proposition 3.4.1. In the following we explain the main steps and ideas of the proof in the case of relative dimension 1. We remind the reader that the proof in the general case makes use of the 1-dimensional case. Some parts work in arbitrary relative dimension without more effort. We consider the case of relative dimension $d \geq 1$ until Example 3.4.26.

A measure for the non-reducedness of a fiber is its geometric multiplicity. The *multiplicity* of an irreducible k-scheme X is the length of $\mathcal{O}_{X,x}$, where x is the generic point of X. The *geometric multiplicity* is the multiplicity of $X \otimes_R k'$, where k'/k is a radicial extension by a perfect field k'.

Let us start with two reduction steps in order to focus on the main problem. As in the proof of the flattening technique [14, Part II], an essential tool is also the étale factorization of a formal morphism through a smooth morphism with geometrically irreducible fibers. For our applications we add some more properties.

Proposition 3.4.9. *In the above situation, let $x \in X_0$ be a point of a special fiber and $f : (X, x) \to (T, t)$ a pointed morphism.*

(a) *Then there exists a commutative diagram of formal S-schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{\ u\ } & (X', x') \\ \Big\downarrow f & & \Big\downarrow g \\ & & (Z', z') \\ & & \Big\downarrow h \\ (T, t) & \xleftarrow{\ v\ } & (T', t') \end{array}$$

where

(i) T', Z', X' *are affine and* u, v *are elementary étale*,
(ii) h *is smooth with irreducible geometric fibers of dimension* d,
(iii) g *is finite and* x' *is the only point above* z'.

(b) *If* f *is flat at* x, *then* g *is flat over the generic point of* $Z'(t')$.
(c) *Assume that* $f : X \to T$ *is flat. Let* x *be the generic point of an irreducible component* $X(t)^0$ *of* $X(t) := X \times_T k(t)$ *and let* N *be the geometric multiplicity of* $X(t)^0$. *Then one can choose the factorization as above, where* u, v *are étale such that* g *is finite flat and* $g_* \mathcal{O}_{X'}$ *is a free* $\mathcal{O}_Z$*-module of rank* N. *In particular,* g_0 *is radicial.*

A *morphism* $(T', t') \to (T, t)$ *of pointed schemes* is a morphism sending t' to t. It is called *elementary étale* if it is étale and the residue field extension $k(t') = k(t)$ is trivial.

Proof. The statement is local on (X, x) and (T, t). Therefore we may assume that $X = \operatorname{Spf} A$ and $T = \operatorname{Spf} B$ are affine.

(a) As a first step, we prove the statement for the reduction $X_0 \to T_0$; i.e., in the case of schemes of finite presentation. This case is settled in [82, 1.1.1]. We sketch its proof in the following; we omit the subindex "0".

First we choose a closed specialization ξ of x in the fiber $X(t)$; so we have $\dim A_\xi \otimes k(t) = d$ and $\dim A_x \otimes k(t) = r \le d$. Then there exists a system of elements $(a_1, \dots, a_d)$ in A such that their images in $A(t) := A \otimes_B k(t)$ is a system of parameters of the local ring $A(t)_\xi$ and $(a_1, \dots, a_r)$ of the ring $A(t)_x$. Now consider the morphism $g : X \to Z := \mathbb{A}^d_T$ given by $(a_1, \dots, a_d)$ and set $z := g(x)$. Then g is quasi-finite at x and ξ.

The closure of $z := g(x)$ in $Z \otimes k(t)$ is geometrically irreducible and reduced, because it is the locus of the coordinate functions $\zeta_1, \dots, \zeta_r$ of $\mathbb{A}^d_T$. Thus, there exists an elementary étale base change $(Z_1, z_1) \to (Z, z)$ such that there exists a connected component (X_1, x_1) of $X_1 := X \times_Z Z_1$ such that $(X_1, x_1) \to (Z_1, z_1)$ is finite and x_1 is the only point above z_1. In particular, $(Z_1, z_1), \to (T, t)$ is smooth and the fiber $Z_1(t)$ is connected. Since the point z and hence the point z' are geometrically integral, the scheme $Z_1(t)$ is geometrically integral of dimension d.

There exists an étale surjective base change $(T_1, t_1) \to (T, t)$ such that there exists a section $\sigma : (T_1, t_1) \to (Z_1, z_1)$. The union of the connected components of the fibers of $Z_1 \times_T T_1 / T_1$, which meet σ, is an open subscheme U_1 of $Z_1 \times_T T_1$ due to [39, IV, 15.6.5]. Furthermore, there exists an elementary étale neighborhood $(T', t') \to (T, t)$ such that $T_1 \times_T T' \to T'$ is finite in a open neighborhood of t_1. Replacing (T_1, t_1) by that neighborhood, we may assume that $(T_1, t_1) \to (T', t')$ is finite étale.

Then the projection $p : Z_1 \times_T T_1 \to Z_1 \times_T T'$ is finite and

$$Z' := Z_1 \times_T T_1 - p(Z_1 \times_T T_1 - U_1 \times_T T_1)$$

is an open neighborhood of $z' := z_1 \times_T T'$. Set $X' := X_1 \times_{Z_1} Z'$ and $x' := (x_1, z')$. So

$$\left(X', x'\right) \to \left(Z', z'\right) \to \left(T', t'\right)$$

fulfills the claim in the case of schemes of finite presentation.

The case of admissible formal schemes follows from the lifting properties of étale and smooth morphisms. In fact, the morphism u_0 and v_0 lift to formal elementary étale morphisms due to the lifting property of étale maps. The morphism h_0 lifts as well, because h_0 is affine. Since $v \circ h : Z' \to T$ is smooth and affine, the morphism $f \circ u : X' \to T$ factorizes through $v \circ h$ by the lifting property of smooth formal morphisms.

(b) If f is flat, then $g : X' \to Z'$ is a morphism of flat formal T-schemes of topologically finite presentation. On the fiber of t the scheme $Z'(t)$ is smooth, then $X'(t)$ is generically flat over $Z(t)$ due to [39, IV$_2$, 6.9.1]. By the criterion of flatness on fibers [39, IV$_3$, 11.3.11], the morphism g is flat at the generic point of $Z'(t)$.

(c) For the proof of (c) we will study the case of schemes $X_0 \to T_0$ more carefully than in (a). In the following we omit the subindex "0" as in the proof of (a). In the following let X be an affine scheme over T, where T is the spectrum of an affine scheme over R_0 of finite presentation. So $X(t)$ is the spectrum of an affine $k(t)$-algebra of finite type. Note that the underlying reduced $k(t)$-scheme $X(t)_{\mathrm{red}}$ of $X(t) := X \times_T \operatorname{Spec} k(t)$ is not necessarily geometrically reduced; but there is a finite radicial extension $k'/k(t)$ such $X(t)'_{\mathrm{red}}$ is geometrically reduced, where $X(t)' = X(t) \otimes_{k(t)} k'$. The multiplicity of $X(t)'$ is the geometric multiplicity of $X(t)$.

Let Ω be the module $\Omega^1_{X/T}$ of relative differential 1-forms and Ω' be the pull-back of Ω to X'. There are canonical surjections

$$\Omega' \twoheadrightarrow \Omega^1_{X(t)'/k'} \twoheadrightarrow \Omega^1_{X(t)'_{\mathrm{red}}/k'}.$$

Thus, after replacing X by a dense open subscheme, there exist functions $a_1, \ldots, a_d$ on X such that the pull-backs of the derivatives $da_1, \ldots, da_d$ to X'_{red} give rise to a basis of the sheaf of k'-differential forms on $X(t)'_{\mathrm{red}}$. Thus, $a_1, \ldots, a_d$ induce a map $a : X \longrightarrow \mathbb{A}^d_T$ such that

$$a_{\mathrm{red}} : X(t)'_{\mathrm{red}} \longrightarrow \mathbb{A}^d_{k(t)} \otimes k'$$

is étale. Let α be the generic point of $\mathbb{A}^d_k$ and let $k(\alpha) \subset \ell \subset k(x)$ be the maximal étale subextension of $k(x)/k(\alpha)$. In particular, $k(x)/\ell$ is radicial.

After shrinking X, these field extensions can be realized by a factorization

$$X(t)_{\mathrm{red}} \xrightarrow{g(t)} Z(t) \xrightarrow{h(t)} \mathbb{A}^d_{k(t)}$$

of $a(t)$ such that $Z(t)$ is irreducible with field of rational functions ℓ and such that $Z(t)$ is étale over $\mathbb{A}^d_k$; in particular smooth over $k(t)$. Since $Z(t)$ is étale over $\mathbb{A}^d_k$, the morphism $X_{\text{red}} \to Z$ lifts to a factorization

$$X(t) \longrightarrow Z(t) \longrightarrow \mathbb{A}^d_{k(t)}.$$

Without loss of generality we can assume that $X(t)$ and $Z(t)$ are affine and so small that $X(t) \to Z(t)$ is finite, and also flat due to the generic flatness over an integral scheme [39, IV_2, 6.9.1]. Thus, $\mathcal{O}_{X(t)}(X(t))$ is a locally free $\mathcal{O}_{Z(t)}(Z(t))$-module of a certain rank r. Then r is equal to the geometric multiplicity N. In particular, $X(t)'_{\text{red}} \to Z \otimes k'$ is an isomorphism.

Now we lift $g(t)$ to a morphism $X \to Z$. We already have the morphism $(X, x) \to (\mathbb{A}^d_T, \alpha)$. Then we perform the étale base change $X' := X \times_{\mathbb{A}^d_T} Z$. Thus, we obtain an étale neighborhood $(X', x') \to (X, x)$ and the projection $(X', x') \to (Z, z)$ which is quasi-finite on the fiber of t. After a further étale base change $(Z', z') \to (Z, z)$ we may assume that $(X', x') \to (Z', z')$ is finite. Thus, we arrive at a factorization

$$(X', x') \xrightarrow{\ g\ } (Z', z') \xrightarrow{\ h\ } (T, t)$$

where h is smooth and g is finite flat at x' due to [39, IV_3, 11.3.11].

After shrinking Z' we may assume that $g_*\mathcal{O}_{X'}$ is a free $\mathcal{O}_{Z'}$-module of rank N. After a final étale base change $(T', t') \to (T, t)$ we may assume that $Z' \to T'$ admits a section and hence, as in the proof of (a), that the fibers of Z'/T' are geometrically irreducible. Since $g(t)$ is radicial, g remains radicial in a neighborhood of z.

Finally, one lifts the whole situation to the formal level as described in the proof of (a) and (b). □

Corollary 3.4.10. *In the situation of Proposition 3.4.9, let $X \to T$ be a flat and quasi-compact. For $N \in \mathbb{N}$ the set of points*

$$E(X/T, N) := \left\{ t \in T_0;\ \begin{matrix} \textit{geometric multiplicity of } X(t) \textit{ is less} \\ \textit{or equal to } N \textit{ at all generic points of } X(t) \end{matrix} \right\}$$

is open in T_0.

Proof. Since the image of a flat map is open, the assertion follows from Proposition 3.4.9(c). □

There are further improvements of Proposition 3.4.9.

Proposition 3.4.11. *Let $f : X \to T$ be a morphism of quasi-compact admissible formal R-schemes. Assume that f is rig-flat and its rig-fibers are geometrically reduced and equidimensional of dimension $d \geq 0$.*

Then, after replacing T by a suitable admissible formal blowing-up and X by the strict transform with respect to this blowing-up, there exists a commutative diagram

$$
\begin{array}{ccccc}
X' & \hookleftarrow & U' & \longleftarrow & Y' \\
\downarrow & & \downarrow & & \downarrow g \\
X & \hookleftarrow & U & & Z \\
\downarrow f & & & & \downarrow h \\
T & = & = & = & T,
\end{array}
$$

where

(i) *$X' \to X$ is an admissible formal blowing-up and $U \subset X$ is a T-dense open*,
(ii) *$U' := X' \times_X U \to U$ is a finite rig-isomorphism*,
(iii) *$Y' \to U'$ is étale with T-dense image*,
(iv) *$g : Y' \to Z$ is finite*, *rig-étale*, *and flat*,
(v) *$h : Z \to T$ is smooth with equidimensional fibers of dimension d*.

An open subset $U \subset X$ is called *T-dense in X* if the fiber $U(t) \subset X(t)$ is dense in $X(t)$ for all points $t \in T$.

Proof. Due to Corollary 3.3.8 we may assume that $f : X \to T$ is flat and hence faithfully flat, because f_{rig} is surjective. Let $\Omega := \Omega^1_{X/T}$ be the sheaf of relative differential forms of degree 1. Since the rig-fibers are geometrically reduced, the points in the fiber $X(t)$, where $\Omega^1_{X(t)/t}$ is generated by d elements, is open dense in $X(t)$, for every rig-point t of T.

Therefore, the subset consisting of the points $x \in X_{\mathrm{rig}}$, where Ω_x is generated by d elements is open dense in every rig-fiber; more precisely its complement is a closed analytic subset D_K of X_{rig} which is rare in every rig-fiber. Again after replacing T by a further admissible formal blowing-up, we may assume that the closure D of D_K in X is flat in dimension d; i.e., it is of relative dimension $\leq (d-1)$. Thus, there exists a T-dense open subscheme U of X such that U/T is rig-smooth. Thus, we may replace U by X and assume that X/T is rig-smooth.

Then Ω is a locally rig-free $\mathcal{O}_X$-module of rank d. By Theorem 3.3.7 there exists an admissible formal blowing-up $p : X' \to X$ such that the strict transform $\mathcal{M}'$ of Ω is a locally free $\mathcal{O}_{X'}$-module of rank d. Note that $\mathcal{M}'$ can be different from $\Omega^1_{X'/T}$; but we still have a surjective map $p^*\Omega^1_{X/T} \to \mathcal{M}'$, which is locally split and rig-bijective, and, moreover, a canonical morphism $p^*\Omega \to \Omega' := \Omega^1_{X'/T}$ which is rig-bijective. After replacing T by an admissible blowing-up $T' \to T$, we may assume that $X' \to T$ is faithfully flat by Corollary 3.3.8. Thus, we have a proper morphism $p : X' \to X$ of formal schemes with equidimensional fibers of dimension d. Then there exists a T-dense open subscheme U of X such that p is finite over U, because the subset, where the dimension of the fibers of p_0 is 0, is open dense in X_0 due to [39, IV_3, 13.1.5]. Now put $U' := X' \times_X U$.

The remaining assertions are local on T and U'. Let t be a point of T_0, x' a generic point of $U'(t)$ and $x := p(x') \in U(t)$. Then there exist functions $a_1, \ldots, a_d$ defined in an open neighborhood of x in U such that the images of $p^*(da_1), \ldots, p^*(da_d)$ in $\mathcal{M}'$ give rise to a $k(x')$-basis of $\mathcal{M}' \otimes_T k(x')$. After shrinking U, we obtain the map

$$a = (a_1, \ldots, a_d) : U \longrightarrow \mathbb{A}^d_T.$$

After modifying $a_1, \ldots, a_d$ by elements of the square of the maximal ideal of x, we may assume that a is quasi-finite at x.

Let $a' : U' \to \mathbb{A}^d_T$ be the induced morphism. By the construction a' is quasi-finite, since $U' \to U$ is finite, and rig-étale. Over a suitable étale neighborhood $Z \to \mathbb{A}^d_T$ of the generic point of $\mathbb{A}^d_{k(t)}$ there exists an open neighborhood Y' of $U' \times_{\mathbb{A}^d_T} Z$ of x' such that the induced map $g : Y' \to Z$ is finite and rig-étale. The morphism $Y' \to Z$ is a morphism of flat formal T-schemes and, moreover, Z is a smooth formal T-schemes. Thus, $Y' \to Z$ is flat on an open dense subset of each fiber; cf. [39, IV_2, 6.9.1]. Due to the criterion of flatness on fibers [39, IV_3, 11.3.11], the map $Y' \to Z$ is flat over a T-dense open subscheme of Z. After shrinking Y' and Z the assertion follows. □

Proposition 3.4.11 has some useful consequences.

Corollary 3.4.12. *In the situation of Proposition 3.4.11 the morphism f admits sections locally for a rig-étale cover.*

Proof. Consider the situation of the assertion in Proposition 3.4.11. The assertion is local on T. The morphism $Z \to T$ admits a section $\sigma : T' \to Z$ after an étale base change $T' \to T$ of a neighborhood of t. Now $T'' := Y' \times_Z T' \to T'$ is finite and rig-étale and σ induces a section $\sigma'' : T'' \to Y'$ via the first projection $T'' \to U'$ composed with $Y' \to X$. Thus, we see that $X \to T$ admits a section after a rig-étale surjective base change. □

Corollary 3.4.13. *In the situation of Proposition 3.4.11 let t be a point of T_0 and let N be the maximum of the geometric multiplicities occurring in the fiber $Y'(t)$. Then there exists an étale neighborhood $T' \to T$ of T such that, after replacing X and Z by T'-dense open subschemes, all conditions of Proposition 3.4.11 are satisfied and, in addition,*

(iv) *g is finite, flat, rig-étale and $g_*\mathcal{O}_{Y'}$ is locally free of rank N,*
(v) *$h : Z \to T$ is smooth with irreducible fibers of dimension d,*
(vi) *there exists a monogenous $\mathcal{O}_Z$-subalgebra of $g_*\mathcal{O}_{Y'}$ which coincides with $g_*\mathcal{O}_{Y'}$ on the rigid part.*

Proof. The new conditions are local on Y', Z and T. Let $t \in T_0$ be a point and $y' \in Y'(t)$ by the generic point of the fiber. Then apply Proposition 3.4.9 to the morphism $g : Y' \to Z$. Thus, we obtain an étale neighborhood $Y'' \to Y'$ of y' and an étale neighborhood $Z' \to Z$ of $g(y')$ such that the induced map $g' : Y'' \to Z'$ satisfies condition (iv). After an étale base change $T' \to T$ and replacing Z' by an open subscheme Z'' of $Z' \times_T T'$, the induced map $h' : Z'' \to T$ satisfies condition (v).

Thus, it remains to show condition (vi). Let $V \to Z$ be the vector bundle corresponding to the locally free $\mathcal{O}_Z$-module $g_*\mathcal{O}_{Y'}$. For every formal Z-scheme $Z' \to Z$ and every section $\sigma : Z' \to Z$ there is a discriminant $\mathfrak{d}(\sigma)$. Indeed, let $b_1, \ldots, b_N$ be a basis of $g_*\mathcal{O}_{Y'}$ over $\mathcal{O}_Z(Z)$ and $b'_1, \ldots, b'_N$ its dual basis. Then σ associates to each b'_i a function $\sigma^* b'_i \in \mathcal{O}_{Z'}$ and hence an element

$$b := \sigma^* b'_1 \cdot b_1 + \cdots + \sigma^* b'_N \cdot b_N \in g_*\mathcal{O}_{Y'} \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z'}.$$

Then the discriminant is given by

$$\mathfrak{d}(\sigma) := \det\bigl(b^0, \ldots, b^{N-1}\bigr) \in \mathcal{O}_{Z'}.$$

The locus of $\mathfrak{d}(\sigma)$ gives rise to a closed subscheme $\Delta(\sigma)$ of Z'.

We can apply this to the universal section $\tau := \mathrm{id} : Z' := V \to V$. The locus $\Delta(\tau)$ is a hypersurface of V, because g is rig-étale. So $\Delta(\tau)$ does not contain any rigid fiber of Y'/Z. Thus, $\Delta(\tau)$ is rig-flat over T.

Due to Corollary 3.3.8 there exists an admissible formal blowing-up $T' \to T$ such that the strict transform of $\Delta(\tau)$ is flat over T'; i.e., the relative dimension of $\Delta(\tau')/Z$ is at most $d - 1$. In particular, $\Delta(\tau)$ is rare in every fiber of V over Z. Since a vector bundle has many local sections, it admits such ones which do not meet $\Delta(\tau)$.

Thus, we see that, locally on T, there exists a T-dense open subscheme Z' of Z and a section $\nu : Z' \to (V - \Delta(\tau))$ depending on Z'. Then $\sigma := \tau \circ \nu : Z' \to V$ gives rise to a section f of $g_*\mathcal{O}_{Y'}$ over $Y' \times_Z Z'$ which generates $g_*\mathcal{O}_{Y'}$ as an $\mathcal{O}_Z$-algebra over the rigid part of Z'. Now we replace Y by the formal spectrum $\operatorname{Spf} \mathcal{O}_Z[f]$. Then $g : Y \to Z$ has all the required properties after shrinking Z to a T-dense open subscheme. □

Remark 3.4.14. Let $W := \operatorname{Spf} \mathcal{O}_Z[f]$ be the admissible formal scheme associated to the subalgebra of $g_*\mathcal{O}_{Y'}$ mentioned in Corollary 3.4.13(vi). If one finally arrives at a morphism $W' \to W$ which is a finite rig-isomorphism with geometrically reduced fibers over T, then W' dominates the Y' mentioned in Theorem 3.4.8 after a suitable admissible blowing-up of T. This will follow from Lemma 3.4.17(a) below.

Proposition 3.4.9 can be used to reduce the proof of Theorem 3.4.8 to an étale cover of X with T-dense open image in X. We start with some preparations.

Lemma 3.4.15. *Let $Z \to T$ be a surjective smooth morphism of admissible formal S-schemes with irreducible geometric fibers and let $\iota : U \hookrightarrow Z$ be the immersion of an open subscheme with a T-dense image in Z.*

(a) *If* $f \in \mathcal{O}_Z(Z)$ *does not vanish identically on any fiber of* Z/T, *then* f *is a non-zero divisor on the infinitesimal neighborhood* Z_λ *for all* $\lambda \in \mathbb{N}$.
(b) *The canonical restriction map* $\mathcal{O}_Z \longrightarrow \iota_*(\mathcal{O}_Z|_U) \cap \mathcal{O}_{Z_{\mathrm{rig}}}$ *is bijective.*

Proof. (a) We may assume that $Z = \operatorname{Spf} C \to T = \operatorname{Spf} B$ is a morphism of affine formal schemes and let $B_\lambda \to C_\lambda$ be the associated morphism. Since we may perform an étale base change, we may assume that there is a section σ_λ of $Z_\lambda \to T_\lambda$ outside the locus of f. Then, by a standard argument on direct limits, we reduce to a Noetherian bases keeping the smoothness and the section. Furthermore, it is possible to keep the connectedness of the fibers due to [39, IV_3, 15.6.5]. Thus, we may assume that B_λ is Noetherian. Since $B_\lambda \to C_\lambda$ is flat with irreducible geometric fibers, the associated prime ideals of C_λ are the generic points of certain fibers of $Z_0 \to T_0$ due to [17, IV_2, no 6, théorème 2]. Thus, f does not belong to the associated prime ideals, and hence f is not a zero-divisor.

(b) We may assume that $Z = \operatorname{Spf} C$ is affine. The injectivity of the restriction follows from (a). In fact, if $g \in C$ with $g|_U = 0$ and hence $g|_{U_\lambda} = 0$ for $\lambda \in \mathbb{N}$, then for each $\lambda \in \mathbb{N}$ there exists an $f \in C$ which does not vanish on a T-dense open subset of U such that $fg|_{Z_\lambda} = 0$. By (a) we obtain that $g|Z_\lambda = 0$. Due to the separateness of the topology on C we obtain $g = 0$.

Now consider a element $h \in \mathcal{O}_Z(U) \cap C_{\mathrm{rig}}$. Then there exists a smallest $\lambda \in \mathbb{N}$ such that $\pi^\lambda h \in C$. If $\lambda = 0$, then there is nothing to show. If $\lambda \geq 1$, then $\pi^\lambda h = 0$ on U_0. Then there exists a function $f \in C$ such that $f\pi^\lambda h \in \pi C$ and f is invertible on a T-dense open subset $U' \subset U$. Since f is a non-zero divisor on Z_0 due to (a), we see $\pi^\lambda h \in \pi C$. Since C has no π-torsion, $\pi^{\lambda-1} h \in C$. This contradicts the minimality of λ. □

Lemma 3.4.16. *Let* $Z \to T$ *be a quasi-compact smooth formal* R*-morphism and* $Y \to Z$ *a proper morphism of admissible formal schemes, which is a rig-isomorphism. Then there exists an admissible formal blowing-up* $T' \to T$ *such that, after the base change by* $T' \to T$ *and replacing* Y *by its strict transform, there exists a* T*-dense open subscheme* U *of* Z *such that* $Y \times_Z U \to U$ *is an isomorphism.*

Proof. There exists an admissible blowing-up $T' \to T$ such that the strict transform Y' of Y is flat over T'. After replacing T by T' we may assume that Y is flat over T. Now look at the morphism $Y(t) \to Z(t)$ of the fibers of a closed point t of T_0. Due to the generic flatness there is a dense open subscheme $V(t)$ of $Z(t)$ such that $Y(t) \to Z(t)$ is flat over $V(t)$; cf. [39, IV_2, 6.9.1]. Due to the criterion of flatness on fibers [39, IV_3, 11.3.11], the map $Y \to Z$ is flat over the generic point of $Z(t)$ of every closed point of T. Since the locus, where $Y \to Z$ is flat, is open due to [39, IV_3, 11.3.1], there exists a T-dense open subscheme $U \subset Z$ such that $Y \to Z$ is flat over Z. In loc. cit. the statement is presented only for morphisms of finite presentation, but by the usual techniques reducing to the map of the level $Y_0 \to Z_0$ it can be carried over to our situation. By reason of dimensions the map $Y \to Z$ is quasi-finite and hence finite over U, because it is proper. Since it is a rig-isomorphism and flat over U, it is an isomorphism over U. Indeed, it can be checked after a faithfully flat base change. Thus, we perform the base change by $U' := Y \times_Z U$. Then we have the tautological section of $U'' := U' \times_U U' \to U'$. Since $U''_{\mathrm{rig}} \to U'_{\mathrm{rig}}$ is an isomorphism, we see that $U'' \to U'$ is an isomorphism. □

Lemma 3.4.17. *Let $X \to T$ be a morphism of admissible formal schemes, which is flat and has reduces geometric fibers. Then we have the results:*

(a) *The smooth locus W of X/T is open and T-dense in X.*
If $U \subset W$ is a T-dense open subscheme, then the canonical map

$$\mathcal{O}_X \longrightarrow \iota_*(\mathcal{O}_X|_U) \cap \mathcal{O}_{X_{\mathrm{rig}}}$$

is bijective, where $\iota : U \to X$ is the inclusion map.
If f is a rigid analytic function on X_{rig} with $|f|_{X_{\mathrm{rig}}} \leq 1$, then there exists an admissible blowing-up $T' \to T$ such that f is defined on $X \times_T T'$.

(b) *If there exists an étale surjective formal morphism $V \to X$ and the assertion in Theorem 3.4.8 holds for V, then it holds also for X.*

(c) *If there exists a T-dense open subset $U \subset X$ such that the assertion in Theorem 3.4.8 holds for U, then the assertion holds for X.*

The additional assertion in (a) is the relative version of Proposition 3.4.1.

Proof. (a) The case, where $X \to T$ is smooth, was settled in Lemma 3.4.15(b).

Since $X \to T$ is flat, the morphism $X \to T$ is smooth at a point x of X if and only if the map of the fibers $X(t) \to \operatorname{Spec} k(t)$ is smooth at x. Since the fiber is geometrically reduced, the smooth locus is open dense in every fiber. Since the smooth locus of X/T is open in X, the set W is T-dense and open in X.

The further assertion can be checked étale locally. So we may assume that $X \to T$ admits a factorization

$$X := \operatorname{Spf} A \xrightarrow{\ g\ } Z = \operatorname{Spf} C \xrightarrow{\ h\ } T = \operatorname{Spf} B$$

as was established by Proposition 3.4.9. Let $V \subset Z$ be a T-dense open subscheme of Z over which g is flat; cf. Proposition 3.4.9(b). We may assume that $U = g^{-1}(V)$. Then A is finite C-module and locally free over V. Set $\mathcal{A} := g_*\mathcal{O}_X$. There exists an exact sequence of C-linear maps

$$0 \to N \to C^r \to A^* := \operatorname{Hom}_C(A, C) \to 0.$$

Then consider the induced exact sequence

$$0 \to A^{**} := \operatorname{Hom}_C\big(A^*, C\big) \to C^r = \operatorname{Hom}_C\big(C^r, C\big) \to N^* := \operatorname{Hom}_C(N, C).$$

Since N^* has no π-torsion, A^{**} is a finitely generated C-module due to Theorem 3.2.1. Composing the injection $A^{**} \hookrightarrow C^r$ with the canonical embedding $A \to A^{**}$, we end up with a map $j : A \to C^r$ and set $Q := C^r/j(A)$. Since

$\mathcal{A}$ is a locally free $\mathcal{O}_Z$-module over V, the morphism $A \to A^{**}$ is an isomorphism over V and Q is locally free over V. Then we assert that $A \to C^r$ is universally injective. To verify the assertion, it suffices to show that the canonical map $A \otimes_T k(t) \to C^r \otimes_T k(t)$ is injective for all closed point t of T_0. The latter is clear, because the fibers of X/T do not have embedded components and the map $g_* \mathcal{O}_X|_V \to \mathcal{O}_Z^r|_V$ splits locally over V. Thus, we see that the quotient $Q := C^r/j(A)$ is flat over T. Thus, the map $A/\pi A \to C^r/\pi C^r$ is injective.

Now the assertion follows easily. Indeed, if $h \in \mathcal{O}_X(g^{-1}(V)) \cap A_{\text{rig}}$, then $j(h) \in C$ by Lemma 3.4.15(b). Furthermore, if $h \neq 0$, then there exists a minimal $r \in \mathbb{N}$ with $\pi^r h \in A$. In particular, we have that $\pi^r j(h) \in \pi^r C$. Since the map $A/\pi A \to C/\pi C$ is injective, we obtain $r = 0$, and hence $h \in A$.

Concerning the last assertion, look at the T-dense open part Z of X, where X is smooth over T. Let $Y \to Z$ be the morphism associated to $\mathcal{O}_Z \to \mathcal{O}_Z[f]$. Then $Y \to Z$ is a finite rig-isomorphism. Due to Lemma 3.4.16 there exists an admissible blowing-up $T' \to T$ and a T-dense open subscheme $U \subset Z$ such that f is defined on $U \times_T T'$. Thus, the assertion follows from what we have proved already.

(b) Assume that there is an étale surjective morphism $V \to X$ such that the assertion of Theorem 3.4.8 holds for $V \to T$. Thus, there exists a rig-étale cover $T' \to T$ and a morphism $W' \to V' := V \times_T T'$ which is a finite rig-isomorphism such that $W' \to T'$ is flat with reduced geometric fibers. Due to the additional assertion in (a), after a suitable admissible formal blowing-up of T there exists a canonical descent datum on the finite morphism $W' \to V'$ with respect to the étale surjective morphism $V' \to X'$. Since this is a descent problem of finitely generated modules, such a descent is effective. Therefore, the finite formal V'-scheme W' descends to a finite formal X'-scheme Y'. Properties like geometrically reduced fibers and flatness descend under étale surjective morphisms.

(c) Due to (b) we are free to work étale locally on X. Likewise as in the proof of (a), we may assume that we have a factorization $X \to Z \to T$. Now we proceed as in (a) till we end up with the C-module $Q := C^r/A$. We may replace U by $g^{-1}(V)$ for some T-dense open subscheme V of Z.

But now we only assert that $A_{\text{rig}} \to C^r_{\text{rig}}$ is universally injective; the latter implies that $C^r_{\text{rig}}/A_{\text{rig}}$ is flat over T_{rig}. To verify the assertion, consider a rigid point $t \in T_{\text{rig}}$. It suffices to show that $A \otimes_T K(t) \to C^r \otimes_T K(t)$ is injective. The injectivity is clear, since X/T has geometrically reduced rigid fibers and the map $g_* \mathcal{O}_X|_V \to \mathcal{O}_Z^r$ splits locally over V. Thus, we see that $Q_{\text{rig}} := C^r_{\text{rig}}/A_{\text{rig}}$ is flat over T_{rig}.

Due to the flattening technique in Theorem 3.3.7 there exists an admissible formal blowing-up $T' \to T$ such that the strict transform Q' of Q is flat over T'. Since Q' is a quotient of Q, we have an exact sequence

$$0 \to A' \to C^r \to Q' \to 0$$

together with a canonical map $A \to A'$. Due to Theorem 3.2.1 the C-module A' is finitely generated. The morphism $A \to A'$ is an isomorphism over V and over Z_{rig}. Moreover, A' is flat over T', since Q' and C^r are flat over T'. Since Z/T has geometrically reduced fibers, the fibers of A' have no embedded components.

To finish the proof, we set $X' := \operatorname{Spf} A'$. The algebra structure, which is given over X_{rig} and over V, extends uniquely to an algebra structure on all of A'. Furthermore, X' is finite over X as it is finite over Z and $X' \times_Z V = X \times_Z V$. Thus, X' corresponds to an admissible formal T-scheme over X which is as required. In fact, the geometric fibers of X'/T are generically reduced, because the fibers of V/T are so, and they do not have embedded components. Thus, we see that the fibers of X'/T are geometrically reduced. □

Using the factorization in Corollary 3.4.13 and the descent arguments in Lemma 3.4.17, to prove Theorem 3.4.8 suffices to consider the following situation; cf. Remark 3.4.14.

Notation 3.4.18. In the sequel we will consider the following special case.

$$X = \operatorname{Spf} A \xrightarrow{g} Z = \operatorname{Spf} C \xrightarrow{h} T = \operatorname{Spf} B \ ,$$

where

(1) the fibers over T_0 have geometric multiplicity $\leq N$,
(2) $X \to Z$ is finite, rig-étale and free as module of rank $N \geq 2$,
(3) $\mathcal{O}_X = \mathcal{O}_Z[f]$,
(4) $Z \to T$ is smooth with irreducible geometric fibers of dimension 1.

Moreover, we will consider the characteristic polynomial

$$F(\xi) = \xi^N + c_1 \xi^{N-1} + \cdots + c_N \in C[\xi]$$

of the multiplication by f on A. Due to [18, Chap. IV, §6, Props. 7 and 11] the discriminant of f is

$$\Delta(f) := (-1)^{N(N-1)/2} \mathrm{N}_{X/Z}\big(F'(f)\big) \in C,$$

where $N_{X/Z}$ is norm of A over C.

Since $X \to Z$ is rig-étale, the discriminant $\Delta(f)$ is rig-invertible. If one alters f by $c \in C$ to $f - c = \beta g$ with $g \in A_{\mathrm{rig}}$ satisfying $|g|_{X_{\mathrm{rig}}} \leq 1$ and $\beta \in B$, then β is rig-invertible and the characteristic polynomial of g is

$$G(\xi) = \frac{1}{\beta^N} F(\beta\xi + c) \in C[\xi]$$

due to Lemma 3.4.17 after replacing T by an admissible blowing-up. The discriminant of g satisfies $\Delta_{X/Z}(f) = \beta^{N(N-1)} \Delta_{X/Z}(g)$. In particular, $\Delta(f)$ is a lower bound of the discriminant of g for every transformation $g := (f - c)/\beta$ of the above type after every base change $T' \to T$.

Remark 3.4.19. Let $c \in C$ and $g := (f - c)/\beta \in A_{\mathrm{rig}}$ as in Notation 3.4.18. Let $t \in T_0$ be a closed point with residue field $k(t)$ and let $\overline{k}(t)$ be the algebraic closure of $k(t)$. Put $A' := A[g]$ and $Y := \operatorname{Spf} A'$. Then the following possibilities can happen:

(I) The canonical morphism $C \otimes_B \overline{k}(t) \to (A' \otimes_B \overline{k}(t))_{\mathrm{red}}$ is not an isomorphism at the generic point and hence the geometric multiplicity of the component $X'(t)$ is less than N.

(II) The canonical morphism $C \otimes_B \overline{k}(t) \to (A' \otimes_B \overline{k}(t))_{\mathrm{red}}$ is an isomorphism at the generic point. In other words, there exists an $\overline{c}' \in A' \otimes_B \overline{k}(t)$ such that the pull-back of the polynomial $G(\xi)$ is of type $\overline{G}(\xi) = (\xi - \overline{c}')^N$. In this case, the fiber $Y(t)$ is irreducible and its geometric multiplicity is still N.

The *idea* is now to choose a generic translation by c in order to obtain a minimal β. Such a translation can be performed only after a suitable rig-étale extension $T' \to T$. If one has such a transformation, then the geometric multiplicities of $Y := \operatorname{Spf} \mathcal{O}_Z[g] \to T$ have dropped, because the minimal polynomial of g is no longer a power $(\xi - u)^N$ over T_0. Thus, proceeding by induction on the geometric multiplicity, we achieve the slightly weaker result that the special fiber is geometrically reduced over a dense open part. The final step to make the whole fiber reduced follows from Lemma 3.4.17(c).

Unfortunately, this approach does not work as easily as it looks like, for two reasons. Firstly, one can perform the minimum only in a finite dimensional space over T. Secondly, one can control only the last coefficient of the characteristic polynomial c_N which is $(-u)^N$ if restricted to a geometric closed fiber in the case of multiplicity N. So one has to extract its N-th root.

The *tame case*, where N is prime to the residue characteristic, can be easily handled by the translation c_1/N and the relative maximum principle of Theorem 3.4.23 further down.

Remark 3.4.20. Suppose that the degree N is invertible on S. Then we can replace f by $(f + c_1/N)$ and $F \in C[\xi]$ by $G(\xi) = F(\xi - c_1/N)$. Thus, we may assume $c_1 = 0$, and hence the coefficient of ξ^{n-1} in G vanishes. Then the reduction $\overline{G}(\xi)$ to every fiber cannot be of type $(\xi - \overline{c})^N$ with $\overline{c} \neq 0$. Thus, if we can adjust the norm of $(f - c_1/N)$ via dividing it by some $\beta \in \mathcal{O}_S(S)$, which represents its "maximum on the fibers", we can drop the geometric multiplicity of the fibers. Therefore, in case of residue characteristic 0, if we provide the relative maximum principle of Theorem 3.4.23, then it is enough to kill the geometric multiplicities as asserted and hereby, to complete the proof of Theorem 3.4.8.

As a second tool we need an important result from commutative algebra.

Proposition 3.4.21 (Gruson-Raynaud). *Consider a smooth morphism $\varphi : Z = \operatorname{Spf}(C) \to T = \operatorname{Spf}(B)$ of admissible formal affine schemes. Assume that φ is faithfully flat with irreducible geometric fibers of dimension $d \geq 1$.*

Then C has a topological basis over B; i.e., there exists a topological generating system $E := (e_i; i \in I) \subset C$ such that every $c \in C$ has a unique representation as a convergent series

$$c = \hat{\sum_{i \in I}} b_i \cdot e_i \quad \text{with } b_i \in B \text{ for } i \in I.$$

Proof. Let $\pi \in \mathfrak{m}_R$ with $\pi \neq 0$. Then $C/C\pi$ is a smooth $(B/B\pi)$-algebra of finite type. Due to a remarkable result of Gruson and Raynaud [82, 3.3.1], the flat $(B/B\pi)$-module $C/C\pi$ is projective and hence free as its rank is not finite due to the assumption on the fibers. Thus, there exists a system $E := (e_i; i \in I) \subset C$ such that E gives rise to an $(B/B\pi)$-basis $E_0 := (\overline{e}_i; i \in I) \subset \overline{C} := C/C\pi$ of $C/C\pi$. Then it is easy to see that E is a topological B-basis of C; cf. [14, II, 1.8]. □

Corollary 3.4.22. *In the situation of Proposition 3.4.21 let $\mathcal{I} \subset C$ be a finitely generated ideal of C such that the locus $V(\mathcal{I})$ of $\mathcal{I}$ does not contain the whole fiber $Z(t)$ of any rigid point of t of T. Then $\mathcal{I}$ admits a finitely generated open ideal of coefficients $\mathfrak{b} \subset B$; i.e., $\mathcal{I}$ contains a power of the parameter π and satisfies the following*:

(i) $\mathcal{I} \subset \mathfrak{b}C$
(ii) *for every formal morphism $B \to B'$ with $\mathcal{I} \cdot C' = 0$ follows $\mathfrak{b}B' = 0$ where $C' := C \otimes_B B'$.*

The ideal of coefficients is uniquely determined and finitely generated.

Proof. Let $\mathcal{I}$ be generated by $f_1, \dots, f_n$. Consider the convergent series

$$f_i = \hat{\sum_{j \in I}} b_{i,j} \cdot e_j \quad \text{with } b_{i,j} \in B$$

of f_i with respect to a topological B-basis of C. Then

$$\mathfrak{b} := (b_{i,j}; j \in I, 1 \leq i \leq n)$$

is finitely generated. In fact, for every rig-point t of T there exists an element $b_{i,j}$ with $b_{i,j}(t) \neq 0$. Thus, finitely many of them generate the unit ideal of $B \otimes_R K$ and hence a power of π belongs to $\mathfrak{b}$. Therefore, $\mathfrak{b}$ is open. Since the series presenting f_i converges, the ideal $\mathfrak{b}$ is finitely generated. It is clear that $\mathfrak{b}$ satisfies the properties (i) and (ii). □

With these tools we are now able to prove the relative maximum principle.

Theorem 3.4.23 (Relative maximum principle). *Let T be an admissible formal R-scheme and $X \to T$ a quasi-compact flat formal morphism with non-empty equidimensional fibers of dimension d. Let f be a global function on X such that f is not nilpotent on any rigid fiber of X/T.*

Then there is a commutative diagram of admissible formal T-schemes

$$\begin{array}{ccccc} X & \longleftarrow & X' & \longleftarrow & Y' \\ \big\downarrow & & \big\downarrow & \swarrow & \\ T & \longleftarrow & T' & & \end{array}$$

with the following properties:

(i) *$T' \to T$ is a rig-flat, rig-quasi-finite cover,*
(ii) *$Y' \to X' := X \times_T T'$ is a finite rig-isomorphism,*
(iii) *there exist a rig-invertible function β on T' and a function g on Y' being invertible on an open subset V' of Y', which maps surjectively to T', such that $f = \beta g$ on Y'.*

If, in addition, the rigid fibers of X/T are geometrically reduced, one can choose $T' \to T$ to be a rig-étale cover.

Proof. After replacing T by an étale cover $T' \to T$ we may assume that X admits an étale cover $X' \to X$ such that X' is covered by finitely many open subschemes $X_1, \ldots, X_r$ where each X_ρ admits a factorization

$$X_\rho = \operatorname{Spf} A_\rho \longrightarrow Z_\rho = \operatorname{Spf} C_\rho \longrightarrow T = \operatorname{Spf} B$$

as in Proposition 3.4.9. Moreover, by shrinking Z_ρ to a T-dense open subscheme, we may assume that A_ρ is a locally free C_ρ-module of a certain rank n_ρ; cf. Proposition 3.4.9(b). Then $f|_{X_\rho}$ has a characteristic polynomial over C_ρ; say

$$\xi^{n_\rho} + c_{1,\rho}\xi^{n_\rho - 1} + \cdots + c_{n_\rho,\rho} \in C_\rho[\xi].$$

Let $n := \operatorname{lcm}\{n_1, \ldots, n_r\}$ be the lowest common multiple of $n_1, \ldots, n_r$. By Lemma 1.4.1 the sup-norm on the fiber $F_\rho(z)$ of $X_{\rho,\mathrm{rig}} \to Z_{\rho,\mathrm{rig}}$ for $z \in Z_{\rho,\mathrm{rig}}$ is given by

$$|f|^{n!}_{F_\rho(z)} = \max\left\{\left|c_{i,\rho}(z)\right|^{n!/i}; i = 1, \ldots, n_\rho\right\}.$$

Put $\gamma_{i,\rho} := c_{i,\rho}^{n!/i}$ for $i = 1, \ldots, n_\rho$ and $\rho = 1, \ldots, r$. The B-algebra C_ρ has a topological basis by Proposition 3.4.21. Thus, we can consider the ideal $\mathfrak{b} \subset B$ of the coefficients of all the $\gamma_{i,\rho}$ for $i = 1, \ldots, n_\rho$ and $\rho = 1, \ldots, r$. Since f is not nilpotent on any rigid fiber of X/T, the ideal $\mathfrak{b}$ is open. Since the coefficients of the $\gamma_{i,\rho}$ converge to 0, the ideal $\mathfrak{b}$ is finitely generated.

Now consider the admissible blowing-up $T' \to T$ of $\mathfrak{b}$. Again, we write T in place of T'. Locally on T, the ideal $\mathfrak{b}$ is principal. Working locally on T, we may assume that $\mathfrak{b}$ is principal. Let β' be local generator of $\mathfrak{b}$. Then we can write

$$\gamma_{i,\rho} = \beta' \cdot \gamma'_{i,\rho} \quad \text{with } \gamma'_{i,\rho} \in C$$

for $i = 1, \dots, n_\rho$ and $\rho = 1, \dots, r$. The ideal of coefficients of $(\gamma'_{i,\rho}; i, \rho)$ is the unit ideal, and hence there is a T-dense open subscheme Z'_ρ of some Z_ρ where $(\gamma'_{i,\rho}; i, \rho)$ coincides with the unit ideal of $\mathcal{O}_{Z_\rho}$.

The base change

$$B \to B' := B[\beta]/\big(\beta^{n!} - \beta'\big)$$

induces a faithfully flat base change $T' \to T$, which is rig-finite but in general not rig-étale. One can extract the $n!$-root of β' over T'. So we can write

$$c_{i,\rho}^{n!/i} = \gamma_{i,\rho} = \beta^{n!}\gamma'_{i,\rho},$$

and hence

$$\big(c_{i,\rho}/\beta^i\big)^{n!/i} = \gamma'_{i,\rho}$$

for $i = 1, \dots, n_\rho$. Since β is rig-invertible, one can view $c_{i,\rho}/\beta^i$ as a function on Z_{rig}. Since sup-norm $|c_{i,\rho}/\beta^i| \leq 1$, we may assume $c_{i,\rho}/\beta^i \in C_\rho$ by Lemma 3.4.17(a).

Now we can look at the rigid analytic function $g := f/\beta$ as an element of $A_{\rho,\mathrm{rig}}$. Since g satisfies the integral equation

$$\xi^{n_\rho} + \big(c_{1,\rho}/\beta^1\big)\xi^{n_\rho - 1} + \dots + \big(c_{n_\rho}/\beta^{n_\rho}\big) \in C_\rho[\xi], \tag{$*$}$$

the extension $A_\rho \to A_\rho[g]$ is finite and hence $Y_\rho := \operatorname{Spf} A_\rho[g]$ satisfies the assertion. This holds for every $\rho = 1, \dots, r$.

We see by Eq. ($*$) above that g is invertible over an open part of Y covering T. In fact, $c_{1,\rho}/\beta^1, \dots, c_{n_\rho}/\beta^{n_\rho}$ generate the unit ideal in $\mathcal{O}_{Z_\rho}$ over the T-dense open part Z'_ρ of Z_ρ covering T for some ρ.

Next let us turn to the additional assumptions. Here we proceed in the same way as above till we arrive at the function β' on T'. Now we avoid to extract the $n!$-root of β'. At this level we can already look at the T-scheme $Y_\rho := \operatorname{Spf} A_\rho[g^{n!}]$. As before let Y'_ρ be the open subscheme, where the absolute value function $|g^{n!}|$ takes the value 1. This scheme covers T.

If the rigid fibers of X_ρ/T and hence of Y_ρ/T are geometrically reduced, then there exists a section $\varepsilon : T' \to Y'_\rho$ after a rig-étale cover $T' \to T$; cf. Corollary 3.4.12. Then $\beta := \varepsilon^* f$ is a rig-invertible function on T', and hence one can proceed as above with $g := f/\beta$. □

The relative maximum principle can be generalized to open ideals.

Corollary 3.4.24. *Let T be an admissible formal R-scheme and $X \to T$ a quasi-compact flat formal morphism with non-empty equidimensional fibers of dimension d. Assume that the rigid fibers are geometrically reduced.*

If $\mathcal{I} \subset \mathcal{O}_X$ is a coherent sheaf of open ideals, then there exist a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ T & \longleftarrow & T' \end{array}$$

and two invertible open ideals $\mathcal{I}_{\min}$ and $\mathcal{I}_{\max}$ of $\mathcal{O}_{T'}$ such that:

(i) *$T' \to T$ is a rig-étale cover,*
(ii) *X' is flat over T' and proper over $X \times_T T'$,*
(iii) *the diagram is rig-Cartesian,*
(iv) *the pull-back $\mathcal{I}'$ of $\mathcal{I}$ on X' is locally principal,*
(v) *$\mathcal{I}_{\min}\mathcal{O}_{X'} \subset \mathcal{I}' \subset \mathcal{I}_{\max}\mathcal{O}_{X'}$, and the open parts of X', where $\mathcal{I}_{\min}\mathcal{O}_{X'} = \mathcal{I}'$ and $\mathcal{I}_{\max}\mathcal{O}_{X'} = \mathcal{I}'$, respectively, covers T'.*

Proof. Let $X' \to X$ be the formal blowing-up of $\mathcal{I}$ on X. Using an admissible formal blowing-up on T we may assume that $X' \to T'$ is flat by Corollary 3.3.8. So we can replace X by X' and assume that $\mathcal{I}$ is invertible on X.

We start with the case where $\mathcal{I}$ is principal, say generated by f. Then we apply Theorem 3.4.23 to f to get $\mathcal{I}_{\max} = \beta \cdot \mathcal{O}_{T'}$. Since $\mathcal{I}$ is an open ideal, there exists an $n \in \mathbb{N}$ with $\pi^n \in \mathcal{I}$ and hence $\pi^n = h \cdot f$. Then let $\mathcal{I}^h_{\max} \subset \mathcal{O}_T$ be the max-ideal associated to h. Then $\mathcal{I}_{\min} := \pi^n \mathcal{I}^h_{\max}$ satisfies the required properties.

In the case where $\mathcal{I}$ is invertible, but not necessarily principal, there is an open cover X^* of X such that the pull-back of $\mathcal{I}$ becomes principal. Then an easy descent argument reduces the assertion to the special case we considered. □

If the ideal $\mathcal{I}$ is principal, then it is not necessary to blow up $\mathcal{I}$, and the assertion of Corollary 3.4.24 is valid for $X' := X \times_T T'$. Moreover, if $X \to T$ is smooth with irreducible geometric fibers, an admissible formal blowing-up $T' \to T$ is enough to obtain the assertion of Corollary 3.4.24.

As a third tool we introduce multiplicative filtrations; cf. [14, IV, 3.1].

Definition 3.4.25. Let X and T be an affine schemes and $f : X \to T$ a flat morphism of finite type. A *geometrically reduced multiplicative filtration* on the $\mathcal{O}_T$-algebra $f_*\mathcal{O}_X$ consists of a filtration

$$\mathcal{O}_T = \mathcal{M}_0 \subset \cdots \subset \mathcal{M}_n \subset \mathcal{M}_{n+1} \subset \cdots \subset f_*\mathcal{O}_X$$

by $\mathcal{O}_T$-modules with the following properties

(i) $f_*\mathcal{O}_X = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$,
(ii) $\mathcal{M}_m\mathcal{M}_n \subset \mathcal{M}_{m+n}$ for all $m, n \in \mathbb{N}$,
(iii) each quotient $\mathcal{M}_n/\mathcal{M}_{n-1}$ is a locally free $\mathcal{O}_T$-module of finite type,
(iv) the graded $\mathcal{O}_T$-algebra $\bigoplus_{n\in\mathbb{N}}(\mathcal{M}_n/\mathcal{M}_{n-1})$ is of finite type and has reduced geometric fibers.

A basis of $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is a basis $(e_{n,\lambda};\lambda \in \Lambda(n))_{n\in\mathbb{N}}$ of $f_*\mathcal{O}_X$ such that, for each $n \in \mathbb{N}$, the system $(e_{n,\lambda};\lambda \in \Lambda(n))$ is a system in $\mathcal{M}_n$ lifting a basis of the quotient $\mathcal{M}_n/\mathcal{M}_{n-1}$.

For example, if $X = \mathbb{A}^1_T$ then the sets $\mathcal{M}_n$ of polynomials of degree less or equal n give rise to a geometrically reduced multiplicative filtration on the $\mathcal{O}_T$-algebra $f_*\mathcal{O}_X$ and the monomials of degree n constitute a basis of the filtration.

Example 3.4.26. Let $\varphi : Q \to T$ be a flat projective morphism of relative dimension 1 whose geometric fibers are irreducible and reduced. Let Δ be a closed subscheme of Q which is contained in the smooth part of Q/T with the following properties:

(i) $\Delta \to T$ is étale.
(ii) Δ meets every fiber of Q/T in at least $2 \cdot \dim H^1(Q(t), \mathcal{O}_{Q(t)}) - 1$ points for every geometric point t of T.

Put $Z := Q - \Delta$ and $f := \varphi|_Z$. Then $f : Z \to T$ is affine and

$$\big(\mathcal{M}_n := \varphi_*\mathcal{O}_Q(n\Delta);\, n \in \mathbb{N}\big),$$

yields a geometrically reduced multiplicative filtration of $f_*\mathcal{O}_Z$ and commutes with base change.

Proof. For a detailed proof see [14, IV, 3.3]. Besides the cohomological facts, which are due to the base change theory [39, III_2, 7.8.7], the main idea for showing the property (iv) of Definition 3.4.25 is the following: Let $t \in T$ be a geometric point. Consider a function h on $(Q - \Delta) \otimes_R k(t)$ with $h \in \mathcal{M}_n \otimes_R k(t)$. If a power h^N belongs to $\mathcal{M}_{Nn-1} \otimes_R k(t)$, then h^N has a pole of an order at most $Nn - 1$ at every point of Δ and hence h has a pole of an order at most $n - 1$ at every point of Δ. Thus, h belongs to $\mathcal{M}_{n-1} \otimes_R k(t)$. □

As a fourth tool we need a compactification theorem for curve fibrations.

Lemma 3.4.27. *In the situation of Notation 3.4.18 assume, in addition, that the morphism $h : Z \to T$ is of relative dimension 1.*

Then, after replacing T by a suitable rig-étale cover $T' \to T$ and Z by T-dense open subscheme, there exists a commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & \widehat{P} \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \hookrightarrow & \widehat{Q} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle \widehat{\varphi}} & \\ & T & \end{array}$$

of flat formal T-schemes with the following properties:

(i) *The horizontal morphisms are open immersions with T-dense image. $\widehat{P}$ and $\widehat{Q}$ are the π-adic completion of P and Q, respectively, where*
(ii) *$\varphi : Q \to T$ is flat projective with irreducible, reduced geometric fibers.*
(iii) *The morphism g is the restriction of a finite morphism $\gamma : P \to Q$.*
(iv) *There exists a closed subscheme Δ of Q, which is contained in the smooth part of Q/T with the properties as on Example 3.4.26 and $Z \subset Q - \Delta$.*

In particular, $\varphi_* \mathcal{O}_Q(Q-\Delta)$ admits a geometrically reduced multiplicative filtration. The function f with $X = \operatorname{Spf} C[f]$ can be replaced by the restriction of an element of $\Gamma(P - \gamma^*\Delta, \mathcal{O}_P)$.

The proof of Lemma 3.4.27 is postponed to Corollary 3.7.6 of Sect. 3.7, because it makes use of the approximation argument of Proposition 3.6.9, which will be explained in Sect. 3.6.

Now we come to the main point:

Proof of Theorem 3.4.8 in the case of relative dimension 1.

We will follow the plan explained after Remark 3.4.19. As explained earlier, we may start with the situation described in Notation 3.4.18.

$$X = \operatorname{Spf} A \xrightarrow{\ g\ } Z = \operatorname{Spf} C \xrightarrow{\ h\ } T = \operatorname{Spf} B \ ,$$

where $A = C[f]$. Let

$$F(\xi) := \xi^N + c_1\xi^{N-1} + \cdots + c_N \in C[\xi]$$

be the characteristic polynomial of f. Proceeding by descending induction on the maximum of geometric multiplicities N in the fibers, our problem is now to reduce the multiplicity N in the given situation.

We are allowed to shrink Z to a T-dense open subscheme; cf. Lemma 3.4.17(c). By Lemma 3.4.27 we may view $X \to Z$ as a morphism induced from $P \to Q$ of projective T-schemes, where the geometric fibers of Q/T are irreducible and reduced. We may replace Z by $Q - \Delta$ and X by $P \times_Q (Q - \Delta)$ or by its completion with respect to the π-adic topology. We have a geometrically reduced multiplicative filtration of $\mathcal{O}_Q(Q-\Delta)$ by Example 3.4.26

$$M_0 \subset M_1 \subset M_1 \subset \cdots \subset C := \mathcal{O}_Q(Q-\Delta).$$

By abuse of notation we denote by C now $\mathcal{O}_Q(Q-\Delta)$.

Since $X \to Z$ is rig-étale, the discriminant $\Delta(f)$ is rig-invertible. We introduced a function $\delta \in B$ which is rig-invertible such that δ represents the minimum of $\Delta(f)$ on the fibers of Z/T. The infinitesimal neighborhood $Z_\lambda \to T_\lambda$ has an induced geometrically reduced multiplicative filtration for every $\lambda \geq 1$.

In the following we denote by "$^-$" the reduction modulo δ. Since δ generates an open ideal of B, there exists an integer $r \in \mathbb{N}$ such that the coefficients c_ν of the characteristic polynomial F satisfy $\overline{c}_\nu \in \overline{M}_r$ for $\nu = 1, \dots, N$, where $(\overline{M}_n, n \in \mathbb{N})$ is the induced filtration of $C/C\delta$ over $B/B\delta$. Let $\underline{e} := (e_1, \dots, e_\ell) \subset M_r$ be a basis of M_r. Thus, we have

$$M_r := Be_1 \oplus \cdots \oplus Be_\ell \subset C.$$

Now consider the characteristic polynomial

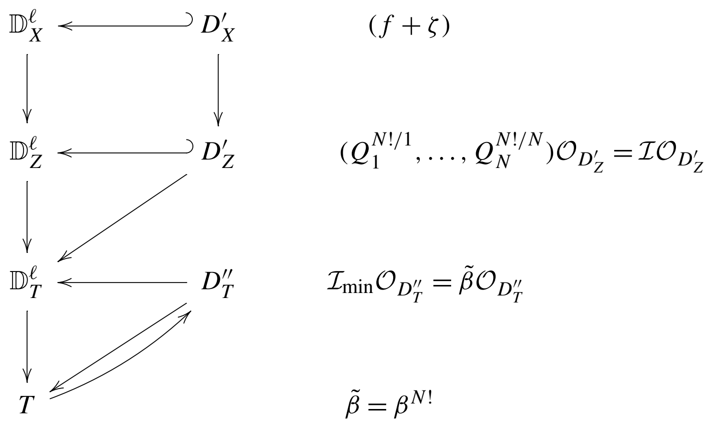
$$F(\eta - \zeta) := \eta^N + Q_1(\zeta)\eta^{N-1} + \cdots + Q_N(\zeta) \in C[\zeta][\eta]$$

of $(f + \zeta)$ over $\mathbb{D}^\ell_Z := Z \times_R \mathbb{D}^\ell_R$, where $\zeta := \zeta_1 e_1 + \cdots + \zeta_\ell e_\ell$.

Then consider the ideal $(Q_1^{N!/1}, \dots, Q_N^{N!/N})$ of $\mathcal{O}_{\mathbb{D}^\ell_Z}$. This ideal is open, because $X \to Z$ is rig-étale. By Corollary 3.4.22 we see that there exists an ideal of coefficients $\mathcal{I} \subset \mathcal{O}_{\mathbb{D}^\ell_T}$. Thus, there exists a $\mathbb{D}^\ell_T$-dense open part $D'_Z \subset \mathbb{D}^\ell_Z$ such that with

$$(Q_1^{N!/1}, \dots, Q_N^{N!/N})\mathcal{O}_{D'_Z} = \mathcal{I}\mathcal{O}_{D'_Z}.$$

Since $\mathcal{I}$ is open, we can apply the minimum principle of Corollary 3.4.24 to $\mathcal{I}$ and the morphism $\mathbb{D}^\ell_T \to T$. Thus, after a suitable rig-étale cover of T, the minimum is given by a function $\tilde{\beta}$ on T which is rig-invertible. The minimum is attained on an open part $D''_T \to \mathbb{D}^\ell_T$ of some admissible blowing-up of $\mathbb{D}^\ell_T$ which covers T. The following diagram presents our situation



The function $\tilde{\beta}$ is minimal for all possible choices of $c \in M_r$ and this remains true after every base change $T' \to T$. Indeed, let $B \to B'$ be an extension of π-adic rings and $c' \in M_r \otimes_B B'$; say $c' = b'_1 e_1 + \cdots + b'_\ell e_\ell$ with $b'_1, \dots, b'_\ell \in B'$. Then $(b'_1, \dots, b'_\ell)$ gives rise to a point $b' \in \mathbb{D}^\ell_{T'}$. Since $D'_{Z'} \subset \mathbb{D}^\ell_{Z'}$ is $\mathbb{D}^\ell_{T'}$-dense open, there exists a point $z' \in D'_{Z'}$ above b' under the map $\mathbb{D}^\ell_{Z'} \to \mathbb{D}^\ell_{T'}$. Let Q'_ν be the coefficients of the characteristic polynomial of $f - c'$; these coefficients coincide with $Q_\nu(z')$. Since $z' \in D'_Z$, we see

$$\left(Q_1^{N!/1}, \ldots, Q_N^{N!/N}\right)\mathcal{O}_{D_{Z',z'}} = \mathcal{I}\mathcal{O}_{D_{Z',z'}}.$$

Therefore,

$$|\tilde{\beta}|^{1/N!} \le \min_{\nu=1}^{N}\{|Q^{1/\nu}(z')|\} = |(f-c')(z')|.$$

After a further suitable rig-étale base change of T, there exists a section into D''_T, so that we can assign values to the variables $\zeta = (\zeta_1, \ldots, \zeta_\ell)$. Thus, we obtain a function $c \in M_r \subset C$, and hence $c \in C$. The relative maximum of $g^{N!}$ for $g := f - c$ is given by $\tilde{\beta}$ as a rig-invertible function of T. By the supplement of Theorem 3.4.23 there exists a further rig-étale cover T' of T such that the minimum function $\tilde{\beta}$ can be replaced by $\beta^{N!}$, where β is a rig-invertible function on T'. Then the function g/β becomes integral over Z after a further admissible blowing-up $T'' \to T'$ by Lemma 3.4.17(a). By replacing T by T'' we obtain a finite rig-isomorphism $Y \to X$, where $\mathcal{O}_Y$ is generated by g/β over $\mathcal{O}_Z$; i.e., $Y = \operatorname{Spf} \mathcal{O}_Z[g/\beta]$.

Now we look at the characteristic polynomial of g/β over Z

$$G(\xi) = \frac{1}{\beta^N} F(\beta\xi + c) = \xi^N + q_1\xi^{N-1} + \cdots + q_N \in C[\xi];$$

cf. Notation 3.4.18. It remains to see that for every fiber $Y(t)$ of Y/T the characteristic polynomial of $g/\beta|_{X(t)}$ over $Z(t)$ cannot be of type $(\eta - u)^N$ with a function u on $Z(t) \otimes_k k'$ for any radical extension k'/k; cf. Remark 3.4.19.

In fact, otherwise we would have

$$q_N = \beta^{-N} Q_N(c) = (-u)^N \quad \text{on } Z(t) \otimes_k k'.$$

Thus, the restriction of $\beta^{-N} Q_N(c)$ to $Z(t)$ and the function u^N belongs to $\overline{M}_{Nr} \otimes k'$ as $\overline{q}_N \in \overline{M}_{Nr}$. Using the geometric reducedness of the graded ring $\bigoplus_{n\in\mathbb{N}} \overline{M}_{n+1}/\overline{M}_n$, we see that $u \in M_r \otimes k'$. Then we choose a finite flat extension $T' \to T$, which lifts k', and perform the base change $T' \to T$. Thus, we can consider a new translation of g/β of the allowed type; i.e., a translation by a linear combination $c' \in M_r$ of $e_1, \ldots, e_\ell$, and obtain a contradiction to the minimality of β around t.

Thus, we see that we can drop the geometric multiplicity. So we succeed by descending induction on the geometric multiplicity of the fibers of X/T. This finishes the proof of the Relative Reduced Fiber Theorem 3.4.8 in the case of relative dimension 1.

For the general case of dimension greater than 1 we refer to [14, Part IV, §4], because it is not essential for this book. The proof is done by reduction to the 1-dimensional case. □

3.5 Complements on Flatness

In the following let R be a complete valuation ring of height 1 with maximal ideal $\mathfrak{m}_R$, not necessarily Noetherian. The purpose of this section is to provide some statements on flatness for extensions of polynomial rings or restricted power series rings by formal power series rings over R. In the Noetherian case all statements of this section are well-known; cf. [17, Chap. III, §5.4].

Lemma 3.5.1. *Let A be a commutative ring and let M be an A-module. Then the following conditions are equivalent*:

(i) *M is A-flat.*
(ii) *The canonical morphism $\mathfrak{a} \otimes_A M \to \mathfrak{a}M$ is injective for every finitely generated ideal $\mathfrak{a} \subset A$.*

Proof. See [17, III, §2.2, Lemma 3]. □

Moreover, we recall some well-known techniques on Weierstraß division for formal power series rings, which follow easily from Euclid's algorithm.

A *formal* power series $f \in R[[\xi_1, \dots, \xi_n]]$ is called *ξ_n-distinguished of order* $b \in \mathbb{N}$ if for the power series expansion

$$f = \sum_{\nu=0}^{\infty} f_\nu \xi_n^\nu \in R[[\xi_1, \dots, \xi_{n-1}]][[\xi_n]]$$

with $f_\nu \in R[[\xi_1, \dots, \xi_{n-1}]]$ the coefficient f_b is a unit and the coefficients f_ν belong to $(\mathfrak{m}_R, \xi_1, \dots, \xi_{n-1})$ for $\nu = 0, \dots, b-1$.

A monic polynomial $\omega \in R[[\xi_1, \dots, \xi_{n-1}]][\xi_n]$ of degree b is called a *formal Weierstraß polynomial* if it is ξ_n-distinguished of order b in $R[[\xi_1, \dots, \xi_{n-1}]][[\xi_n]]$.

Theorem 3.5.2 (Weierstraß division theorem). *Let g be a power series in $R[[\xi_1, \dots, \xi_n]]$. If g is distinguished in ξ_n of order b, then for every f in $R[[\xi_1, \dots, \xi_n]]$ there exists a power series $q \in R[[\xi_1, \dots, \xi_n]]$ and a polynomial $r \in R[[\xi_1, \dots, \xi_{n-1}]][\xi_n]$ with $\deg r < b$ such that $f = qg + r$.*

The elements q, r are uniquely determined.

Theorem 3.5.3 (Weierstraß preparation theorem). *Let g be a power series in $R[[\xi_1, \dots, \xi_n]]$. If g is distinguished in ξ_n of order b, then there exist a uniquely determined Weierstraß polynomial $\omega \in R[[\xi_1, \dots, \xi_{n-1}]][\xi_n]$ of degree b and a unit $e \in R[[\xi_1, \dots, \xi_n]]$ such that $g = e\omega$.*

If $g \in R[[\xi_1, \dots, \xi_{n-1}]][\xi_n]$ is a polynomial, then $e \in R[[\xi_1, \dots, \xi_{n-1}]][\xi_n]$.

Proposition 3.5.4. *Let $\xi_1, \dots, \xi_n$ be variables. Then the following ring extensions are flat*:

(i) $R[\xi_1, \dots, \xi_n] \to R[[\xi_1, \dots, \xi_n]]$,

(ii) $R\langle\xi_1,\dots,\xi_n\rangle \to R[[\xi_1,\dots,\xi_n]]$,
(iii) $R[\xi_1,\dots,\xi_n] \to R\langle\xi_1,\dots,\xi_n\rangle$.

Proof. (i) We make use of Lemma 3.5.1. So let $\mathfrak{a} = (a_1,\dots,a_\ell) \subset R[\xi_1,\dots,\xi_n]$ be a finitely generated ideal, $\mathfrak{a} \neq 0$. We have to show that the canonical morphism

$$\mathfrak{a} \otimes_{R[\xi_1,\dots,\xi_n]} R[[\xi_1,\dots,\xi_n]] \longrightarrow \mathfrak{a}R[[\xi_1,\dots,\xi_n]]$$

is injective. We will prove this by induction on the number n of variables. For $n = 0$ there is nothing to show. Now let $n \geq 1$ and assume that the assertion is true for $n-1$. Since $\mathfrak{a}$ is finitely generated, there exists an element $t \in R$ such that

$$|t| = \max\{|a|; a \in \mathfrak{a}\},$$

where $|a|$ is the Gauss norm of a. Thus we can write every $a \in \mathfrak{a}$ as

$$a = t \cdot \alpha \quad \text{for some } \alpha \in R[\xi_1,\dots,\xi_n].$$

Moreover, there exists a polynomial $p \in R[\xi_1,\dots,\xi_n]$ with $tp \in \mathfrak{a}$ and $|p| = 1$. Similarly as in Remark 1.2.3 one shows that there exists a transformation of variables such that, in the new coordinates, p is distinguished in ξ_n as an element of $R[[\xi_1,\dots,\xi_n]]$. Due to Proposition 3.5.3 we can write

$$p = e \cdot \omega,$$

where $e, \omega \in R[[\xi_1,\dots,\xi_{n-1}]][\xi_n]$ are polynomials, e is a unit in the power series ring $R[[\xi_1,\dots,\xi_n]]$ and ω is a Weierstraß polynomial in $R[[\xi_1,\dots,\xi_{n-1}]][\xi_n]$ of finite degree b.

Now consider an element

$$\sum_{\lambda=1}^{\ell} a_\lambda \otimes f_\lambda \in \mathfrak{a} \otimes_{R[[\xi_1,\dots,\xi_{n-1}]][\xi_n]} R[[\xi_1,\dots,\xi_n]]$$

such that

$$\sum_{\lambda=1}^{\ell} a_\lambda \otimes f_\lambda \longmapsto \sum_{\lambda=1}^{\ell} a_\lambda \cdot f_\lambda = 0.$$

Then we use Weierstraß division in Theorem 3.5.2

$$f_\lambda = q_\lambda \cdot \omega + r_\lambda \quad \text{with } q_\lambda \in R[[\xi_1,\dots,\xi_n]],\ r_\lambda \in R[[\xi_1,\dots,\xi_{n-1}]][\xi_n],$$

and Euclid's algorithm

$$a_\lambda r_\lambda = t \cdot h_\lambda \cdot \omega + b_\lambda \quad \text{with } h_\lambda, b_\lambda \in R[[\xi_1,\dots,\xi_{n-1}]][\xi_n],\ \deg b_\lambda < b.$$

Then the relation can be rewritten as

$$\begin{aligned} 0 = \sum_{\lambda=1}^{\ell} a_\lambda \cdot f_\lambda &= \sum_{\lambda=1}^{\ell} a_\lambda \cdot q_\lambda \cdot \omega + a_\lambda \cdot r_\lambda \\ &= \left(\sum_{\lambda=1}^{\ell} a_\lambda \cdot q_\lambda + t \cdot h_\lambda \right) \cdot \omega + \sum_{\lambda=0}^{\ell} b_\lambda. \end{aligned}$$

Due to the uniqueness of the decomposition in Theorem 3.5.2, we obtain

$$\sum_{\lambda=0}^{\ell} b_\lambda = 0 \quad \text{and} \quad \sum_{\lambda=1}^{\ell} a_\lambda \cdot q_\lambda + t \cdot h_\lambda = 0. \qquad (*)$$

Due to the induction hypothesis the canonical homomorphism

$$\mathfrak{a} \otimes_{R[\xi_1,\dots,\xi_n]} R[[\xi_1,\dots,\xi_{n-1}]][\xi_n] \xrightarrow{\sim} \mathfrak{A} := \mathfrak{a}R[[\xi_1,\dots,\xi_{n-1}]][\xi_n]$$

is an isomorphism. Thus, it remains to see that

$$\mathfrak{A} \otimes_{R[[\xi_1,\dots,\xi_{n-1}]][\xi_n]} R[[\xi_1,\dots,\xi_n]] \longrightarrow \mathfrak{a}R[[\xi_1,\dots,\xi_n]] \qquad (**)$$

is injective. We can write $a_\lambda = t\alpha_\lambda$ with $\alpha_\lambda \in R[\xi_1,\dots,\xi_n]$. Then look at the relation of the tensors

$$\begin{aligned} \sum_{\lambda=1}^{\ell} a_\lambda \otimes f_\lambda &= \sum_{\lambda=1}^{\ell} a_\lambda \otimes q_\lambda \omega + \sum_{\lambda=1}^{\ell} a_\lambda \otimes r_\lambda \\ &= \sum_{\lambda=1}^{\ell} a_\lambda \omega \otimes q_\lambda e e^{-1} + \sum_{\lambda=1}^{\ell} a_\lambda r_\lambda \otimes e e^{-1} \\ &= \sum_{\lambda=1}^{\ell} \alpha_\lambda t e \omega \otimes q_\lambda e^{-1} + \sum_{\lambda=1}^{\ell} (te\omega h_\lambda + e b_\lambda) \otimes e^{-1} \\ &= \sum_{\lambda=1}^{\ell} te\omega \alpha_\lambda \otimes q_\lambda e^{-1} + \sum_{\lambda=1}^{\ell} te\omega h_\lambda \otimes e^{-1} \\ &= te\omega \otimes \left(\sum_{\lambda=1}^{\ell} \alpha_\lambda q_\lambda + h_\lambda \right) e^{-1} \\ &= te\omega \otimes 0 = 0. \end{aligned}$$

In the fourth line we used the relation $\sum_{\lambda=1}^{\ell} e b_\lambda = 0$. The last line is true, because the multiplication by t on $R[[\xi_1,\dots,\xi_n]]$ is injective and because the relation $(*)$ implies

$$t \cdot \left(\sum_{\lambda=1}^{\ell} \alpha_\lambda q_\lambda + h_\lambda \right) = \sum_{\lambda=1}^{\ell} a_\lambda q_\lambda + t h_\lambda = 0.$$

This shows that the map $(**)$ is injective.

(ii) and (iii) follow in the same way by using Weierstraß division for the ring of restricted power series. □

A similar result holds in the mixed case.

Proposition 3.5.5 (Gabber). *Let $\underline{\xi} = (\xi_1, \ldots, \xi_m)$ and $\underline{\eta} = (\eta_1, \ldots, \eta_n)$ be sets of variables. Then the extension $R\langle\underline{\xi}\rangle[\underline{\eta}] \longrightarrow R\langle\underline{\xi}, \underline{\eta}\rangle$ is flat.*

It is not clear how to handle this case by the Weierstraß theory. Therefore, we make use of the following lemma due to Gabber.

Lemma 3.5.6. *Let A be a flat R-algebra and M an A-module. The following conditions are equivalent*:

(a) *M is A-flat.*
(b) *$M \otimes_R K$ is $A \otimes_R K$-flat, $M/\pi M$ is $A/\pi A$-flat and the π-torsion of M is* 0.

Proof. (a) → (b): Flatness commutes with base change. Since A is R-flat, so M is R-flat, and hence M has no π-torsion.

(b) → (a): Let N be an A-module. We will show $\mathrm{Tor}_q^A(M, N) = 0$ for all $q \geq 1$. Then this implies that M is A-flat.

If $N = A/\pi A$, then $0 \to A \xrightarrow{\pi} A \to N \to 0$ is a projective resolution of N. Moreover, $0 \to M \xrightarrow{\pi} M \to M \otimes_A N \to 0$ is exact due to (b). Thus, we have that $\mathrm{Tor}_q^A(M, N) = 0$ for $q \geq 1$.

If $\pi N = 0$, then consider a projective resolution $P_* \to M \to 0$. Then the sequence $P_*/\pi P_* \to M/\pi M \to 0$ is exact, because $\mathrm{Tor}_q^A(M, A/\pi A)$ vanishes for $q \geq 1$, as shown above. Since $M/\pi M$ is a flat $A/\pi A$-module, the sequence $P_* \otimes_A N \to M \otimes_A N \to 0$ is exact, and hence $\mathrm{Tor}_q^A(M, N) = 0$ for $q \geq 1$.

If $\pi^n N = 0$ for some $n \in \mathbb{N}$, then it follows $\mathrm{Tor}_q^A(M, N) = 0$ for $q \geq 1$ by induction from the preceding result by looking at the long exact $\mathrm{Tor}_*^A(M, \_)$-sequence associated to the sequence

$$0 \to \pi^{n-1} N \to N \to N/\pi^{n-1} N \to 0.$$

If N consists of $\pi^{\mathbb{N}}$-torsion exclusively, then N is the direct limit of the submodules of π^n-torsion for $n \in \mathbb{N}$. Since Tor commutes with filtered direct limits, the vanishing of $\mathrm{Tor}_q^A(M, N) = 0$ for $q \geq 1$ holds as well.

If N has no π-torsion, the sequence

$$0 \to N \xrightarrow{\varphi} N \otimes_R K \to Q \to 0$$

is exact, where Q is the cokernel of φ. In particular, Q is $\pi^{\mathbb{N}}$-torsion. Since $\mathrm{Tor}_q^A(M, Q) = 0$ for $q \geq 1$, as shown above, and $M \otimes_R K$ is flat over A, we obtain the vanishing of $\mathrm{Tor}_q^A(M, N) = 0$ for $q \geq 1$ by looking at the long exact Tor-sequence.

Finally, in the general case we consider the exact sequence

$$0 \to \left(N : \pi^{\mathbb{N}}\right) \to N \to F \to 0$$

defining F, where $(N : \pi^{\mathbb{N}})$ is the submodule of $\pi^{\mathbb{N}}$-torsion. Thus, F is free of $\pi^{\mathbb{N}}$-torsion. By considering the long exact $\mathrm{Tor}_*^A(M, \_)$-sequence, the vanishing of $\mathrm{Tor}_q^A(M, N) = 0$ for $q \geq 1$ follows from the preceding results. Thus, we see that M is A-flat. □

Proof of Proposition 3.5.5. Apply Lemma 3.5.6 to $A = R\langle\underline{\xi}\rangle[\underline{\eta}]$ and $M = R\langle\underline{\xi}, \underline{\eta}\rangle$. Then M fulfills the condition (b) in Lemma 3.5.6. Indeed, A and M are free of π-torsion, $A/\pi A = M/\pi M$, and $A \otimes_R K \to M \otimes_R K$ is flat. The latter holds, because every maximal ideal $\mathfrak{n}$ of $M \otimes_R K$ is generated by the maximal ideal $\mathfrak{m} := \mathfrak{n} \cap A \otimes_R K$ by Corollary 1.2.7 and the $\mathfrak{m}$-adic completion of $A \otimes_R K$ coincides with the $\mathfrak{n}$-adic completion of $M \otimes_R K$. □

Finally let us discuss applications of these results.

Corollary 3.5.7. *Let K be a non-Archimedean algebraically closed field, X a reduced formal analytic space of pure dimension d and reduction map $\rho : X \to \widetilde{X}$. Consider X also as an admissible formal scheme over* $\mathrm{Spf}\, R$. *For a closed point $\tilde{x} \in \widetilde{X}$ one has the local ring $\mathcal{O}_{X,\tilde{x}}$ and the ring $\mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ of 1-bounded holomorphic functions on the formal fiber $X_+(\tilde{x}) := \rho^{-1}(\tilde{x})$. Then the canonical map $\mathcal{O}_{X,\tilde{x}} \longrightarrow \mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ is faithfully flat.*

Before we start with the proof, let us note the following.

Remark 3.5.8. The ring $\mathring{\mathcal{O}}_{\mathbb{D}^d}(\mathbb{D}^d_+(\tilde{0}))$ of 1-bounded functions is canonically isomorphic to $R[[\xi_1, \ldots, \xi_d]]$. This ring is local and Henselian.

Proof. It is evident that the ring of 1-bounded functions on $\mathbb{D}^d_+(\tilde{0})$ is $R[[\xi_1, \ldots, \xi_d]]$. If the valuation of K is discrete, then this ring is the completion of $R[\xi_1, \ldots, \xi_d]$ with respect to $(\pi, \xi_1, \ldots, \xi_d)$, and hence it is local and Henselian. In the general case it is also true. In fact, one easily shows the lifting of simple zeros of monic polynomials; cf. [78, VII, §3, Prop. 3]. □

Proof of Corollary 3.5.7. In our application in Lemma 5.4.3 we only need the case where X is smooth. So let us treat this case first. We may assume that $X = \mathrm{Spf}\, A$ is affine and smooth of dimension d at $\tilde{x}$. Then there exists a general projection $\tilde{\xi} : \mathrm{Spec}\, \widetilde{A} \to \mathbb{A}^d_k$ which is finite and étale at $\tilde{x}$. Thus, we obtain functions $\xi_1, \ldots, \xi_d \in A$ which induce $\tilde{\xi}$. Then $(\xi_1, \ldots \xi_d)$ give rise to a morphism $\xi : X \to \mathbb{D}^d_R$, which is finite and étale at $\tilde{x}$. Since the ring $\mathring{\mathcal{O}}_{\mathbb{D}^d_K}(\tilde{0})$ is Henselian, $\mathring{\mathcal{O}}_{\mathbb{D}^d}(\mathbb{D}^d_+(\tilde{0})) \to \mathcal{O}_{X,\tilde{x}} \otimes_{\mathcal{O}_{\mathbb{D}^d_R,\tilde{0}}} \mathring{\mathcal{O}}_{\mathbb{D}^d}(\mathbb{D}^d_+(\tilde{0}))$ is an isomorphism. Then it is easy to see that $\mathring{\mathcal{O}}_{\mathbb{D}^d_K}(\tilde{0}) \to \mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ is an isomorphism, and hence that

$\mathcal{O}_{X,\tilde{x}} \otimes_{\mathcal{O}_{\mathbb{D}^d_R,\tilde{0}}} \mathring{\mathcal{O}}_{\mathbb{D}^d}(\mathbb{D}^d_+(\tilde{0})) = \mathring{\mathcal{O}}_X(X_+(\tilde{x}))$. Thus $\mathcal{O}_{X,\tilde{x}} \to \mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ is faithfully flat as follows by base change from Proposition 3.5.4(ii).

The general case is more complicated. We may assume that $X = \operatorname{Spf} A$ is affine. Then there exists a Noether normalization $T := R\langle\xi_1, \dots, \xi_d\rangle$ such that $T \to A$ is finite; cf. Corollary 1.2.6 and Theorem 3.1.17. Let $\varphi : X \to Z := \operatorname{Spf} T$ be the associated map of formal schemes and assume that the origin $\tilde{z}$ is the image of $\tilde{x}$. Then, the ring of 1-bounded functions on $Z_+(\tilde{z})$ is the formal power series ring $R[[\xi_1, \dots, \xi_d]]$. The pre-image of $\tilde{z}$ under $\widetilde{\varphi} : \widetilde{X} \to \widetilde{Z}$ consists of finitely many points $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_r$. Thus, one obtains a decomposition

$$\mathring{\mathcal{O}}_X\big(\widetilde{\varphi}^{-1}\big(Z_+(\tilde{z})\big)\big) = \mathring{\mathcal{O}}_X\big(X_+(\tilde{x}_1)\big) \oplus \cdots \oplus \mathring{\mathcal{O}}_X\big(X_+(\tilde{x}_r)\big),$$

by $\mathring{\mathcal{O}}_Z(Z_+(\tilde{z}))$-modules. Then one can show by the methods of [7, §5] that the canonical homomorphism

$$A \otimes_T \mathring{\mathcal{O}}_Z\big(Z_+(\tilde{z})\big) \longrightarrow \mathring{\mathcal{O}}_X\big(X_+\big(\widetilde{\varphi}^{-1}(\tilde{z})\big)\big)$$

is an isomorphism. Since $T \to \mathring{\mathcal{O}}_Z(Z_+(\tilde{z}))$ is flat by Proposition 3.5.4(ii), the extension $A \to \mathring{\mathcal{O}}_X(X_+(\widetilde{\varphi}^{-1}(\tilde{z})))$ is flat, because flatness commutes with base change. Thus, the morphism $A \to \mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ is flat, and hence $\mathcal{O}_{X,\tilde{x}} \to \mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ is flat. The latter extension is faithfully flat, since the ring $\mathcal{O}_{X,\tilde{x}}$ is local and its maximal ideal is a proper ideal of $\mathring{\mathcal{O}}_X(X_+(\tilde{x}))$. □

Corollary 3.5.9. *Let A be an admissible formal R-algebra and let $\mathcal{I} \subset A$ be an open ideal. Let $X_{\mathcal{I}}^{\text{alg}} \to X := \operatorname{Spf} A$ be the blowing-up of $\mathcal{I}$ on X and let $X_{\mathcal{I}}^{\text{for}}$ be the formal blowing-up of $\mathcal{I}$ on X. Then the canonical map $X_{\mathcal{I}}^{\text{for}} \to X_{\mathcal{I}}^{\text{alg}}$ is faithfully flat.*

Proof. The flatness follows from Proposition 3.5.5 by base change. The extension is faithful, since the canonical map is an isomorphism on the special fibers. □

Remark 3.5.10. One can also ask if $R\langle\underline{\xi}\rangle[\underline{\eta}] \to R\langle\underline{\xi}\rangle[[\underline{\eta}]]$ is flat. Probably, the assertion is true, but it is not covered by our results. One knows that $R\langle\underline{\xi}\rangle[[\underline{\eta}]]$ is flat over $R\langle\underline{\xi}\rangle$, since $R\langle\underline{\xi}\rangle$ is a coherent ring due to Corollary 3.2.2. More generally, one can ask if $A \to A[[\underline{\eta}]]$ is flat for every commutative ring A. The latter is true if A is a coherent ring; see the article of Soublin [91, Prop. 12]. But it is false in general. A counterexample is given by the following:

Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Assume that $\mathfrak{m}$ is not finitely generated and satisfies $\mathfrak{m}^2 = 0$. Then the canonical homomorphism

$$\rho : A[[\eta]]/\mathfrak{m}A[[\eta]] \longrightarrow (A/\mathfrak{m})[[\eta]] = k[[\eta]]$$

is not injective. We assert that $A[[\eta]]$ is not flat over A.

Every flat A-module M is free, because a lifting $(e_i; i \in I)$ of a basis $(\overline{e}_i; i \in I)$ of $M \otimes_A k$ generates M due to Nakayama's lemma and the flatness of M implies that $(e_i; i \in I)$ is linearly independent. Now assume that $M = A[[\eta]]$ is flat over A.

Thus, choose a basis $(\overline{e}_h; h \in H)$ of $A[[\eta]]/\mathfrak{m}A[[\eta]]$ over the field $k := A/\mathfrak{m}$ in the following way. First, choose a basis $(\overline{e}_i; i \in I)$ of $k[[\eta]]$ and liftings $e_i \in A[[\eta]]$. Since the map ρ is not injective, $(e_i; i \in I)$ does not generate $A[[\eta]]/\mathfrak{m}A[[\eta]]$. Thus, one has to add elements $(\overline{e}_j; j \in J)$. Finally, choose a lifting $(e_h; h \in H)$ of this system with $H = I \cup J$. This is a basis of $A[[\eta]]$ over A as seen above. Thus, every $b \in A[[\eta]]$ has a unique representation $b = \sum_{h \in H} a_h e_h$, where $a_h \in A$ and almost all $a_h = 0$. Now $b \in \mathfrak{m}A[[\eta]]$ if and only if $a_h \in \mathfrak{m}$ for all $h \in H$. On the other hand, an element $b \in \mathfrak{m}A[[\eta]]$ is a finite sum $m_1 b_1 + \cdots + m_r b_r$ with $m_\rho \in \mathfrak{m}$ and $b_\rho \in A[[\eta]]$. Since $\mathfrak{m}^2 = 0$, one can view b_ρ as an element of $k[[\eta]]$. Thus, b is linear combination $b = \sum_{i \in I} n_i e_i$ with $n_i \in \mathfrak{m}$. Thus, if $b \in A[[\eta]]$ and not in $\bigoplus_{i \in I} A e_i$, we obtain two different representation of b. Thus, we arrive at contradiction.

3.6 Approximation in Smooth Rigid Spaces

The main result of this section is an approximation theorem for rigid analytic morphisms. Consider an affinoid space T_K and two separated T_K-spaces Z_K and X_K. Let U_K be an open subspace of Z_K and let $\varphi : U_K \to X_K$ be an T_K-morphism. Then we want to approximate φ by a T_K-morphism $\varphi' : U'_K \to X_K$, where U'_K is an open subspace of Z_K such that U_K is relatively Z_K-compact in U'_K; i.e., $U_K \Subset_{Z_K} U'_K$, cf. Definition 3.6.1 below for the precise definition.

As a natural condition we have to assume that the image $\varphi(U_K)$ is relatively T_K-compact in X_K. One expects that such a statement is valid under mild conditions; for instance, if U_K is affinoid, $\varphi(U_K)$ is relatively T_K-compact in X_K and $\mathcal{O}(Z_K)$ is dense in $\mathcal{O}(U_K)$. We can show such a statement under the additional condition that X_K is rig-smooth over T_K.

Without the smoothness there is only a local result proved under the additional condition that U_K is relatively compact in Z_K in the absolute sense and that X_K is affinoid; this can be shown by adapting Artin's method [3] to the rigid analytic case; cf. [66].

Our proof in the smooth case involves mainly two methods. We use the main result of [61] to reduce to the case, where X_K is affinoid. Then we adapt Elkik's approximation technique [24] to the rigid analytic case. Before we start with precise statements, let us recall the notion of relative T_K-compactness.

In the following, when we consider formal R-schemes X, we denote by R_n the ring $R/R\pi^{n+1}$ and by X_n the R_n-scheme $X \otimes_R R_n$ for $n \in \mathbb{N}$. As usual, π is an non-zero element in the maximal ideal $\mathfrak{m}_R$ of R.

Definition 3.6.1. Let T_K be a rigid space assumed to be quasi-separated and quasi-compact. Thus, T_K is the generic fiber of an admissible formal R-scheme T; cf. Theorem 3.3.3. Let X_K be a quasi-separated, quasi-compact T_K-space. An open subspace U_K of X_K is called *relatively T_K-compact in X_K* if there exists a model $X \to T$ of $X_K \to T_K$, where U_K is induced by an open subscheme U of X such that the schematic closure $\overline{U}_0$ of U_0 in X_0 is proper over T_0; i.e., U is relatively

T-proper in X. In sign we will write $U_K \Subset_{T_K} X_K$. In the case $T_K = \mathrm{Sp}(K)$ we only write $U_K \Subset X_K$.

A subset M_K of X_K is called relatively T_K-compact in X_K if it is contained in an open subspace which is relatively T_K-compact.

If all spaces under consideration are affinoid, this notion of relative T_K-compactness coincides with Kiehl's notion $U_K \Subset_{T_K} X_K$; cf. Definition 1.6.3.

If there exists one model X such that U_K is induced by a relatively proper T-subscheme U of X, then it holds for every other model X' of X_K on which U_K is induced by an open subscheme; cf. [61, 2.5 and 2.6].

If X_K is proper over T_K, every open subspace U_K of X_K is relatively T_K-compact in X_k; this applies in particular to $X_K = T_K$.

We mention some elementary properties which are easy to verify. In the following we simplify the notation by omitting the index "K".

Remark 3.6.2. In the above situation let $V \subset X$ be a further open subscheme of X. Then the following holds:

(a) If $U \subset V$ and $V \Subset_T X$, then $U \Subset_T X$.
(b) If $U \Subset_X V \subset X$ and $U \Subset_T X$, then $U \Subset_T V$.
(c) If $U_i \Subset_{T_i} X_i$ for $i = 1, 2$, then $U_1 \times U_2 \Subset_{T_1 \times T_2} X_1 \times X_2$.

Proof. It is only necessary to explain (b). In terms of models $U \Subset_X V \subset X$ means that the schematic closure $\overline{U}_0^V$ of U_0 in V_0 is closed in X_0. Thus, $\overline{U}_0^V$ equals the schematic closure $\overline{U}_0^X$ of U_0 in X_0. Since $U \Subset_T X$, we see $U \Subset_T V$. □

In order to make the notion more transparent we add the following lemma.

Lemma 3.6.3. *Let $c_i, c'_i \in \sqrt{|K^\times|}$ with $c_i \le c'_i \le 1$ for $i = 1, \dots, n$. Put*

$$\mathbb{D}^n_K(c) := \mathrm{Sp}\, K\langle \xi_1/c_1, \dots, \xi_n/c_n\rangle.$$

Let $r \in \mathbb{N}$ with $r \le n$. Then the following conditions are equivalent:

(a) $\mathbb{D}^n_K(c) \Subset_{\mathbb{D}^r_K(1)} \mathbb{D}^n_K(c')$, *where $\mathbb{D}^n_K(1) \to \mathbb{D}^r_K(1)$ is the projection to the first coordinates.*
(b) $c_i < c'_i$ *if* $c'_i < 1$ *for* $i \le r$ *and* $c_i < c'_i$ *for* $i \ge r + 1$.

Proof. (b) → (a): This is clear in the case $n = 1$. Indeed, let $X \subset X'$ be a model of $\mathbb{D}^1_K(c) \subset \mathbb{D}^1_K(c')$, where $X' \to T' = \mathrm{Spf}\, R\langle \xi/c'\rangle$ is an admissible blowing-up.

If $r = 0$, then ξ/c' takes values on X_{rig} with norm less than 1. Thus, the scheme X_0 is mapped to the origin under the map $X'_0 \to T'_0$, and hence the closure $\overline{X}_0$ is proper over R_0; i.e., $\mathbb{D}^n_K(c) \Subset \mathbb{D}^n_K(c')$.

If $r = 1$ and $c' < 1$ then $c < c'$ and the assertion follows as above.

If $r = 1$ and $c' = 1$, the assertion is evident.

The general case follows by taking products; see Remark 3.6.2(c).

(a) → (b): By fixing $(n-1)$ coordinates of $\mathbb{D}^n_K(1)$ one can reduce to the case $n=1$.

If $r=0$, there exists R-models $X \subset X'$ of the discs $\mathbb{D}^1_K(c) \subset \mathbb{D}^1_K(c')$ where X is an open subscheme of X' and X' is an admissible blowing-up of $T' := \operatorname{Spf} R\langle \xi/c' \rangle$. The closure $\overline{X}_0$ of the scheme X_0 is mapped to a proper R_0-subscheme of T'_0 under the map $X'_0 \to T'_0$. Since it is connected and contains the origin, it consists of a single point, which is the origin. Due to the maximum principle it follows $c < c'$.

If $r=1$ and $c'=1$, there is nothing to show.

If $r=1$ and $c \le c' < 1$, then $\mathbb{D}^1_K(c) \Subset \mathbb{D}^1_K(1)$. Thus we see by Remark 3.6.2(b) that $\mathbb{D}^1_K(c) \Subset \mathbb{D}^1_K(c')$, and hence we are reduced to the case $r=0$. □

For further use, we mention the following result on properness which is much stronger than Theorem 3.3.12. This result is proved in [61, Theorem 5.1] if the valuation is discrete and in [94, Theorem 5.1] for a general valuation of rank 1.

Theorem 3.6.4. *Let $f: Y \to X$ be a separated morphism of admissible formal schemes, where $X = \operatorname{Spf}(B)$ is affine. Let V be an open subscheme of Y such that its associated rigid analytic space V_{rig} is affinoid and the schematic closure $\overline{V}_0$ of V_0 in Y_0 is proper over X_0. Then there exists an admissible formal blowing-up $Y' \to Y$ and an open subscheme V' of Y' such that the following holds*:

(i) *V'_{rig} is affinoid and the schematic closure $\overline{V}'_0$ of V'_0 in Y'_0 is proper over $\overline{V}_0$ and hence proper over X_0.*
(ii) *The schematic closure of $(V \times_Y Y')_0$ in Y'_0 is contained in V'_0. In particular, V_{rig} is relatively compact in V'_{rig} over X_{rig}.*
(iii) *V_{rig} is a Weierstraß domain in V'_{rig}.*

Definition 3.6.5. A separated rigid analytic space X_K is said to have *no boundary* if every admissible open affinoid subvariety U_K of X_K is relatively K-compact in X_K; i.e., $U_K \Subset X_K$.

Proposition 3.6.6. *Let X_K/K be a separated scheme of finite type over K. Then the associated rigid-analytic variety has no boundary.*

Proof. By Nagata's compactification theorem we can view X_K as a dense open subscheme of a proper K-scheme Y_K. Now Y_K is a proper rigid analytic space by Remark 1.6.10 and it has no boundary by Theorem 3.6.4. Thus, it remains to see that every admissible open affinoid subvariety U_K of X_K admits an affinoid enlargement U'_K disjoint from $Y_K - X_K$.

Indeed, let $V \to \operatorname{Spf} R$ be an admissible formal R-scheme, whose generic fiber V_{rig} is an admissible open subvariety of Y_K, and let U_K be associated to a formal open subscheme U of V such that the schematic closure $\overline{U}_0$ of U_0 in V_0 is proper over R_0. Now let $\mathcal{J}_0 \subset \mathcal{O}_V$ be a coherent sheaf of ideals with locus $V_0 - U_0$. Note that $\mathcal{J}_0|_{U_0} = \mathcal{O}_{U_0}$. Then look at the coherent sheaf of ideals given by $\mathcal{J} := \ker(\mathcal{O}_V \to \mathcal{O}_V/\mathcal{J}_0)$.

Let $W^{(n)} \to V$ be the admissible formal blowing-up of the sheaf of open ideals $\mathcal{I}^{(n)} := (\pi, \mathcal{J}^n)$ for $n \in \mathbb{N}$ and let $V^{(n)}$ be the open subscheme of $W^{(n)}$, where π belongs to $\mathcal{J}^n$. For large $n \in \mathbb{N}$ the generic fiber of $V^{(n)}$ is disjoint from $Y_K - X_K$, since the system $(V_{\text{rig}}^{(n)}; n \in \mathbb{N})$ is a filter of admissible neighborhoods of U_K. Moreover, U_K is relatively compact in $V_{\text{rig}}^{(n)}$. Now one can apply Theorem 3.6.4 to the situation $U_K \Subset V_{\text{rig}}^{(n)}$. □

Theorem 3.6.7 (Approximation theorem). *Let $T = \operatorname{Spf} A$ be an admissible affine formal scheme over* $\operatorname{Spf}(R)$. *Let X be a separated admissible formal T-scheme and assume that $X \to T$ is rig-smooth. Let $Z \to T$ be an admissible formal scheme and U an open subscheme of Z. Let $\varphi : U \to X$ be a T-morphism. Assume that the following conditions are satisfied*:

(i) Z_{rig} *is affinoid and* U_{rig} *is a Weierstraß domain in* Z_{rig}.
(ii) $\varphi(U) \subset V \Subset_T X$, *where* V_{rig} *is affinoid.*

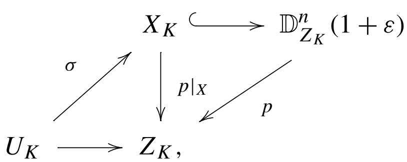
Let $\lambda_0 \in \mathbb{N}$. Then there exists an admissible formal blowing-up $Z' \to Z$ which is finite over U and an open subscheme U' of Z' such that

(a) *the schematic closure of $(Z' \times_Z U)_0$ in Z'_0 is contained in U'_0*,
(b) *the schematic closure $\overline{U}'_0$ is proper over $\overline{U}_0$*,
(c) *there exists a morphism $\varphi' : U' \to X$ such that $\varphi'|_U$ coincides with φ up to the level λ_0.*

For the notion of a Weierstraß domain see Definition 1.3.1. A more general approximation theorem is shown in [66, 5.1.1]; actually one can avoid the assumption that X_{rig} is smooth if in addition $U_{\text{rig}} \Subset Z_{\text{rig}}$ is assumed. Moreover, one can show a smoothening result [66, 5.2.1] in the style of [15, 3.6.12]. Some explanations are necessary to illustrate the assertions in down-to-earth terms.

Remark 3.6.8. One can rephrase the assertion in terms of sections after the base change $Z \to T$. We replace $X \to T$ by $p : X \times_T Z \to Z$, where p is the projection. The morphism φ gives rise to a section of p over U.

Since the generic fibers $Z_K := Z_{\text{rig}}$, $U_K := U_{\text{rig}}$, and $X_K := X_{\text{rig}}$ are affinoid, one can rewrite the problem in terms of solutions of a finite system of equations. Thus, consider the following situation



where $X_K = V(\underline{f})$ is a closed rig-smooth subvariety of $\mathbb{D}^n_{Z_K}(1+\varepsilon)$ for an $\varepsilon > 0$ and σ is a section of $\overline{p}|_X$ with $\sigma(U_K) \subset X_K \cap \mathbb{D}^n_{Z_K}(1)$. The system $\underline{f} = (f_1, \ldots, f_m)$ is a finite family of holomorphic functions on $\mathbb{D}^n_{Z_K}(1+\varepsilon)$. The section σ is equivalent to a solution $\underline{y} = (y_1, \dots, y_n) \in \mathcal{O}_{Z_K}(U_K)^n$ such that $\underline{f}(\underline{y}) = 0$. Theorem 3.6.7 asserts that one can approximate the solution $\underline{y}$ by a solution $\underline{y}' \in \mathcal{O}_{Z_K}(U'_K)^n$ on a domain U'_K which is strictly larger than U_K relatively to Z_K; i.e., $U_K \Subset_{Z_K} U'_K$.

In return, $\underline{y}'$ gives rise to a section $\sigma' : U'_K \to X_K$. The composition with first projection $X \times Z \to X$ yields a morphism φ' as required.

Before we start the proof, let us recall the lemma of Elkik, which is of interest for itself; cf. [24, Lemma 1, p. 555] and [24, Remark on p. 560].

Proposition 3.6.9. *Let $T = \mathrm{Spf}(A)$ be an admissible affine formal scheme and let $V = \mathrm{Spf}(B)$ be an admissible formal T-scheme. Assume that $V \to T$ is rig-smooth. Then there exists an integer $\lambda_1 \in \mathbb{N}$ with the following property:*

Let $U = \mathrm{Spf}(C)$ be an admissible affine formal scheme over T. Let λ be an integer with $\lambda \geq \lambda_1$ and let $\tau_\lambda : U_\lambda \to V_\lambda$ be a T-morphism. Then there exists an T-morphism $\sigma : U \to V$ which coincides with τ_λ on $U_{\lambda-\lambda_1}$.

In particular, if τ_λ is an isomorphism, then σ is an isomorphism also.

The result of Elkik asserts that a nearby solution $\underline{y}$, i.e., $|\underline{f}(\underline{y})| \leq |\pi|^\lambda$, over U can be improved to a true solution defined over U if λ is large enough. The last condition depends only on the determinant of certain minors of the Jacobi matrix $(\partial \underline{f}/\partial \underline{Y})$. For the convenience of the reader we reproduce the statement of Elkik here, which obviously implies Proposition 3.6.9.

Let us introduce a more general setting:

Let A be a commutative ring, and let $B = A[Y]/I$ be an A-algebra of finite presentation with $I = (f_1, \dots, f_q) \subset A[Y] := A[Y_1, \dots, Y_N]$ and polynomials $f_1, \dots, f_q \in A[Y]$. For every $p \in \{1, \dots, q\}$ and for every index $\alpha = (\alpha_1, \dots, \alpha_p)$ with $1 \leq \alpha_1 < \dots < \alpha_p \leq q$ put

$$F_\alpha := (f_{\alpha_1}, \dots, f_{\alpha_p}) \subset A[Y]$$

$$K_\alpha := \left\{ g \in A[Y];\, gI \subseteq F_\alpha \right\}$$

$$M_\alpha := \left(\frac{\partial f_{\alpha_i}}{\partial Y_j} \right)_{i=1,\dots,p,\, j=1,\dots,N}$$

$$M := \left(\frac{\partial f_i}{\partial Y_j} \right)_{i=1,\dots,q,\, j=1,\dots N}$$

and let $\Delta_\alpha \subset A[Y]$ be the ideal generated by the p-minors of the matrix M_α. Let $H_B \subset A[Y]$ be an ideal with

$$H_B \subseteq \sum_{p,\alpha} K_\alpha \Delta_\alpha,$$

where the sum runs over all $p \in \{1, \dots, q\}$ and all indices α associated to p as above. For an N-tuple $\mathrm{a} \in A^N$ consider the ideal

$$I(\mathrm{a}) := \left\{ g(\mathrm{a});\, g \in I \right\} \subset A.$$

We will write f for the column vector

$$\mathrm{f} := (f_1, \dots, f_q)^{\mathrm{t}} \in A[Y]^q .$$

A solution $\mathrm{a} \in A^N$ of the system of equations $\mathrm{f}(Y) = 0$ means $I(\mathrm{a}) = 0$. A nearby solution $\mathrm{a} \in A^N$ means $I(\mathrm{a}) \subset (\pi^n)$ for a chosen parameter π and an integer $n \geq 1$.

Theorem 3.6.10 (Elkik). *In the above situation of above let $\pi \in A$ be an element of A such that A is complete with respect to the (π)-adic topology. Assume that the $\pi^{\mathbb{N}}$-torsion of A is killed by π^k for some integer k. Let $h, n \in \mathbb{N}$ such that*

$$n > \max\{2h, h+k\}.$$

If $\mathrm{a} = (a_1, \dots, a_N)^{\mathrm{t}} \in A^N$ *satisfies*

$$H_B(\mathrm{a}) \supset (\pi)^h \quad \textit{and} \quad I(\mathrm{a}) \subset \left(\pi^n\right),$$

then there exists a true solution $\widehat{\mathrm{a}} = (\widehat{a}_1, \dots, \widehat{a}_N)^{\mathrm{t}} \in A^N$ *with*

$$I(\widehat{\mathrm{a}}) = 0 \quad \textit{and} \quad \widehat{\mathrm{a}} \equiv \mathrm{a} \bmod \pi^{n-h}.$$

The condition $H_B(\mathrm{a}) \supset (\pi)^h$ means that the localization B_π is smooth over A_π. In the case of an admissible R-algebra A one can replace the polynomial ring $A[Y]$ by the R-algebra of restricted power series without changing anything in the following proof. The element π is the usual parameter $\pi \in \mathfrak{m}_R - \{0\}$. The condition on the ideal H_B can be assured by asking B to be rig-smooth over A. Then the condition on the $\pi^{\mathbb{N}}$-torsion is automatically fulfilled. We start the proof with a technical lemma.

Lemma 3.6.11. *Let $\alpha = (\alpha_1, \dots, \alpha_p)$ be an index as above and δ a p-minor of M_α. Consider an element $g \in A[Y]$ and* $\mathrm{a} \in A^N$ *with*

$$gI \subset F_\alpha + \pi^{2n} A[Y] \quad \textit{and} \quad I(\mathrm{a}) \subset \left(\pi^n\right).$$

Then there exists an N-tuple $\mathrm{z} \in A^N$ *with*

$$\mathrm{z} \equiv 0 \bmod \pi^n \quad \textit{and} \quad g(\mathrm{a})\delta(\mathrm{a})\mathrm{f}(\mathrm{a}) \equiv M(\mathrm{a})\mathrm{z} \bmod \pi^{2n}.$$

Proof. We may assume $\alpha = (1, \dots, p)$ and that δ is given by

$$\delta = \det\left(\frac{\partial f_i}{\partial Y_j}\right)_{i,j=1,\dots,p}.$$

Due to the assumption for $j = p+1, \dots, q$ there is a relation

$$g f_j \equiv \sum_{i=1}^{p} P_{ij} f_i \bmod \pi^{2n} A[Y]$$

with suitable $P_{ij} \in A[Y]$. Taking derivatives yields

$$g\frac{\partial f_j}{\partial Y_l} \equiv \sum_{i=1}^{p} P_{ij}\frac{\partial f_i}{\partial Y_l} \bmod I + \pi^{2n}A[Y]$$

for $j = p+1, \ldots, q$ and $l = 1, \ldots, N$ because $f_i, f_j \in I$. Inserting a yields

$$g(\mathrm{a})f_j(\mathrm{a}) \equiv \sum_{i=1}^{p} P_{ij}(\mathrm{a})f_i(\mathrm{a}) \bmod \pi^{2n}, \tag{3.1}$$

$$g(\mathrm{a})\frac{\partial f_j}{\partial Y_l}(\mathrm{a}) \equiv \sum_{i=1}^{p} P_{ij}(\mathrm{a})\frac{\partial f_i}{\partial Y_l}(\mathrm{a}) \bmod \pi^{n}, \tag{3.2}$$

because $I(\mathrm{a}) \subseteq (\pi^n)$. Thus, if $\mathrm{b} = (b_1, \ldots, b_q)^{\mathrm{t}} \in M(\mathrm{a})A^N$, then we have

$$g(\mathrm{a})b_j \equiv \sum_{i=1}^{p} P_{ij}(\mathrm{a})b_i \bmod \pi^{n} \tag{3.3}$$

for $j = p+1, \ldots, q$. If $\mathrm{c} = (c_1, \ldots, c_N) \in \pi^n A^N$ and

$$\mathrm{b} := M(\mathrm{a}) \cdot \mathrm{c}, \tag{3.4}$$

then the congruence (3.3) even holds mod π^{2n}. Now we put

$$M_0 := \left(\frac{\partial f_i}{\partial Y_j}(\mathrm{a})\right)_{i,j=1,\ldots,p}$$

and let $N_0 \in A^{p\times p}$ be the adjoint matrix of M_0. Thus, we obtain

$$M_0 \cdot N_0 = N_0 \cdot M_0 = \delta(\mathrm{a}) \cdot I_p,$$

since $\delta(\mathrm{a}) = \det M_0$, where I_p is the $(p \times p)$-unit matrix. Add zero rows to N_0 to get a matrix $N_0' \in A^{N\times p}$. Then

$$M(\mathrm{a}) \cdot N_0' \cdot \begin{pmatrix} f_1(\mathrm{a}) \\ \vdots \\ f_p(\mathrm{a}) \end{pmatrix} = \begin{pmatrix} \delta(\mathrm{a})f_1(\mathrm{a}) \\ \vdots \\ \delta(\mathrm{a})f_p(\mathrm{a}) \\ u_{p+1} \\ \vdots \\ u_q \end{pmatrix} =: \mathrm{b}$$

with some elements $u_{p+1}, \dots, u_q \in A$. Thus, b fulfills the condition (3.4) because of $I(\mathrm{a}) \subset A\pi^n$, and hence (3.3) implies for $j = p+1, \dots, q$

$$g(\mathrm{a})u_j \equiv \delta(\mathrm{a}) \sum_{i=1}^{p} P_{ij}(\mathrm{a}) f_i(\mathrm{a}) \bmod \pi^{2n}$$
$$\equiv \delta(\mathrm{a}) g(\mathrm{a}) f_j(\mathrm{a}) \bmod \pi^{2n};$$

the last equation follows from (3.1). Thus, we obtain that

$$M(\mathrm{a}) \cdot N_0' \cdot g(\mathrm{a}) \begin{pmatrix} f_1(\mathrm{a}) \\ \vdots \\ f_p(\mathrm{a}) \end{pmatrix} = g(\mathrm{a})\mathrm{b} \equiv g(\mathrm{a})\delta(\mathrm{a})\mathrm{f}(\mathrm{a}) \bmod \pi^{2n}.$$

Now by putting

$$\mathrm{z} := N_0' \cdot g(\mathrm{a}) \begin{pmatrix} f_1(\mathrm{a}) \\ \vdots \\ f_p(\mathrm{a}) \end{pmatrix}$$

the assertion follows. □

Now we turn to the *proof of Theorem 3.6.10*.
It suffices to show that there exists a vector $\mathrm{y} \in A^N$ with

$$I(\mathrm{a} - \mathrm{y}) \subseteq \left(\pi^{2n-2h}\right) \quad \text{and} \quad \mathrm{y} \equiv 0 \bmod \pi^{n-h}. \tag{3.5}$$

Because of $2n - 2h > n$ we can iterate the process. Due to the completeness of A we obtain a true solution $\widehat{\mathrm{a}} \in A^N$. First, we will show

$$H_B(\mathrm{a} - \mathrm{y}) \supset \left(\pi^h\right) \quad \text{if } \mathrm{y} \equiv 0 \bmod \pi^{n-h}.$$

The Taylor expansion yields

$$g(\mathrm{a}) = g(\mathrm{a} - \mathrm{y} + \mathrm{y}) \in H_B(\mathrm{a} - \mathrm{y}) + (\pi)^{n-h}$$

for every $g \in H_B$. Since $n - h \geq h + 1$,

$$\pi^h \in H_B(\mathrm{a}) \subseteq H_B(\mathrm{a} - \mathrm{y}) + (\pi)^{h+1}.$$

Thus, there exists an element $x \in A$ with $\pi^h(1 - \pi x) \in H_B(\mathrm{a} - \mathrm{y})$. Since $1 - \pi x \in A^\times$ is a unit, we see that $\pi^h \in H_B(\mathrm{a} - \mathrm{y})$.

Next we want to show that there exists a vector $\mathrm{y} = (y_1, \dots, y_N)^{\mathrm{t}} \in A^N$ satisfying the condition (3.5). The Taylor expansion yields for every $\mathrm{y} \in A^N$

$$\mathrm{f}(\mathrm{a} - \mathrm{y}) = \mathrm{f}(\mathrm{a}) - M(\mathrm{a})\mathrm{y} + \sum_{i,j} \mathrm{Q}_{ij}(\mathrm{a}, \mathrm{y}) y_i y_j$$

with q-tuples $\mathrm{Q}_{ij}(\mathrm{a}, \mathrm{y}) \in A[a_1, \ldots, a_N, y_1, \ldots, y_N]^q$. Thus, it suffices to find a vector $\mathrm{y} \in A^N$ satisfying the conditions

$$\mathrm{y} \equiv 0 \bmod \pi^{n-h},$$
$$\mathrm{f}(\mathrm{a}) \equiv M(\mathrm{a})\mathrm{y} \bmod \pi^{2n-2h}.$$

Consider a $p \in \{1, \ldots q\}$, an index $\alpha = (\alpha_1, \ldots, \alpha_p)$, a p-minor δ of M_α and an element $g \in K_\alpha$. For such a triple (α, δ, g) there exists by Lemma 3.6.11 a vector $\mathrm{z} \in A^N$ with

$$\mathrm{z} \equiv 0 \bmod \pi^n,$$
$$g(\mathrm{a})\delta(\mathrm{a})\mathrm{f}(\mathrm{a}) \equiv M(\mathrm{a})\mathrm{z} \bmod \pi^{2n}.$$

Because of $\pi^h \in H_B(\mathrm{a})$ the element π^h is a sum of suitable $g(\mathrm{a})\delta(\mathrm{a})$'s. Thus, we see

$$\pi^h \mathrm{f}(\mathrm{a}) \equiv M(\mathrm{a})\mathrm{z}' \bmod \pi^{2n}$$

for a suitable $\mathrm{z}' \in A^N$ with $\mathrm{z}' \equiv 0 \bmod \pi^n$. Writing $\mathrm{z}' = \pi^h \mathrm{y}$ with $\mathrm{y} \in \pi^{n-h} A^N$, we obtain

$$\pi^h \mathrm{f}(\mathrm{a}) - \pi^h M(\mathrm{a})\mathrm{y} = \pi^{2n}\mathrm{x}$$

for some N-tuple $\mathrm{x} \in A^N$. Then $\mathrm{f}(\mathrm{a}) - M(\mathrm{a})\mathrm{y} - \pi^{2n-h}\mathrm{x}$ is annihilated by π^h and it is a multiple of π^{n-h}, as $\mathrm{y} \in \pi^{n-h} A^N$ and $\mathrm{f}(a) \in \pi^n A^N$. Since the $\pi^{\mathbb{N}}$-torsion is already killed by π^k and $k < n - h$, it follows

$$\mathrm{f}(\mathrm{a}) - M(\mathrm{a})\mathrm{y} - \pi^{2n-h}\mathrm{x} = 0.$$

Thus we see $\mathrm{f}(\mathrm{a}) \equiv M(\mathrm{a})\mathrm{y} \bmod \pi^{2n-h}$. □

For the proof of Theorem 3.6.7 we need further preparations.

Lemma 3.6.12. *Let A be an admissible formal R-algebra. Consider A as a subring of the associated affinoid K-algebra A_{rig} and let $f_1, \ldots, f_n \in A_{\mathrm{rig}}$ be elements with $|f_i|_{\sup} \le 1$ for $i = 1, \ldots, n$.*

Then $B := A[f_1, \ldots, f_n]$ is an admissible R-algebra and finite over A.

If $r \in \mathbb{N}$ is chosen such that $\pi^r f_i \in A$ for all i, then the canonical morphism $\mathrm{Spf}\, B \to \mathrm{Spf}\, A$ *can be viewed as the blowing-up of the coherent sheaf of the open ideal generated by $\pi^r, \pi^r f_1, \ldots, \pi^r f_n$ on* $\mathrm{Spf}\, A$.

Proof. Chose a finite set of variables ξ and an epimorphism $R\langle\xi\rangle \to A$. The associated morphism $K\langle\xi\rangle \to A_{\mathrm{rig}}$ is an epimorphism also. Then we see by Proposition 3.1.1 that B is integral over $R\langle\xi\rangle$, and hence finite over A. Thus, B is an admissible R-algebra with $A_{\mathrm{rig}} = B_{\mathrm{rig}}$. Now consider the blowing-up $Y' \to \mathrm{Spec}\, A$ of the ideal $\mathfrak{a} := (\pi^r, \pi^r f_1, \ldots, \pi^r f_n)$ on $\mathrm{Spec}\, A$. Since the pull-back of $\mathfrak{a}$ to B is generated by π^r, and hence invertible, there is a canonical factorization $\mathrm{Spec}\, B \to Y' \to \mathrm{Spec}\, A$ of the morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$.

The first map identifies Spec B with the open part V' of Y', where $\mathfrak{a}\mathcal{O}_{Y'}$ is generated by π^r. We assert that $V' = Y'$. To verify this, we have to show that, over each open part Spec $A_i \subset Y'$, where

$$A_i = A\big[\pi^r/\pi^r f_i, \pi^r/\pi^r f_1, \dots, \pi^r f_n/\pi^r f_i\big]/(\pi\text{-torsion}),$$

the pull-back $\mathfrak{a}A_i$ is generated by π^r. In fact, A_i contains the inverse of f_i, and the extension $A_i \to A_i[f_i]$ is integral. But then, using an integral equation of f_i over A_i and multiplying with a suitable power of f_i^{-1}, we see that $f_i \in A_i$. Thus, the pull-back of $\mathfrak{a}$ to A_i, which is generated by $\pi^r f_i$, is also generated by π^r. This shows that the blowing-up $Y' \to$ Spec A coincides with the morphism Spec $B \to$ Spec A. Then it follows by π-adic completion, which is trivial in this case, that Spf $B \to$ Spf A is the formal blowing-up of $\mathfrak{a}$ on Spf A. This concludes the proof. □

Lemma 3.6.13. *Let Z be a quasi-compact admissible formal scheme and let U be an open subscheme of Z.*

(a) *Let $\mathcal{J} \subset \mathcal{O}_Z$ be a coherent ideal and $n \in \mathbb{N}$ with $\mathcal{J}|_U \subset \pi^{n+1}\mathcal{O}_U$. Let $Z' \to Z$ be the blowing-up of $(\pi^n, \mathcal{J})$ and let U' be the open subscheme of Z', where $\mathcal{J}$ is contained in $\pi^n\mathcal{O}_{Z'}$. Then $U \Subset_Z U'$.*
(b) *Let $\mathcal{J} \subset \mathcal{O}_Z$ be a coherent ideal and $n \in \mathbb{N}$ with $\pi^n \subset \pi\mathcal{J}$ over U. Let $Z' \to Z$ be the blowing-up of $(\pi^n, \mathcal{J})$ and let U' be the open subscheme of Z', where π^n is contained in $\mathcal{J}\mathcal{O}_{Z'}$. Then $U \Subset_Z U'$.*

Proof. (a) Assume that there is a point $x \in \overline{(Z' \times_Z U)_0} - U'_0$. Thus, we have that $\pi^n\mathcal{O}_{Z',x} \subset \mathcal{J}\mathcal{O}_{Z',x}$. Then, by topological reasons, there exists also a point $z \in (Z' \times_Z U)_0$ which satisfies $\pi^n\mathcal{O}_{Z',z} \subset \mathcal{J}\mathcal{O}_{Z',z}$. Thus, we have that $\pi^n\mathcal{O}_{Z',z} \subset \mathcal{J}\mathcal{O}_{Z',z} \subset \pi^{n+1}\mathcal{O}_{Z',z}$. The latter is impossible. Thus, we that $\overline{(Z' \times_Z U)_0} \subset U'_0$.

(b) Assume that there is a point $x \in \overline{(Z' \times_Z U)_0} - U'_0$. Thus, we have that $\mathcal{J}\mathcal{O}_{Z',x} \subset \pi^n\mathcal{O}_{Z',x}$. Then, by topological reasons, there exists a point z in $(Z' \times_Z U)_0$ with $\mathcal{J}\mathcal{O}_{Z',z} \subset \pi^n\mathcal{O}_{Z',z}$. Thus, we see that

$$\pi^n\mathcal{O}_{Z',z} \subset \pi\mathcal{J}\mathcal{O}_{Z',z} \subset \pi^{n+1}\mathcal{O}_{Z',z}.$$

The latter is impossible. Thus, we that $\overline{(Z' \times_Z U)_0} \subset U'_0$. □

Now we come to the *proof of Theorem 3.6.7.*

We start with the special case, where φ factorizes through some affine open subspace V of X such that $\varphi_0(U_0)$ is relatively T_0-compact in V_0. Let $\iota : V \hookrightarrow \mathbb{D}_T^N$ be a closed immersion. Set $\lambda = \lambda_0 + \lambda_1$, where λ_1 is as in Proposition 3.6.9 with respect to $V \to T$; note the remark just after Proposition 3.6.9. The morphism φ composed with the closed immersion ι yields a morphism

$$\psi : U \xrightarrow{\varphi} V \xrightarrow{\iota} \mathbb{D}_T^N.$$

Let $\eta_1, \ldots, \eta_N$ be the coordinates of $\mathbb{D}_T^N$ and set

$$y_i = \psi^* \eta_i \in C := \mathcal{O}_Z(U) \quad \text{for } i = 1, \ldots, N.$$

Since $\mathcal{O}(Z_{\mathrm{rig}})$ is dense in $\mathcal{O}(U_{\mathrm{rig}})$, there exists an $\overline{y}_i \in \mathcal{O}_Z(Z_{\mathrm{rig}})$ such that

$$y_i - \overline{y}_i|_U \in \pi^{\lambda+2} C.$$

In particular, $\overline{y}_i|_U \in C$. Since $\varphi(U) \Subset_T V$, the schematic closure of U_0 under φ_0 in V_0 is proper over T_0. As V_0 is affine over T_0, the schematic closure is finite over T_0. Hence, there exists a monic polynomial $F_i(\zeta)$ in $\mathcal{O}_T(T)[\zeta]$ such that $F_i(y_i) \in \pi C$. In particular, we obtain $F_i(\overline{y}_i|_U) \in \pi C$. Let r be an integer such that $\pi^r F_i(\overline{y}_i)$ lies in $\mathcal{O}_Z(Z)$. Then look at

$Z' \to Z$ the formal blowing-up of $\mathcal{J} = \big(\pi^r F_i(\overline{y}_i), \pi^r\big)_{i=1,\ldots,N}$,

W' the open subscheme of Z', where $\mathcal{J}\mathcal{O}_{Z'}$ is generated by π^r.

Due to Lemma 3.6.13(a) we obtain $U \Subset_Z W'$, because $\pi^r F_i(\overline{y}_i|_U) \in \pi^{r+1} C$ because of $F_i(\overline{y}_i|_U) \in \pi C$. Thus, we may replace Z by W'. In particular, $\overline{y}_i$ has supnorm ≤ 1 because of $|F_i(\overline{y}_i)|_{W'_{\mathrm{rig}}}| \leq 1$. By Lemma 3.6.12 we see that there is a blowing-up $Z' \to Z$, which is finite over U such that $\overline{y}_i$ is defined on Z'. Thus, we may assume that $\overline{y}_i$ is defined on Z. Now look at the map

$$\overline{\psi} : Z \longrightarrow \mathbb{D}_T^N \quad \text{given by } \overline{y}_1, \ldots, \overline{y}_N.$$

Let $\mathcal{I}$ be the ideal defining V as a closed subscheme of $\mathbb{D}_T^N$. We have

$$\overline{\psi}^* \mathcal{I} \mathcal{O}_U \subset \pi^{\lambda+2} \mathcal{O}_U,$$

because $\overline{\psi}|U$ and ψ coincide up to the level $\pi^{\lambda+2}$. Now let

$Z' \to Z$ be the formal blowing-up of $\mathcal{J} = \big(\pi^{\lambda+1}, \overline{\psi}^* \mathcal{I}\big)\mathcal{O}_Z$,

U' the open subscheme, where $\mathcal{J}\mathcal{O}_{Z'}$ is generated by $\pi^{\lambda+1}$.

In particular, $U \Subset_Z U'$ by Lemma 3.6.13(a). Moreover, U'_{rig} is affinoid, and hence U' is proper over some affine formal R-scheme, which is an R-model of U'_{rig}. Thus, we can perform the Stein-factorization

$$U' \longrightarrow Y' := \operatorname{Spf} \Gamma\big(U', \mathcal{O}_{Z'}\big)$$

in the category of admissible formal R-schemes. Then $\Gamma(U', \mathcal{O}_{Z'})$ is an admissible formal R-algebra. In fact, we may assume that U' is obtained by an admissible formal blowing-up of $\operatorname{Spf} A'$, where A' is an admissible R-algebra contained in $\mathcal{O}_{Z_{\mathrm{rig}}}(U'_{\mathrm{rig}})$ and hence $\Gamma(U', \mathcal{O}_{Z'})$ is a finitely generated A'-module. The map $\overline{\psi}$ factorizes through the Stein factorization. Thus, we obtain a map

$$\widetilde{\psi} : Y' \longrightarrow \mathbb{D}_T^N.$$

Since $\mathcal{I}|_{U'} \subset \pi^{\lambda+1}\mathcal{O}_{U'}$, we have $\mathcal{I}|_{Y'} \subset \pi^{\lambda+1}\mathcal{O}_{Y'}$. Thus, the map $\widetilde{\psi}$ gives rise to a morphism $Y'_\lambda \to V_\lambda$. By Proposition 3.6.9 this map lifts to a map $Y' \to V$ up to the level $\lambda_0 = \lambda - \lambda_1$. Composing it with $U' \to Y'$, we obtain a morphism $\varphi' : U' \to V$ we are looking for.

For technical reasons we add the following supplement to our special case, where V_{rig} may be induced by some $\widetilde{V}$ which is not necessarily affine:

If $\psi : \widetilde{U} \to \widetilde{X}$ is another model of φ_{rig}, where $\widetilde{U}$ is an open subscheme of an R-model $\widetilde{Z}$ of Z_{rig}, then there exist an admissible formal blowing-up $\widetilde{Z}' \to \widetilde{Z}$, an open subscheme $\widetilde{U}'$ of $\widetilde{Z}'$ with the properties (a) *and* (b) *of Theorem* 3.6.7 *and a morphism $\psi' : \widetilde{U}' \to \widetilde{X}$ such that $\psi'\rfloor_{\widetilde{U}}$ coincides with ψ up to the level λ_0.*

Indeed, consider the model $\psi : \widetilde{U} \to \widetilde{V}$ of φ_{rig}. We may assume that $\widetilde{V} \to V$ is an admissible formal blowing-up of V; say of an open ideal $\mathcal{H}$ such that $\pi^{n-1} \in \mathcal{H}$. We may assume $n \le \lambda_0$. Thus, we obtain a map $\tau : \widetilde{U} \to V$. Due to the previous assertion there exists a blowing-up $\widetilde{Z}' \to \widetilde{Z}$ and an open subscheme $\widetilde{U}'$ with the above asserted properties (a) and (b) as well as a morphism $\tau' : \widetilde{U}' \to V$ such that τ' and τ coincide on $\widetilde{U}_\lambda$. Since we may assume that $((\tau')^*\mathcal{H})\mathcal{O}_{\widetilde{U}'}$ is invertible, we obtain a lifting $\psi' : \widetilde{U}' \to \widetilde{V}$ of τ'. Now we have

$$\left(\tau'\right)^*\mathcal{H}|_{\widetilde{U}} = (\tau)^*\mathcal{H}|_{\widetilde{U}},$$

because τ' and τ coincide mod π^{λ_0} and π^{n-1} is contained in $\mathcal{H}$. Since $\widetilde{V} \to V$ is the blowing-up of $\mathcal{H}$, the restriction of ψ' to $\widetilde{U}$ and ψ coincide on $\widetilde{U}_{\lambda_0}$.

Now we turn to the *general case*. We have that $\varphi_{\mathrm{rig}}(U_{\mathrm{rig}}) \subset V_{\mathrm{rig}}$, where $V_{\mathrm{rig}} \subset X_{\mathrm{rig}}$ is an open affinoid subdomain of X_{rig}, which is relatively compact X_{rig} with respect to $X_{\mathrm{rig}} \to T_{\mathrm{rig}}$. Then we obtain by Theorem 3.6.4 that there exists an open affinoid subspace V'_{rig} of X_{rig} such that $V_{\mathrm{rig}} \Subset_{T_{\mathrm{rig}}} V'_{\mathrm{rig}}$. Now there exists an admissible formal blowing-up $X' \to X$ such that V'_{rig} is induced by an open subscheme $V' \subset X'$ and an admissible blowing-up $Z' \to Z$ such that φ is induced by a morphism $\varphi' : U \times_Z Z' \to V'$. Thus, we are in a situation we considered above.

So we are also done in the general situation. □

Now we want to study how certain properties of the given map φ_{rig} are transmitted to the approximation.

Corollary 3.6.14. *In the situation of Theorem* 3.6.7 *assume, in addition $U \Subset_T Z$. Then there exist an admissible formal blowing-up $Z' \to Z$, an open subscheme U' of Z' with $U \Subset_T U'$ resp. an admissible formal blowing-up $X' \to X$ and an open subscheme V' of X' with $V \Subset_T V'$ and a T-morphism $\varphi' : U' \to V'$ such that its restriction to U yields a morphism $\varphi'|_U : U \to V$ which coincides with φ up to the level λ with the following properties*:

(a) *If φ_{rig} is finite, so is φ'_{rig}.*
(b) *If φ_{rig} is a closed immersion* (*resp. an isomorphism*), *so is φ'_{rig}.*

Proof. (a) Since a map $\varphi : \mathrm{Spf}(A) \to \mathrm{Spf}(B)$ is finite if and only if the map $B_0 \to A_0$ is finite, it is clear that we can arrange the approximation that the restriction $(\varphi'|_U)_{\mathrm{rig}}$ of φ' is finite. Then (a) follows from Lemma 3.6.17 below.

(b) As in (a) we may assume that $(\varphi'|_U)_{\mathrm{rig}}$ is a closed immersion or an isomorphism. Thus, (b) follows from Lemma 3.6.17 below. □

Remark 3.6.15. The statement of Corollary 3.6.14 is also true for open immersions without the condition $U \Subset_T Z$.

Proof. It follows from Lemma 3.1.4 that a good approximation φ' of φ yields an open immersion $\varphi'|_{U_{\mathrm{rig}}}$ as well. Thus, φ'_{rig} is flat on U_{rig}. Since the flat locus is open due to [52, 3.3], one can choose U' so small that $\varphi'_{\mathrm{rig}}|_{U'_{\mathrm{rig}}}$ is also flat. Due to Theorem 3.3.4 there exists an admissible blowing-up $X' \to X$ such that $\varphi' : U' \to V'$ is flat after replacing U' by the strict transform. Since the image of a flat morphism is open, the image $W' := \varphi'(U')$ is an open subscheme of V'. It remains to see that $\varphi' : U' \to W'$ is an isomorphism. This can be checked after the faithfully flat base change $W'' := U' \to W'$. Then we have the tautological section $\sigma : W'' \to U'' := U' \times_{W'} U'$ which is an isomorphism over $\varphi'(U)$, because it is an isomorphism on the rigid part. Now $\sigma(W'')$ is a closed subscheme of U'' and it does not meet the rigid part of $U \times_W U'$. Then look at the exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{U''} \xrightarrow{\lambda} \mathcal{O}_{\sigma(W'')} \to 0$$

where $\mathcal{K}$ is the kernel of the canonical epimorphism λ. The support of $\mathcal{K}$ is disjoint from $U \times_W W'$. Then, after a admissible blowing-up and shrinking U' we may assume that $\mathcal{K}$ vanishes. Thus, we see that $\varphi' : U' \to W'$ is an open immersion. □

Lemma 3.6.16. *Let $\varphi : Z \to X$ be a morphism of quasi-compact admissible formal schemes and V an open subscheme of X. Let $\mathcal{I}$ be a coherent sheaf of open ideals of $\mathcal{O}_X$ with vanishing locus $V(\mathcal{I}\mathcal{O}_{X_0}) = X_0 - V_0$. Let $X^n \to X$ be the admissible formal blowing-up of $(\pi, \mathcal{I}^n)$ and V^n the open subscheme of X^n, where π is contained in $\mathcal{I}^n\mathcal{O}_{X^n}$. Assume that $\varphi^{-1}(V) = U \cup W$ is a disjoint union of open subschemes. Then there exists an $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, there exists a disjoint union*

$$\varphi^{-1}\left(V^n\right) = U^n \cup W^n$$

of open subschemes with $U^n \cap \varphi^{-1}(V) = U$ and $W^n \cap \varphi^{-1}(V) = W$.

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be open coherent ideals of $\mathcal{O}_Z$ such that

$$V(\mathcal{F}\mathcal{O}_{Z_0}) = Z_0 - U_0 \quad \text{and} \quad V(\mathcal{G}\mathcal{O}_{Z_0}) = Z_0 - W_0.$$

Let $Z^{(n)} \to Z$ be the admissible formal blowing-up of $(\pi, \mathcal{F}^n, \mathcal{G}^n)$. Set

$$Z^{(n)} \supset U^{(n)} = \left\{z \in Z^{(n)}; \pi \in \mathcal{F}^n\mathcal{O}_{Z^{(n)},z}\right\} \supset U$$
$$Z^{(n)} \supset W^{(n)} = \left\{z \in Z^{(n)}; \pi \in \mathcal{G}^n\mathcal{O}_{Z^{(n)},z}\right\} \supset W$$

Since $\mathcal{F}$ and $\mathcal{G}$ are topologically nilpotent outside of U_0 and W_0, respectively, there exists an $n_1 \in \mathbb{N}$ such that $U^{(n)} \cap W^{(n)} = \emptyset$ for $n \geq n_1$. Since $\mathcal{J} := \mathcal{I}\mathcal{O}_{Z^{(n)}}$ is nilpotent on $Z^{(n)} - (V^{(n)} \cup W^{(n)})$, there exists an index $n_0 \geq n_1$ such that $\mathcal{J}^n \subset \mathcal{F}^{n_1} + \mathcal{G}^{n_1}$ on $Z^{(n_1)}$ for all $n \geq n_0$. This implies $\varphi^{-1}(V^n_{\mathrm{rig}}) \subset U^{(n_1)}_{\mathrm{rig}} \cup W^{(n_1)}_{\mathrm{rig}}$ for all $n \geq n_0$.

Then the assertion is evident. □

Lemma 3.6.17. *Let $\varphi : Z \to X$ be a morphism of quasi-compact separated admissible formal T-schemes. Let U be an open subspace of Z with $U \Subset_T Z$ and let V be an open quasi-compact subspace of X with $V \Subset_T X$. Assume that the restriction of φ to U yields a morphism $\varphi|_U : U \to V$ and that $(\varphi|_U)_{\mathrm{rig}} : U_{\mathrm{rig}} \to V_{\mathrm{rig}}$ satisfies one of the following conditions*:

(a) *proper*,
(b) *finite*,
(c) *isomorphism*,
(d) *closed immersion*.

Then there exist an admissible formal blowing-up $Z' \to Z$ and an open subscheme U' of Z' with $U \Subset_T U'$ resp. $X' \to X$ and an open subscheme V' of X' with $V \Subset_T V'$ such that $(\varphi_{\mathrm{rig}}|_{U'})_{\mathrm{rig}} : U_{\mathrm{rig}} \to V_{\mathrm{rig}}$ extends to a T-morphism $\varphi' : U' \to V'$ and satisfies the same condition.

Proof. (a) Let $\mathcal{J}$ be a coherent sheaf of open ideals of $\mathcal{O}_X$ such that the locus of $\mathcal{J}\mathcal{O}_{X_0}$ is $X_0 - V_0$. As in Lemma 3.6.16, we consider the admissible formal blowing-up $X^n \to X$ of the sheaf of ideals $(\pi, \mathcal{J}^n)$ for $n \in \mathbb{N}$ on X and the open subscheme $V^n := \{x \in X^n; \pi \in \mathcal{J}^n \mathcal{O}_{X^n,x}\}$. For $n \geq 2$ it is clear that the subscheme V^n is also relatively T-proper in X^n, because $\overline{V}^n_0 \to \overline{V}_0$ is proper. Similarly one defines Z^n and U^n with respect to $(\pi, \mathcal{J}^n)\mathcal{O}_Z$. Thus, we obtain a canonical commutative diagram

$$\begin{array}{ccc} U^n & \longrightarrow & V^n \\ \downarrow & & \downarrow \\ Z^n & \longrightarrow & X^n. \end{array}$$

Now $\varphi|_U : U \to V$ is proper, because $(\varphi|_U)_{\mathrm{rig}}$ is proper; cf. Theorem 3.3.12. Thus, the inclusion $U \to \varphi^{-1}(V)$ is proper [39, II, 5.4.3]. Then U is a sum of connectedness components of $\varphi^{-1}(V)$. If we choose n large enough, then by Lemma 3.6.16 there is a sum of connectedness components U^n of $\varphi^{-1}(V^n)$ with $U^n \cap \varphi^{-1}(V) = U$. Such a U^n is also relatively T-proper in Z. After replacing Z by U^{n_0} for some n_0, we may assume $U = \varphi^{-1}(V)$. The upper horizontal map is proper, because the schematic closure of U^n_0 in Z^n_0 is proper over $\overline{U}_0$ and hence proper over T_0. Then $\varphi|_{U^n}$ induces a proper morphism $U^n_{\mathrm{rig}} \to V^n_{\mathrm{rig}}$ by Theorem 3.3.12.

(b) Now assume that $(\varphi|_U)_{\mathrm{rig}}$ is finite. Then it is proper. Due to (a) we may assume that φ_{rig} and by Theorem 3.3.12 that φ is proper. As in (a) we may assume

$\varphi^{-1}(V) = U$. Set

$$B_K := \{x \in X_{\text{rig}}; \dim \varphi^{-1}(x) \geq 1\}.$$

B_K is a closed analytic subset of X_{rig} which is disjoint from V_{rig}. With the above notion there exists an integer $n \in \mathbb{N}$ such that B_K does not meet V^n. We may assume that $X = V^n$. Thus, φ_{rig} has finite fibers. Then it follows from Corollary 3.3.8 that there exists a model of φ'_{rig} which is finite. Since such a model can be obtained by an admissible blowing-up of X and taking strict transforms of Z, the assertion follows.

(c) Due to (b) we may assume that φ_{rig} is finite and, as explained above, that φ is finite. Moreover, as above we may assume $\varphi^{-1}(V) = U$. Then look at the morphism $\lambda : \mathcal{O}_{X_{\text{rig}}} \longrightarrow \varphi_* \mathcal{O}_{Z_{\text{rig}}}$. The support of its kernel resp. of its cokernel is disjoint from V_{rig}. Thus, there exists an integer $n \in \mathbb{N}$ such that the support does not meet V^n. We may replace X by V^n. Then φ is an isomorphism.

(d) Again we may assume that φ is finite. Then we replace X by the schematic image of φ and reduce to (c). □

Finally, let us rewrite a special case of Corollary 3.6.14 in a more down-to-earth formulation which will be used later.

Corollary 3.6.18. *Let $\varphi : \mathbb{D}^d_K(1) \hookrightarrow X_K$ be an open immersion of the d-dimensional unit polydisc into a separated smooth rigid space. Assume that the image of φ is relatively compact in X_K. Then there exists an approximation $\varphi' : \mathbb{D}^d_K(c) \hookrightarrow X_K$ with $c > 1$ of φ over a strictly larger polydisc $\mathbb{D}^d_K(c)$ with $\mathbb{D}^d_K(1) \Subset \mathbb{D}^d_K(c)$ such that φ' is an open immersion.*

In particular, one can choose the approximation in such a way that the images of φ and $\varphi'|_{\mathbb{D}^d_K(1)}$ coincide.

Proof. We may assume that X_K is quasi-compact. Due to Theorem 3.3.4 the map φ is induced by a formal model. Then the assertion follows from Corollary 3.6.14. □

3.7 Compactification of Smooth Curve Fibrations

There are several types of compactification results. Let us begin with the absolute case, where the base is a non-Archimedean field.

Proposition 3.7.1. *Let R be a complete valuation ring of height 1 with perfect residue field k. Let $X \to S := \operatorname{Spf} R$ be a quasi-compact flat relative formal R-curve with reduced geometric fibers.*

If there exists an effective relative Cartier divisor Δ of X which meets every connected component of the smooth part of X/R, then there exists a flat projective formal R-curve $\widehat{P} \to S$ such that X can be embedded into $\widehat{P}$ with R-dense image. Moreover, $\widehat{P}$ is the completion of a projective flat R-curve P with respect to its special fiber and one can choose P to be smooth at infinity; i.e., *P is smooth over R at all points in $P_0 - X_0$.*

Proof. Consider the reduction $X_k := X \otimes_R k$ which is a reduced curve. Then there exists a projective closure P_k of X_k; i.e., a projective curve which contains X_k as an open dense subscheme and P_k is normal at all points in $P_k - X_k$. Since k is perfect, P_k is smooth over k at infinity.

Thus, for each point $z \in P_k - X_k$ there exists an affine neighborhood $Z_k \subset P_k$ of z such that the locus $Z_k = V(\overline{f}_2, \ldots, \overline{f}_n)$ in $\mathbb{A}^n_k$ is defined by $(n-1)$ functions and $(\overline{f}_2, \ldots, \overline{f}_n)$ satisfies the Jacobian criterion of smoothness. Now choose liftings $f_2, \ldots, f_n \in R[\xi_1, \ldots, \xi_n]$ and let Z be their locus $V(f_2, \ldots, f_n)$ in $\mathbb{A}^n_R$. After an eventual shrinking, Z is a smooth affine R-curve and Z_k intersects X_k in an affine open subset.

Since the schemes are of finite presentation, one can choose the parameter $\pi \in \mathfrak{m}_R$ such that the intersection $V_k := Z_k \cap X_k$ is defined over $R/R\pi$; i.e., it is induced from an open affine subscheme V_Z of Z and V_X of X such that the gluing datum is defined over $R/R\pi$. Due to the lifting property of smoothness the gluing lifts to an isomorphism

$$Z \supset V_Z \xrightarrow{\sim} V_X \subset X.$$

Thus, by a gluing process around every point at infinity we obtain a formal proper R-curve $\widehat{P}$. A multiple $N \cdot \Delta$ of the effective Cartier divisor Δ gives rise to a relative very ample divisor on $\widehat{P}/R$. Then it follows from Grothendieck's existence theorem [39, III_1, 5.4.5] that $\widehat{P}$ is induced from a flat projective R-curve P and P_0 is smooth at all points in $P_0 - X_0$. □

The next goal is to finish the proof of the Relative Reduced Fiber Theorem. It remained to prove Lemma 3.4.27. Let us fix the situation for the following: Let $Z = \operatorname{Spf} C \to T = \operatorname{Spf} B$ be a smooth morphism of formal R-schemes with irreducible geometric fibers of dimension 1. First we will look at a similar algebraic situation.

Lemma 3.7.2. *Let k be a field and T be a reduced affine k-scheme of finite type over a field k. Let $Z \to T$ be a smooth morphism of affine k-schemes with irreducible geometric fibers of dimension* 1.

Then there exists a finite stratification by closed reduced subschemes

$$T = T^0 \supset T^1 \supset \cdots \supset T^{m+1} = \emptyset$$

with the following property for $i = 0, \ldots, m$: Over $V^i := T^i - T^{i+1}$ there exists a V^i-compactification of $Z^i := Z \times_T V^i$

$$\begin{array}{ccc} Z^i & \hookrightarrow & P^i \\ & \searrow \qquad \swarrow & \\ & V^i & \end{array}$$

where $P^i \to V^i$ is flat projective with irreducible and generically reduced geometric fibers. There exists an effective V^i-ample relative Cartier divisor on P^i with support $(P^i - Z^i)$.

Proof. Consider a generic point t of T. Then let $P(t)$ be a normal projective closure of $Z(t)$. Since $Z(t)$ is affine, the complement $P(t) - Z(t)$ is supported by an ample Cartier divisor. Since only finitely many coefficients are involved, $P(t)$ and the embedding $Z(t) \to P(t)$ are defined over an open neighborhood V of t. Since we have this for every generic point of T, we may assume that V is dense open in T. Since the geometric fibers of Z/T are irreducible and reduced, we obtain a projective scheme $P_V \to V$ as desired. Then set $T^0 := V$ and proceed similarly with $T - T^0$. Thus, one succeeds by induction, because the space T is Noetherian. □

Lemma 3.7.3. *In the situation of Lemma 3.7.2 put $V = V^0$.*

(a) *There exists a blowing-up $T' \to T$ with center in $T - V$ and a T'-dense open subscheme U' of $Z' := Z \times_T T'$ such that, étale locally on T', there exists a T'-flat projective compactification Q' of U' with irreducible and generically reduced geometric fibers which extends $P_V \to V$.*
(b) *With the notations of* (a) *there exists an étale cover $\widetilde{T} \to T'$ such that there exists a finite morphism $\widetilde{Q}' := Q' \times_{T'} \widetilde{T} \to \widetilde{Q}$, where $\widetilde{Q}$ is a relatively planar $\widetilde{T}$-curve with irreducible and reduced geometric fibers such that $\widetilde{Q}' \to \widetilde{Q}$ is $\widetilde{T}$-birational. In particular, after shrinking $\widetilde{U}' := U' \times_T \widetilde{T}$ to a $\widetilde{T}$-dense open subscheme, $\widetilde{Q}$ is a $\widetilde{T}$-compactification of $\widetilde{U}'$.*

Proof. (a) Keep the result of Lemma 3.7.2. Because of the ample divisor, there exists a closed immersion

$$f_V := (f_1, \ldots, f_n) : Z_V \longrightarrow \mathbb{A}^n_V.$$

After chasing denominators, we may assume that f_V extends to a morphism

$$f : Z \longrightarrow \mathbb{A}^n_T \subset \mathbb{P}^n_T,$$

which is a closed immersion over V. Then let $P \subset \mathbb{P}^n_T$ be the schematic image of f.

Thus, we have a morphism $f : Z \to P$ of Z to a projective T-scheme P such that $Z_V \to P_V := P \times_T V$ is an open immersion. By the algebraic version of the flattening [82, 5.2.2] there exists a blowing-up $P' \to P$ with center in $P - P_V$ such that the strict transform $Z' \to P'$ is flat and, hence an open immersion. After replacing T by a suitable blowing-up $T' \to T$ with center in $T - V$ we may assume that $P' \to T$ and $Z' \to T$ are flat. Next, consider the morphism $Z' \to Z$ between T-flat schemes of finite type. Since the fibers of Z/T are geometrically irreducible and reduced, there exists a T-dense open subscheme U of Z such that $Z' \to Z$ is an isomorphism by similar arguments as in the proof of Lemma 3.4.16.

In the following we replace Z by U and P by P'. Since $Z \to T$ is smooth and surjective, there exists an étale cover $T' \to T$ and a section $\sigma : T' \to Z \times_T T'$. Now we replace T' by T. A multiple of σ gives rise to an effective relative Cartier divisor Δ of P such that $\mathcal{O}_P(\Delta)$ is generated by global sections.

These sections induce a morphism $\varphi : P \to \mathbb{P}^N_T$ such that it contracts all irreducible components of the fibers of P/T, which do not meet Z, and induces an open immersion Z into the schematic closure Q of φ. Thus, Q is a projective T-scheme with irreducible and generically reduced geometric fibers, because Z is T-dense in Q. Again by the flattening technique [82, 5.2.2] we can arrange Q to be flat over T. Here we have used the minor argument that étale base change and blowing-up commute which easily follows by the flattening technique; cf. [14, IV, 5.7].

(b) Let t be a point of T'. Now we work étale locally around t. In our notation we do not make the base change explicit and retain the old symbols. Étale locally around t there exists an embedding of Q' into some $\mathbb{P}^n_T$ and there exists a projection

$$p : Q' \longrightarrow \mathbb{P}^2_T,$$

which is finite and T-birational onto its schematic image. Now consider the coherent sheaf $p_*\mathcal{O}_{Q'}$. Due to [39, III_1, 22.1] there exists a short exact sequence

$$0 \to \mathcal{K} \xrightarrow{\alpha} \mathcal{E} \to p_*\mathcal{O}_{Q'} \to 0$$

of coherent sheaves on $\mathbb{P}^2_T$ with a vector bundle $\mathcal{E}$ of a certain rank r. The kernel is T-flat, because $p_*\mathcal{O}_{Q'}$ and $\mathcal{E}$ are T-flat. Since the depth of every fiber $Q'(t)$ is 1, the depth of $\mathcal{K} \otimes_T k(t)$ is 2. Since $\mathcal{K} \otimes_T k(t)$ is a coherent sheaf on a non-singular scheme of dimension two, $\mathcal{K} \otimes_T k(t)$ has projective dimension 0. Thus, $\mathcal{K} \otimes_T k(t)$ is a vector bundle of rank r equal to the rank of $\mathcal{E}$. Since $\mathcal{K}$ is T-flat, it is $\mathbb{P}^2_T$-flat; cf. [17, III, §5, no. 4, Prop. 3] and hence $\mathcal{K}$ is a vector bundle of rank r. Therefore, the determinant

$$\det(\alpha) := \wedge^r \alpha \in \Gamma\big(\mathbb{P}^2_T, \bigwedge^r \mathcal{E} \otimes \bigwedge^r \mathcal{K}^{-1}\big) = \Gamma\big(\mathbb{P}^2_T, \mathcal{O}_{\mathbb{P}^2_T}(m)\big)$$

is a homogeneous polynomial of a certain degree m if T is connected. Then let $Q = V(\det \alpha) \subset \mathbb{P}^2_T$ be the locus of $\det \alpha$, which is a planar curve. Due to the construction p factorizes through Q. Since p is T-birational, the polynomial $\det \alpha$ has no multiple irreducible components on every fiber $\mathbb{P}^2_t$ and hence the geometric fibers of Q/T are irreducible and reduced. □

Lemma 3.7.4. *Let $Z \to T$ be a smooth morphism of affine formal R-schemes with irreducible geometric fibers of dimension 1. Let V_0 be a dense open subscheme of T_0 such that $Z \times_T V_0$ admits a flat projective V_0-compactification $P_{V_0} \to V_0$ with irreducible and generically reduced geometric fibers. Assume that $P_{V_0} - Z_{V_0}$ is supported by an effective V_0-ample relative Cartier divisor. Set $A_0 := T_0 - V_0$.*

Then there exists an admissible formal blowing-up $T' \to T$ with center in A_0 such that T' is a union of open subschemes T^1 and T^2 with the following *properties*:

(i) *There exists an étale cover* $\widetilde{T}^1 \to T^1$ *and a* $\widetilde{T}^1$*-dense open subscheme* $\widetilde{U}^1$ *of* $\widetilde{Z}^1 := Z \times_T \widetilde{T}^1$ *which admits a flat projective compactification by a relatively planar* $\widetilde{T}^1$*-curve* $\widetilde{Q}^1$ *with irreducible and reduced geometric fibers.*
(ii) T_0^2 *lies over* A_0.

Proof. The assumption means that we start with a situation as established by Lemma 3.7.3(a). Let $T_k := (T_0)_{\mathrm{red}}$ be the maximal reduced subscheme of T_0. This is a k-scheme of finite type. Now we can apply the results of Lemma 3.7.3. There we had the base extensions

$$\widetilde{T}_k \longrightarrow T_k' \to T_k,$$

where $T_k' \to T_k$ is blowing-up with center in A_0 and $\widetilde{T}_k \to T_k'$ is an étale cover. Let $\mathcal{J}_0 \subset \mathcal{O}_{T_0}$ be a coherent sheaf of ideals which induce the blowing-up $T_k' \to T_k$ and let $\mathcal{J} := \ker(\mathcal{O}_T \to \mathcal{O}_{T_0}/\mathcal{J}_0)$. Let $T' \to T$ be the blowing-up of $\mathcal{J}$ on T.

In particular, there is a canonical morphism $T_k' \to T_0'$. The special fiber of T' is a union of two closed subschemes $A^1 \cup A^2$, where A^1 is T_k' and A^2 is the union of all the irreducible components which are not contained in T_k'. In particular $A^1 \cap A^2$ is rare in T_0^1 and T_0^2. Thus, there exists an admissible blowing-up $T'' \to T'$ with center in $A^1 \cap A^2$ such that T'' is a union of open subschemes $T^1 \cup T^2$, where T_0^i factorizes through A^i for $i = 1, 2$. In particular, T_0^2 lies above A_0. Note that the composition $T'' \to T' \to T$ is an admissible blowing-up as well.

Next we want to verify that T^1 satisfies the assertion (i). The étale cover $\widetilde{T}_k \to T_k'$ uniquely lifts to an étale cover $\widetilde{T} \to T^1$. The T_k'-dense open subscheme U_k' of $Z_k^1 := Z \times_T T_k^1$ lifts to a T^1-dense open subscheme U^1 of $Z^1 := Z \times_T T^1$. Eventually one has to replace π by a bigger $\pi' \in \mathfrak{m}_R$ in order to have the algebraic situation defined over $T \otimes_R R/R\pi'$, but then it follows by the étale lifting property.

The relatively planar projective $\widetilde{T}_k$-curve $\widetilde{Q}_k$ lifts to a relatively planar projective T^1-curve $\widetilde{Q}^1$. In fact, we can lift the homogeneous polynomial in $\mathcal{O}_{\widetilde{T}_k}[\xi_0, \xi_1, \xi_2]$ which defines $\widetilde{Q}_k$ as a subscheme of $\mathbb{P}^2_{\widetilde{T}}$ to a homogeneous polynomial in $\mathcal{O}_{\widetilde{T}^1}[\xi_0, \xi_1, \xi_2]$. Finally, the open embedding

$$\widetilde{U}_k^1 = U^1 \times_{T^1} \widetilde{T}_k \hookrightarrow \widetilde{Q}_k^1$$

lifts to an open immersion $\widetilde{U}^1 \hookrightarrow \widetilde{Q}^1$, which satisfies our assertion. □

Proposition 3.7.5. *Let* $Z \to T$ *be a smooth morphism of quasi-compact formal* R*-schemes with irreducible geometric fibers of dimension* 1*. Then, after a suitable base change by a rig-étale cover of* T *and replacing* Z *by a* T*-dense open subscheme, there exists an open immersion* $Z \hookrightarrow \widehat{Q}$ *into a flat projective formal* T*-scheme* $\widehat{Q}$ *with the following properties*:

(i) $\widehat{Q}$ *is the* π*-adic completion of a flat projective* T*-scheme* Q.
(ii) *The geometric fibers of* $\varphi : Q \to T$ *are irreducible and reduced.*

(iii) *There exists an effective relative Cartier divisor Δ of Q/T with support in Z such that $\Delta \to T$ is finite and étale.*
(iv) $\varphi_* \mathcal{O}_Q(Q - \Delta)$ *admits a geometrically reduced multiplicative filtration.*

In particular, the compactification $Q \to T$ can be chosen relatively planar.

Proof. (i) and (ii): Consider the induced smooth relative curve

$$Z_k := Z \times_T T_k \longrightarrow T_k := (T_0)_{\mathrm{red}}.$$

By Lemma 3.7.2 there exists a stratification

$$T_k = T_k^0 \supset T_k^1 \supset \cdots \supset T_k^{m+1} = \emptyset$$

by closed subscheme such that $Z_k \times V^i$ admits a V^i-compactification as in Lemma 3.7.2, where $V^i := T_K^i - T_k^{i+1}$ for $i = 0, \ldots, m$. Then we proceed by induction on m. For $m = 0$ the assertion follows from Lemma 3.7.3. If $m \geq 1$, the assertion can be reduced to the case $m - 1$ by Lemmas 3.7.4 and 3.7.3.

(iii) and (iv) follow from Example 3.4.26, because $Z \to T$ admits a section after an étale surjective base change. □

Corollary 3.7.6. *Consider a situation as in Notation 3.4.18; i.e., $Z \to T$ is as in Proposition 3.7.5 and $g : X \to Z$ is a rig-étale finite flat morphism such that $g_* \mathcal{O}_X$ is generated by a function f over $\mathcal{O}_Z$.*

Then, after replacing T by a suitable rig-étale cover $T' \to T$ and by replacing Z and X by T-dense open subscheme, there exists a commutative diagram of flat formal T-schemes

$$\begin{array}{ccc} X & \hookrightarrow & \widehat{P} \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \hookrightarrow & \widehat{Q} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & T, & \end{array}$$

where the horizontal maps are open immersions with T-dense image, $\widehat{P}$ and $\widehat{Q}$ are the π-adic completion of P and Q, respectively. Here Q is a flat projective T-scheme as in Proposition 3.7.5 and $P \to Q$ is finite. There is a global section on $P - g^*\Delta$ generating $g_* \mathcal{O}_X$ over $\mathcal{O}_Z$, where $\Delta \subset Q$ is Cartier divisor as in Proposition 3.7.5.

Proof. First, after shrinking Z, one constructs $Z \hookrightarrow \widehat{Q}$ after a rig-étale base change as in Proposition 3.7.5, where Q/T is relatively planar curve with irreducible and reduced geometric fibers. We may assume that $X = \operatorname{Spf} A$, $Z = \operatorname{Spf} C$ and

$T = \operatorname{Spf} B$ are affine. Moreover, we may assume that the complement $Q_0 - Z_0$ is the supported by a hyperplane section H_0 on Q_0. We can lift H_0 to a hyperplane $H \subset Q$ with $Z_0 = Q_0 - H_0$ by lifting the coefficients of the equation defining H_0. Now look at the characteristic polynomial

$$F(\eta) = \eta^N + c_1\eta^{N-1} + \cdots + c_N \in C[\eta]$$

of f. The ring C is the π-adic completion of $\mathcal{C} := \mathcal{O}_Q(Q - H)$.

Since $X \to Z$ is rig-étale, $X \to T$ is rig-smooth. Then there exists an integer λ_1 which satisfies the assumption of Proposition 3.6.9. One can approximate the coefficients c_ν by functions γ_ν in $\mathcal{C}$ such that there is an isomorphism

$$\sigma_\lambda^* : \mathcal{A}/\pi^\lambda\mathcal{A} \xrightarrow{\sim} A/\pi^\lambda A$$

for some $\lambda \geq \lambda_1$, where

$$\mathcal{A} := \mathcal{C}[\eta]/(\Phi) \quad \text{with } \Phi(\eta) := \eta^N + \gamma_1\eta^{N-1} + \cdots + \gamma_N \in \mathcal{C}[\eta].$$

By Proposition 3.6.9 the morphism σ_λ^* lifts to an isomorphism $\sigma^* : \widehat{\mathcal{A}} \xrightarrow{\sim} A$ to the π-adic completion of $\mathcal{A}$. We may assume that each γ_ν is a global section of the invertible sheaf $\mathcal{O}_Q(H)$. If $L \to Q$ is the total space of $\mathcal{O}_Q(H)$, then $V(\Phi)$ gives rise to a closed subscheme $P \subset L$ which is finite flat over Q.

Let α be the image of the residue class of η in A. Then α generates A over C. There exists a global function γ on $Q - \Delta$ which is generically invertible on the fibers of Q/T such that $\gamma \cdot \alpha$ satisfies an integral equation over $Q - \Delta$. Moreover, $\gamma \cdot \alpha$ generates $g_*\mathcal{O}_X$ as $\mathcal{O}_Z$-algebra over a T-dense open subscheme of Z. This settles the assertion. □

Chapter 4
Rigid Analytic Curves

The main objective of this chapter is the Stable Reduction Theorem 4.5.3 for smooth projective K-curves X_K. Its proof is split into two problems. In a first step, dealt with in Sect. 3.4, we provide a projective R-model X of X_K such that its special fiber $X \otimes_R k$ is reduced. In a second step we will now analyze the singularities of $X \otimes_R k$. This part is related to the resolution of singularities in dimension 2.

For each point $\tilde{x}$ of the special fiber $X \otimes_R k$ we have the formal fiber $X_+(\tilde{x})$; cf. Definition 3.1.6(d). A cornerstone towards the Stable Reduction Theorem is the presentation of the periphery of $X_+(\tilde{x})$ in Proposition 4.1.11. This is a precise identification how the interior of the formal fiber is connected to the remainder of the curve. Noteworthy, we do not make use of a desingularization result [59] as the usual proofs do in [5] or [21], see also [83, Chap. 5].

In Sect. 4.2 the result on the periphery is used to constitute a genus formula in Proposition 4.2.6 which relates the genus of a projective rigid analytic curve to geometric data of the reduction. The formula allows us to define the genus of a formal fiber which serves as a measure for the quality of the singularity in the reduction. From these results we deduce the Stable Reduction Theorem in Sect. 4.4 for smooth projective curves by studying the behavior of meromorphic functions in Sect. 4.3. Blowing-up and blowing-down of components in the reduction can easily be handled by changing formal analytic structures.

Finally, the Stable Reduction Theorem leads in Sect. 4.6 to a construction of a universal covering of a curve. In the case of a split rational reduction the universal covering can be embedded into the projective line and its deck transformation group is a subgroup of $\mathrm{PGL}(2, K)$, which actually is a Schottky group. Finally we obtain a characterization of Mumford curves by conditions on its stable reduction.

We want to mention that there is also a rigid-analytic proof of the Stable Reduction Theorem by van der Put [96].

In the sections Sect. 4.1 till Sect. 4.3 we assume that our non-Archimedean field K is algebraically closed.

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4.1 Formal Fibers

In this section we prefer the notion of reduction which was introduced in Definition 3.1.6. In the following let X be a formal analytic space over K which is reduced. Note that we omit the subindex "K". A reduction is a map $\rho : X \to \widetilde{X}$ in the sense of Definition 3.1.6(c). We will analyze the formal fibers of points $\tilde{x} \in \widetilde{X}$; cf. Definition 3.1.6(d). This was first studied by Bosch in [7]. In the case of curves, we give here a simplified proof and a concrete approach to the result [11, 2.4].

If $X = \operatorname{Sp} A_K$ is a reduced affinoid space, then we put

$$A := \left\{ f \in A_K;\, \left| f(x) \right| \le 1 \text{ for all } x \in X \right\}.$$

Let $\widetilde{A}$ be the reduction of A_K, cf. Definition 1.4.4. There is a canonical surjective map

$$\rho : X := \operatorname{Sp} A_K \longrightarrow \widetilde{X} := \operatorname{Spec} \widetilde{A},\ x \longmapsto \rho(x) := \tilde{x}.$$

Definition 4.1.1. The pre-image of a point $\tilde{x}$ under ρ is called the *formal fiber* of $\tilde{x}$. An open formal affinoid variety $U \subset X$ with $\rho^{-1}(\tilde{x}) \subset U$ is called a *formal neighborhood* of the formal fiber.

In the following let X be an affinoid space which is reduced of pure dimension 1 with canonical reduction $\widetilde{X}$. Due to Corollary 1.2.6 there is a finite map

$$\varphi : X \longrightarrow \mathbb{D}$$

from X to the unit disc $\mathbb{D}$ such that its reduction $\widetilde{\varphi} : \widetilde{X} \to \widetilde{\mathbb{D}} = \mathbb{A}^1_k$ is finite and generically étale.

Let $T_K := K\langle\zeta\rangle$ be the Tate algebra in one variable ζ. So T_K coincides with the ring of holomorphic functions on $\mathbb{D}$. Set

$$T := R\langle\zeta\rangle = \left\{ f \in T_K;\, |f| \le 1 \right\}$$

and let

$$\widetilde{T} := T/\mathfrak{m}_R T = k[\tilde{\zeta}]$$

be its canonical reduction, where $\mathfrak{m}_R \subset R$ is the maximal ideal of R and $\tilde{\zeta}$ the reduction of ζ. The essential tool for our proof is the so-called *Gradgleichung* of Theorem 3.1.16

$$n := [A : T] = [\widetilde{A} : \widetilde{T}],$$

where the brackets indicate the degree of the associated extension of their total fields of fractions. Since $\widetilde{A}$ is a finite $\widetilde{T}$-module via $\widetilde{\varphi}^*$ and free of $\widetilde{T}$-torsion, $\widetilde{A}$ is a free $\widetilde{T}$-module. Then it follows from Theorem 3.1.17 that there is a T-basis $a_1, \ldots, a_n$ such that

$$A = T \cdot a_1 \perp \ldots \perp T \cdot a_n, \tag{4.1}$$

where $\perp$ indicates that the sum is orthonormal; i.e., for the sup-norm holds

$$|t_1a_1 + \cdots + t_na_n|_X = \max\bigl\{|t_1|_{\mathbb{D}}, \ldots, |t_n|_{\mathbb{D}}\bigr\}.$$

In particular, for every open admissible $U \subset \mathbb{D}$ and every $f \in \mathcal{O}_X(\varphi^{-1}(U))$ there is a unique representation

$$f = t_1a_1 + \cdots + t_n \cdot a_n \quad \text{with } t_i \in \mathcal{O}_{\mathbb{D}}(U).$$

We remind the reader that such a decomposition is in general no longer orthonormal if U is not formal open. However, we will show that the orthonormal decomposition (4.1) remains valid for formal fibers. Set

$$\begin{aligned} \mathbb{D}_+ &:= \bigl\{z \in \mathbb{D};\, \bigl|\zeta(z)\bigr| < 1\bigr\}, \\ X_+ &:= \varphi^{-1}(\mathbb{D}_+) = \bigl\{x \in X;\, \bigl|\varphi(x)\bigr| < 1\bigr\}. \end{aligned}$$

Let us go back to $f \in \mathcal{O}_X(\varphi^{-1}(U))$ from above. We can consider the characteristic polynomial

$$\chi_f(\eta) = (-1)^n\eta^n + \chi_1 \cdot \eta^{n-1} + \cdots + \chi_n \in \mathcal{O}_{\mathbb{D}}(U)[\eta]$$

of the multiplication by f on $\mathcal{O}_X(\varphi^{-1}(U))$. The sup-norm of f over $\varphi^{-1}(U)$ is given by

$$|f|_U = \max\bigl\{\sqrt[\nu]{|\chi_\nu|_U};\, \nu = 1, \ldots, n\bigr\} \tag{4.2}$$

by Lemma 1.4.1. In the following, for $\rho \in |K^\times|$ with $0 < \rho \le 1$, set

$$\begin{aligned} \mathbb{D}_\rho &:= \bigl\{z \in \mathbb{D};\, \bigl|\zeta(z)\bigr| \le \rho\bigr\}, \\ X_\rho &:= \bigl\{x \in X;\, \bigl|\varphi(x)\bigr| \le \rho\bigr\} = \varphi^{-1}(\mathbb{D}_\rho). \end{aligned} \tag{4.3}$$

For $f \in \mathcal{O}_X(X)$ the behavior of the sup-norm $|f|_\rho := |f|_{X_\rho}|_{X_\rho}$ on X_ρ can be described by a Newton polygon. This is a piecewise log-linear, continuous function which is monotone increasing; i.e., there are finitely many breaks

$$c_i \in \bigl|K^\times\bigr| \quad \text{with } 0 < c_1 < \cdots < c_{r+1} = 1,$$

exponents $\nu_i \in \mathbb{Q}$ with $0 \le \nu_1 \le \cdots \le \nu_r$ and $b_i \in |K^\times|$ such that

$$|f|_\rho := |f|_{X\rho} = b_i \cdot \rho^{\nu_i} \quad \text{for } c_i \le \rho \le c_{i+1} \text{ for } i = 1, \ldots, r.$$

Indeed, this follows from (4.2), because the coefficients $\chi_\nu \neq 0$ have only finitely many zeros and the spectral norm of an invertible function on an annulus $A(c_i, c_{i+1})$ behaves like a power of ρ, as follows from Proposition 1.3.4.

On $\widetilde{T} = k[\tilde{\zeta}]$ we have the (additively written) valuation given by the ideal $(\tilde{\zeta})$. We denote this valuation by $\mathfrak{o}(\tilde{t})$ for $\tilde{t} \in \widetilde{T} - \{0\}$, in particular we have that $\mathfrak{o}(\tilde{0}) = \infty$.

This gives rise to the spectral norm

$$\mathfrak{o} : \widetilde{A} \longrightarrow \mathbb{Q}, \ \tilde{f} \longmapsto \mathfrak{o}(\tilde{f}) := \min\left\{ \frac{\mathfrak{o}(\tilde{\chi}_\nu)}{\nu}; \nu = 1, \dots, n \right\},$$

on $\widetilde{A}$, where $(\tilde{\chi}_1, \dots, \tilde{\chi}_n)$ are the coefficients of the characteristic polynomial of $\tilde{f}$, which is obviously the reduction of the characteristic polynomial of $f \in A$ as A is a free T-module and $\widetilde{A} = A \otimes_R k = A \otimes_T \widetilde{T}$.

Now we are prepared to verify the following result.

Lemma 4.1.2. *Let $t_1, \dots, t_n$ be elements of T_K and set*

$$f = t_1 \cdot a_1 + \dots + t_n \cdot a_n.$$

(a) *Then there exists a constant $c(f) \in |K^\times|$ with $c(f) < 1$ and an exponent $\sigma(f) \in \mathbb{Q}$ such that*

$$|f|_\rho = s(f) \cdot \rho^{\sigma(f)} \quad \textit{for all } \rho \in \mathbb{R} \textit{ with } c(f) \le \rho \le 1,$$

where $s(f) := \max\{|t_1|_1, \dots, |t_n|_1\}$.

(b) *If $s(f) = 1$, then $\sigma(f) = \mathfrak{o}(\tilde{f})$.*

Proof. (a) We may assume $s(f) = 1$. The Newton polygon of f shows that there exist $c(f)$ and $\sigma(f)$ such that $|f|_\rho = \overline{s}(f) \cdot \rho^{\sigma(f)}$ for all ρ with $c(f) \le \rho \le 1$ and a constant $\overline{s}(f)$. It remains to show that $\overline{s}(f) = 1$. The latter follows from the fact that $\tilde{a}_1, \dots, \tilde{a}_n$ are linearly independent over $\widetilde{T}$.

(b) This follows from the fact that the reduction of the characteristic polynomial of f is the characteristic polynomial of the reduction $\tilde{f}$. □

In the following we will modify the basis $a_1, \dots, a_n$ in such a way that their reductions are part of an orthonormal basis with respect to the order function $\mathfrak{o}$. We will build such an orthonormal basis of the k-vector space $\widetilde{A}$ in a constructive way.

Notation 4.1.3. Consider the normalization $\widetilde{A}'$ of $\widetilde{A}$ over the ideal $\tilde{\zeta}\widetilde{A}$. Then we introduce the following notations:

- d — number of points $\tilde{x}_j$ in $\widetilde{X} = \operatorname{Spec} \widetilde{A}$ above the origin $\tilde{0} \in \widetilde{\mathbb{D}}$,
- d' — number of points $\tilde{x}'_i$ in $\widetilde{X}' = \operatorname{Spec} \widetilde{A}'$ above the origin $\tilde{0} \in \widetilde{\mathbb{D}}$,
- e_i — ramification index of $\tilde{\zeta}$ in $\widetilde{A}'$ at $\tilde{x}'_i$,
- $\tilde{\zeta}_i$ — uniformizer of $\widetilde{A}'$ at $\tilde{x}'_i$ such that $\tilde{\zeta} \equiv \tilde{\zeta}_i^{e_i} \bmod (\tilde{x}'_i\widetilde{A}')^{e_i+1}$,
- $\tilde{\varepsilon}_i$ — idempotent to single out the localization of $\widetilde{A}'$ at $\tilde{x}'_i \bmod \widetilde{A}' \cdot \tilde{\zeta}$

for $i = 1, \dots, d'$ and

$$\tilde{b}_{i,\nu} := \tilde{\varepsilon}_i \cdot \tilde{\zeta}_i^\nu \quad \text{for } \nu = 0, \dots, e_i - 1.$$

In particular, we have that

$$n = e_1 + \cdots + e_{d'}.$$

There exists a power $\tilde{\zeta}^N$ such that $\zeta^N \cdot \widetilde{A}' \subset \widetilde{A}$. Then

$$\widetilde{A}/\widetilde{A}'\tilde{\zeta}^N = k \cdot \overline{\ell}_1 \oplus \cdots \oplus k \cdot \overline{\ell}_r$$

is a finite dimensional vector space over k. The basis $(\overline{\ell}_1, \ldots, \overline{\ell}_r)$ is induced by a system $\tilde{\ell}_1, \ldots, \tilde{\ell}_r$ in $\widetilde{A}$ which is chosen to be orthonormal with respect to the order function $\mathfrak{o}$; i.e.

$$\mathfrak{o}\left(\sum_{j=1}^{r} \tilde{c}_j \tilde{\ell}_j\right) = \min\bigl\{\mathfrak{o}(\tilde{c}_1\tilde{\ell}_1), \ldots, \mathfrak{o}(\tilde{c}_r\tilde{\ell}_r)\bigr\}.$$

Such a system can be constructed by stepwise choosing liftings of linearly independent systems of ascending order. In particular, each $\tilde{\ell}_j$ has order $\mathfrak{o}(\tilde{\ell}_j) < N$. The completion of $\widetilde{A}'$ with respect to the ideal $\widetilde{A}'\tilde{\zeta}$ can be presented in the form

$$\widehat{\widetilde{A}}' = \bigoplus_{i=1}^{d'} k[[\tilde{\zeta}_i]] = \bigoplus_{i=1}^{d'} \bigl(k[[\tilde{\zeta}]]\tilde{\varepsilon}_i\tilde{\zeta}_i^0 \oplus \cdots \oplus k[[\tilde{\zeta}]]\tilde{\varepsilon}_i\tilde{\zeta}_i^{e_i-1}\bigr).$$

Thus, the system

$$\bigl(\tilde{\zeta}^{\mu+N}\tilde{b}_{i,\nu};\, \mu \in \mathbb{N}, \nu = 0, \ldots, e_i - 1, i = 1, \ldots, d'\bigr)$$

is a Schauder basis of $\tilde{\zeta}^N\widehat{\widetilde{A}}'$. Since $\widehat{\widetilde{A}}'/\tilde{\zeta}^N\widehat{\widetilde{A}}'$ has an orthogonal basis with respect to our order function $\mathfrak{o}$, the same is true for the subspace $\widehat{\widetilde{A}}/\tilde{\zeta}^N\widehat{\widetilde{A}}'$. Hereby, we see that the basis of $\tilde{\zeta}^N\widehat{\widetilde{A}}'$ can be extended to an orthogonal basis of $\widehat{\widetilde{A}}$ by adding the elements $\tilde{\ell}_1, \ldots, \tilde{\ell}_r$ which can be chosen in $\widetilde{A}$. Then the system

$$(\tilde{h}_\lambda)_{\lambda\in\Lambda} := (\tilde{\ell}_1, \ldots, \tilde{\ell}_r) \cup \bigcup_{\mu\in\mathbb{N}} \left(\tilde{\zeta}^\mu \cdot \tilde{\zeta}^N\tilde{b}_{i,\nu};\ \begin{matrix} \nu = 0, \ldots, e_i - 1 \\ i = 1, \ldots, d' \end{matrix}\right)$$

indexed by a set Λ is a topological k-basis of the completion $\widehat{\widetilde{A}}$; i.e., every $\tilde{f} \in \widehat{\widetilde{A}}$ can be represented in a unique way as a convergent series

$$\tilde{f} = \sum_{\lambda\in\Lambda} \tilde{c}_\lambda \tilde{h}_\lambda \quad \text{with } \tilde{c}_\lambda \in k.$$

Let $h_\lambda \in A$ be a lifting of $\tilde{h}_\lambda$ for $\lambda \in \Lambda$; they are chosen in such a way that we first choose liftings $\ell_j \in A$ of the $\tilde{\ell}_j$ and $a_{i,\nu} \in A$ of the $\tilde{\zeta}^N\tilde{b}_{i,\nu}$ and multiply the $a_{i,\nu}$ with the monomials ζ^μ. Thus, we obtain the system

$$(h_\lambda; \lambda \in \Lambda) = (\ell_1, \ldots, \ell_r) \cup \bigcup_{\mu\in\mathbb{N}} \left(\zeta^\mu \cdot a_{i,\nu};\ \begin{matrix} \nu = 0, \ldots, e_i - 1 \\ i = 1, \ldots, d' \end{matrix}\right) \subset A.$$

We have $\tilde{a}_{i,\nu} = \tilde{\zeta}^N\tilde{b}_{i,\nu}$ for the reduction of $a_{i,\nu}$.

Lemma 4.1.4. *There exists an element $\varrho_0 \in |K^\times|$ with $0 < \varrho_0 < 1$ with the property: If $\Lambda' \subset \Lambda$ is a finite subset and $c_\lambda \in K$ for $\lambda \in \Lambda'$, then*

$$\left|\sum_{\lambda\in\Lambda'} c_\lambda \cdot h_\lambda\right|_\rho = \max_{\lambda\in\Lambda'}\left\{|c_\lambda| \cdot \rho^{\mathfrak{o}(\tilde{h}_\lambda)}\right\} \quad \textit{for } \varrho_0 \le \rho \le 1.$$

Proof. In our system $(h_\lambda; \lambda \in \Lambda)$ there are only finitely many elements $\ell_j, a_{i,\nu}$ involved which do not belong to T. For these elements there exists a common ϱ_0 such that their Newton polygon over $[\varrho_0, 1]$ is log-linear. Then it is log-linear for all h_λ over $[\varrho_0, 1]$, since every product $\zeta^\mu \cdot a_{i,\nu}$ is also log-linear over $[\varrho_0, 1]$. Thus, the inequality "$\le$" follows from Lemma 4.1.2.

For the converse inequality, set $f := \sum_{\lambda\in\Lambda'} c_\lambda \cdot h_\lambda$. Let us first consider the case where $\mathfrak{o}(h_\lambda) = \alpha$ for all $\lambda \in \Lambda'$. Obviously, we may assume that $\max\{|c_\lambda|; \lambda \in \Lambda'\} = 1$. Then look at the characteristic polynomial of f

$$\chi_f(\eta) = (-1)^n \eta^n + \chi_1 \cdot \eta^{n-1} + \cdots + \chi_n \in T[\eta].$$

Its reduction is the characteristic polynomial of $\tilde{f}$. Then the spectral norm of $f|_{X_\rho}$ is given by

$$|f|_\rho = \max\left\{\sqrt[\nu]{|\chi_\nu|_\rho}; \nu = 1, \ldots, n\right\}.$$

For $g \in T$ with $|g|_1 = 1$ we have that $|g|_\rho \ge \rho^{\mathfrak{o}(\tilde{g})}$. Thus, we see that

$$|f|_\rho \ge \max_{\nu=1}^{n} \sqrt[\nu]{|\chi_\nu|_\rho} \ge \rho^{\min_{\nu=1}^{n} \mathfrak{o}(\tilde{\chi}_\nu)/\nu} \ge \rho^{\mathfrak{o}(\tilde{f})}.$$

Since the system $(\tilde{h}_\lambda; \lambda \in \Lambda)$ is orthogonal with respect to $\mathfrak{o}$, we have that $\mathfrak{o}(f) = \alpha$. Thus, the assertion is true in our special case.

In the general case, we arrange the sum with respect to $\mathfrak{o}$. Let

$$V := \left\{\alpha \subset \mathbb{Q}; \alpha = \mathfrak{o}(h_\lambda) \text{ for some } \lambda \in \Lambda' \text{ with } c_\lambda \ne 0\right\}$$

be the set of all possible orders. Thus, we obtain a decomposition

$$f = \sum_{\alpha\in V} f_\alpha \quad \text{with } f_\alpha := \sum_{\mathfrak{o}(\lambda)=\alpha} c_\lambda h_\lambda,$$

where $\mathfrak{o}(\lambda) := \mathfrak{o}(\tilde{h}_\lambda)$. From our special case we know that

$$|f_\alpha|_\rho = s_\alpha \cdot \rho^\alpha \quad \text{with } s_\alpha := \max_{\mathfrak{o}(\lambda)=\alpha} |c_\lambda|$$

for all $\rho \in |K^\times|$ with $\varrho_0 \le \rho \le 1$. The set

$$S := \left\{\rho \in [\varrho_0, \varrho_1]; s_\alpha \rho^\alpha = s_\beta \rho^\beta \text{ for some } \alpha, \beta \in V \text{ with } \alpha \ne \beta\right\}$$

is finite. Thus, using the ultrametric inequality, we see

$$|f|_\rho = \max_{\lambda \in \Lambda'} \{ |c_\lambda| \cdot \rho^{\mathfrak{o}(\tilde{h}_\lambda)} \}$$

for all $\rho \in [\varrho_0, 1] - S$. Since $|f|_\rho$ is a continuous function on ρ, the assertion follows. □

Corollary 4.1.5. *Let $\sum_{\lambda \in \Lambda} c_\lambda h_\lambda$ be a convergent series in $\mathcal{O}_X(X_+)$, then*

$$\left| \sum_{\lambda \in \Lambda} c_\lambda h_\lambda \right| = \sup_{\lambda \in \Lambda} |c_\lambda|.$$

Proposition 4.1.6. *In the above situation let $\pi \in R$ with $0 < |\pi| < 1$. Set*

$$T_+ := \left\{ t \in \mathcal{O}_{\mathbb{D}}(\mathbb{D}_+);\ \left| t(z) \right| \le 1 \text{ for all } z \in \mathbb{D}_+ \right\},$$
$$A_+ := \left\{ f \in \mathcal{O}_X(X_+);\ \left| f(z) \right| \le 1 \text{ for all } z \in X_+ \right\}.$$

Then we have the following results:

(a) *A_+ is complete with respect to the $(\pi A_+ + \zeta A_+)$-adic topology.*
(b) *Every holomorphic function f on X_+ has a unique representation in the form $f = \sum_{\lambda \in \Lambda} c_\lambda \cdot h_\lambda$ which converges on X_ρ for all $\rho \in |K^\times|$ with $\rho < 1$.*

Moreover, $|f| \le 1$ if and only if $c_\lambda \in R$ for all $\lambda \in \Lambda$.

(c) *The canonical morphism $A \to A_+$ yields an isomorphism from the $(\pi A + \zeta A)$-adic completion of A to A_+. In particular, $A_+ = A \otimes_T T_+$.*

Proof. (a) If $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in A_+, then the restriction $(f_n|_{X_\rho})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{O}_X(X_\rho)$ for every $\rho \in |K^\times|$ with $\rho < 1$, and converges to a unique element in $\mathcal{O}_X(X_\rho)$. Thus, we see that A_+ is complete.

(b) Since A is a finitely generated T-module, the $T/\zeta^m T$-module $A/\zeta^m A$ is also finitely generated, and hence $A/\zeta^m A$ is a finitely generated R-module. Then there exists a finite subset $\Lambda_m \subset \Lambda$ such that the residue classes $\{\overline{h}_\lambda;\ \lambda \in \Lambda_m\}$ is k-basis of the vector space $A/\zeta^m A \otimes_R k = \widetilde{A}/\tilde{\zeta}^m \widetilde{A}$. By the lemma of Nakayama we see that $A/\zeta^m A$ is generated by the set of the residue classes $\{\overline{h}_\lambda;\ \lambda \in \Lambda_m\}$ over R. Thus, we have that

$$A = A\zeta^m \oplus \bigoplus_{\lambda \in \Lambda_m} R h_\lambda. \qquad (*)$$

Now consider the K-vector space

$$B_K := \left\{ \sum_{\lambda \in \Lambda} c_\lambda h_\lambda;\ \lim_{\lambda \in \Lambda} |c_\lambda| \cdot \rho^{\mathfrak{o}(h_\lambda)} = 0 \text{ for all } \rho < 1 \right\}.$$

It is evident that B_K is a subspace of $\mathcal{O}_X(X_+)$, and that B_K is complete and hence closed in the Frechet space $\mathcal{O}_X(X_+)$.

Put $B := B_K \cap \mathring{\mathcal{O}}_X(X_+) = B \cap A_+$, which is an R-submodule of A_+. This R-module is complete and hence closed in A_+. Then the canonical map $A \to A_+$ factorizes through B. In fact, if $f \in A$, then by ($*$) for every $m \in \mathbb{N}$ there exists an element $b_m \in \bigoplus_{\lambda \in \Lambda_m} Rh_\lambda$ such that $f - b_m = \zeta^m f_m \in \zeta^m A$. Then $(b_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in B. Since B is complete, we see that $f \in B$.

Thus, we see that the canonical map $A_K \to \mathcal{O}_X(X_+)$ factorizes through B_K. Since A_K has dense image in $\mathcal{O}_X(X_+)$, the K-vector space B_K is a dense closed subset of $\mathcal{O}_X(X_+)$, and hence $B_K = \mathcal{O}_X(X_+)$. This verifies our assertion. The additional assertion follows from Corollary 4.1.5.

(c) As in the proof of (b) we see that the canonical isomorphism

$$A/\big(\pi^n A + \zeta^m A\big) \xrightarrow{\sim} A_+/\big(\pi^n A_+ + \zeta^m A_+\big) \quad \text{for all } m, n \in \mathbb{N}$$

is bijective. This implies the assertion by (a). □

One can introduce the notion of the order of a function $f \in A_+$ as well. For such an f there is also a Newton polygon, but it can have infinitely many breaks; i.e., there exist real numbers $0 < c_1 < c_2 < \cdots < 1$ such that

$$|f|_\rho = b_i(f) \cdot \rho^{\sigma_i(f)} \quad \text{for } c_i \le \rho \le c_{i+1}.$$

The polygon is strictly increasing and thus $\sigma_i(f) < \sigma_{i+1}(f)$ if it is not constant. We set $\mathfrak{o}(f) = \infty$ if $|f|_1 < 1$ or $\sigma_i(f) \to \infty$.

Proposition 4.1.7. *In the above situation we have the following*:

(a) *There is a canonical map* $\rho : A_+ \to \widehat{\widetilde{A}}$ *from* A_+ *to the completion of* $\widetilde{A}$ *with respect to the ideal* $\widetilde{A}\tilde{\zeta}$. *Its kernel is*

$$\ker \rho = \mathfrak{m}_R[[\zeta]] \cdot a_1 \oplus \cdots \oplus \mathfrak{m}_R[[\zeta]] \cdot a_n,$$

where $\mathfrak{m}_R$ *is the maximal ideal of* R. *Moreover,* $\ker \rho$ *is just the set of those* $f \in A_+$ *with* $\mathfrak{o}(f) = \infty$.

(b) *The map* ρ *is a surjective homomorphism of rings. In particular,* $\widehat{\widetilde{A}}$ *is the separated completion of* A_+ *with respect to the order function* $\mathfrak{o}$.

Proof. (a) Due to Proposition 4.1.6 the canonical residue maps

$$A/(A\pi + A\zeta)^j \longrightarrow \widetilde{A}/\widetilde{A}\tilde{\zeta}^j \quad \text{for } j \in \mathbb{N}$$

imply a surjective morphism $A_+ \to \widehat{\widetilde{A}}$. If $A = T$, then the map is given by

$$T_+ \xrightarrow{\sim} R[[\zeta]] \longrightarrow k[[\tilde{\zeta}]] = \widehat{\widetilde{T}}.$$

The kernel of ρ is as stated, since $(a_1, \ldots, a_n)$ is a T-basis of A. Thus, the description of $\ker \rho$ by $\mathfrak{o}(f)$ follows as well.

(b) The ideal $\mathfrak{a}_m := \{f \in A_+; \mathfrak{o}(f) \geq m\}$ is equal to $A_+\mathfrak{m}_R + A_+\zeta^m$. So $A_+/\mathfrak{a}_m \xrightarrow{\sim} \widetilde{A}/\widetilde{A}\tilde{\zeta}^m$ is an isomorphism for all $m \in \mathbb{N}$. □

Our next goal is to show that the completion of $\widetilde{A}$ depends intrinsically on the analytic structure of X_+ and is independent of the ambient space.

Proposition 4.1.8. *Let $\widetilde{X} = \operatorname{Spec} \widetilde{A}$ and $\tilde{x} \in \widetilde{X}$ be a closed point. Then we have the following results*:

(a) *The restriction $\rho : \mathring{\mathcal{O}}_X(X_+(\tilde{x})) \to \widehat{\widetilde{A}}_{\tilde{x}}$ is surjective, where $\mathring{\mathcal{O}}_X$ is the sheaf of functions bounded by 1 and $\widehat{\widetilde{A}}_{\tilde{x}}$ is the completion of $\widetilde{A}$ at $\tilde{x}$.*

(b) *The ring $\mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ is local and its maximal ideal is*

$$\mathfrak{m}_{\tilde{x}} = \left\{f \in \mathring{\mathcal{O}}_X\big(X_+(\tilde{x})\big); \left|f(x)\right| < 1 \text{ for all } x \in X_+(\tilde{x})\right\}.$$

(c) *The separated $\mathfrak{m}_{\tilde{x}}$-adic completion of $\mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ equals $\widehat{\widetilde{A}}_{\tilde{x}}$.*

(d) *The completion $\widehat{\widetilde{A}}_{\tilde{x}}$ depends on $X_+(\tilde{x})$ but not on the ambient space.*

(e) *$X_+(\tilde{x})$ is connected.*

Proof. There exists a finite homomorphism $\widetilde{\varphi} : \widetilde{A} \to \widetilde{T}$ such that $\tilde{x}$ lies above the origin $\tilde{0} \in \operatorname{Spec} \widetilde{T}$. Let $\tilde{x}_1 := \tilde{x}, \tilde{x}_2, \ldots, \tilde{x}_d$ be all the points above $\tilde{0}$. Due to Proposition 4.1.7 the restriction map induces a surjective morphism

$$\widehat{\mathring{\mathcal{O}}}_X\big(X_+(\tilde{x}_1)\big) \times \cdots \times \widehat{\mathring{\mathcal{O}}}_X\big(X_+(\tilde{x}_d)\big) \longrightarrow \widehat{\widetilde{A}} = \widehat{\widetilde{A}}_{\tilde{x}_1} \times \cdots \times \widehat{\widetilde{A}}_{\tilde{x}_d},$$

where the decomposition corresponds to the canonically given idempotents. Then all the assertions follow from Proposition 4.1.7. For assertion (c) note that $|K^\times|$ is divisible. Thus, $\mathfrak{m}_R = \mathfrak{m}_R^2$ and hence $\mathfrak{m}_{\tilde{x}}^m$ coincides with the set of functions of order $\geq m$ at $\tilde{x}$ for all $m \in \mathbb{N}$. □

The statements of Propositions 4.1.6, 4.1.7 and 4.1.8 are also true in higher dimensions but their proofs are more involved; cf. [7].

Lemma 4.1.9. *Let $\varphi : Y := \operatorname{Sp} B \to X := \operatorname{Sp} A$ be a morphism of reduced affinoid spaces. Let $\tilde{y} \in \widetilde{Y}$ and $\tilde{x} := \widetilde{\varphi}(\tilde{y}) \in \widetilde{X}$ be points in the reduction. Assume that the map $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$ is étale at $\tilde{y}$, then the induced morphism*

$$\varphi : Y_+(\tilde{y}) \xrightarrow{\sim} X_+(\tilde{x})$$

of the formal fibers is an isomorphism.

Proof. The statement is local with respect to the formal topology. We may assume that $\widetilde{A} \to \widetilde{B}$ is standard étale; cf. [15, 2.3/3]. There is an open immersion $\widetilde{Y} \hookrightarrow \widetilde{Z} = V(\widetilde{P}) \subset \mathbb{A}^1_{\widetilde{X}}$, where $\widetilde{P} \in \widetilde{A}[\tilde{\zeta}]$ is a monic polynomial and its derivative $\widetilde{P}'$ has no zeros on $\widetilde{Y}$. Moreover, $\widetilde{Y}$ is mapped isomorphically to $\widetilde{Z}_{\widetilde{Q}}$, where $\widetilde{Q} \in \widetilde{A}[\tilde{\zeta}]$. We can lift the polynomial $\widetilde{P}$ to a monic polynomial $P \in A[\zeta]$, and hence $Z = V(P) \subset \mathbb{D}_X$ is a closed subvariety of the relative disc $\mathbb{D}_X$. In particular, $V(P) \to X$ is formally étale at $\tilde{y}$. We also lift $\widetilde{Q}$ to a polynomial $Q \in A[\zeta]$.

By Remark 3.2.5 the formal analytic structures gives rise to formal R-models. Note that the isomorphism $\widetilde{Y} \xrightarrow{\sim} \widetilde{Z}_{\widetilde{Q}}$ can be defined at a level modulo π, where we eventually have to replace π some π' with $|\pi| < \pi'| < 1$. Thus, we can apply the lifting property of étale maps. So, the isomorphism $\widetilde{Y} \xrightarrow{\sim} \widetilde{Z}_{\widetilde{Q}}$ lifts to an isomorphism $Y \xrightarrow{\sim} Z_{|Q|=1}$ to the open part $Z_{|Q|=1}$ of Z, where Q takes absolute value 1. Thus, we can replace Y by Z. Since the residue field extension $k(\tilde{x}) \hookrightarrow k(\tilde{y})$ is trivial, we can consider $\tilde{y}$ as a simple zero of $\widetilde{P}(\tilde{x})(\tilde{\zeta})$. Then $Z \cap Y_+(\tilde{y}) \to X_+(\tilde{x})$ is an isomorphism. □

Corollary 4.1.10. *Let $X = \operatorname{Sp} A$ be a reduced affinoid space with reduction $\widetilde{X} = \operatorname{Spec} \widetilde{A}$. If $\tilde{x}$ is a smooth point of $\widetilde{X}$ of dimension n and if the residue field extension $k = k(\tilde{x})$ is trivial, then the formal fiber $X_+(\tilde{x})$ is isomorphic to the open unit ball $\mathbb{D}^n_+$.*

Proof. Since $\widetilde{X}$ is smooth at $\tilde{x}$ of dimension n, there exists functions $f_1, \ldots, f_n$ vanishing at $\tilde{x}$ such that their differentials $df_1, \ldots, df_n$ generate the module differential forms. Then the map $f := (f_1, \ldots, f_n) : X \to \mathbb{D}^n$ is formally étale at $\tilde{x}$ and hence the assertion follows from Lemma 4.1.9. □

Proposition 4.1.11. *Let $X = \operatorname{Sp} A_K$ be smooth of dimension 1. Let A be the set of all $f \in A_K$ with $|f|_X \leq 1$ and $\widetilde{A} = A \otimes_R k$ the canonical reduction of A_K. Let $\tilde{x}$ be a point in the reduction $\widetilde{X} = \operatorname{Spec} \widetilde{A}$. Let f be a function in A such that $\tilde{x}$ is an isolated zero of $\tilde{f} \in \widetilde{A}$. Let $\tilde{x}'_1, \ldots, \tilde{x}'_{d'}$ be all the points in the normalization $\widetilde{X}' = \operatorname{Spec} \widetilde{A}'$ of $\widetilde{X}$ lying over $\tilde{x}$.*

If now $\rho \in |K^\times|$ with $\rho < 1$ is close to 1, then the rigid analytic variety $\{x \in X_+(\tilde{x}); |f(x)| \geq \rho\}$ decomposes into d' connected components $R_1, \ldots, R_{d'}$ which are semi-open annuli.

More precisely, let ξ be a coordinate function on a disc $\mathbb{D}$, and denote by $e_i = \operatorname{ord}_{\tilde{x}'_i}(\tilde{f})$ the vanishing order of $\tilde{f}$ in $\widetilde{A}'$ at $\tilde{x}'_i$, for $i = 1, \ldots, d'$. Then there are isomorphisms

$$\varphi_i : R_i \xrightarrow{\sim} \left\{z \in \mathbb{D}; \rho^{1/e_i} \leq \left|\xi(z)\right| < 1\right\}$$

such that $f|_{R_i}$ coincides with $\varphi_i^*(\xi^{e_i})$, up to a unit in $\mathring{\mathcal{O}}_X(R_i)$.

Furthermore, if the image $\tilde{h} \in \widehat{\mathcal{O}}_{\widetilde{X},\tilde{x}}$ of an element $h \in \mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ has order $\tau_{i_0} < \infty$ at $\tilde{x}'_{i_0}$ for some index i_0, and if ρ is close to 1, then $h|_{R_{i_0}}$ coincides with $\varphi_{i_0}^*(\xi^{\tau_{i_0}})$, up to a unit in $\mathring{\mathcal{O}}_X(R_{i_0})$.

Proof. First assume that f gives rise to a finite map $\widetilde{\varphi} : \widetilde{X} \to \widetilde{\mathbb{D}} = \mathbb{A}^1_k$ such that $\widetilde{\varphi}^*(\tilde{\zeta}) = \tilde{f}$. In this case we are in a situation as already discussed in the whole section. Recall the notations from Notation 4.1.3. We introduced idempotents $\tilde{\varepsilon}_i$ and uniformizers $\tilde{\zeta}_i$ of the local ring $\widetilde{A}'_{\tilde{x}'_i}$ for $i = 1, \ldots, d'$. Therefore, $\tilde{\varepsilon}_i \cdot \tilde{\zeta}_i \in \widetilde{A}'$ is a uniformizer of $\widetilde{A}'_{\tilde{x}'_i}$ which vanishes at the points $\tilde{x}'_j$ of order greater than e_j for $j \neq i$. For $i = 1, \ldots, d'$ set

$$\varepsilon_i := a_{i,0}/\zeta^N \quad \text{and} \quad \xi_i := a_{i,1}/\zeta^N.$$

The functions $\varepsilon_1, \ldots, \varepsilon_{d'}$ behave like liftings of the idempotents $\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{d'}$ and $\xi_1, \ldots, \xi_{d'}$ behave like liftings of the uniformizers $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_{d'}$. By our choice in Notation 4.1.3 we have that

$$\begin{aligned} &\operatorname{ord}_{\tilde{x}'_i} \tilde{\varepsilon}_i = 0 \quad \text{and} \quad \operatorname{ord}_{\tilde{x}'_j} \tilde{\varepsilon}_i > e_j \quad \text{for } j \neq i \\ &\operatorname{ord}_{\tilde{x}'_i} \tilde{\xi}_i = 1 \quad \text{and} \quad \operatorname{ord}_{\tilde{x}'_j} \tilde{\xi}_i > e_j \quad \text{for } j \neq i \\ &\operatorname{ord}_{\tilde{x}'_j} (\tilde{\xi}_i^{e_i} - \tilde{\zeta}\tilde{\varepsilon}_i) > e_j \quad \text{for all } j = 1, \ldots, d'. \end{aligned} \tag{4.4}$$

Although ε_i and ξ_i eventually do not belong to A, they are defined as holomorphic functions on $X - V(\zeta)$, so they have Newton polygons. Let $\varrho_0 < 1$ be chosen as in Lemma 4.1.4, then we see

$$\begin{aligned} &|\varepsilon_i|_\rho = 1 && \text{for all } \rho \in [\varrho_0, 1], \\ &|\xi_i|_\rho = \rho^{1/e_i} && \text{for all } \rho \in [\varrho_0, 1], \end{aligned} \tag{4.5}$$

because $\mathfrak{o}(\tilde{a}_{i,0}) = N$ and $\mathfrak{o}(\tilde{a}_{i,1}) = N + 1/e_i$. Furthermore, we have

$$\begin{aligned} &|\varepsilon_i \cdot \varepsilon_j|_\rho < 1 && \text{for all } i, j \text{ with } i \neq j, \rho \in [\varrho_0, 1) \\ &\big|\varepsilon_1(x) + \cdots + \varepsilon_{d'}(x)\big| = 1 && \text{for all } x \in A(\rho, \beta), \end{aligned} \tag{4.6}$$

where, for $\varrho_0 \leq \rho < \beta < 1$, we put

$$\begin{aligned} A(\rho, \beta) &:= \big\{z \in \mathbb{D}; \rho \leq \big|\zeta(z)\big| \leq \beta\big\}, \\ R(\rho, \beta) &:= \varphi^{-1}\big(A(\rho, \beta)\big). \end{aligned}$$

Moreover, for $i = 1, \ldots, d'$ set

$$R_i(\rho, \beta) := \big\{x \in R(\rho, \beta); \varepsilon_i(x) = 1\big\}.$$

By (4.5) and (4.6) we see that, for $\rho, \beta \in [\rho_0, 1)$ with $\rho < \beta < 1$,

$$R(\rho, \beta) = R_1(\rho, \beta) \dot{\cup} \cdots \dot{\cup} R_{d'}(\rho, \beta)$$

is a disjoint union of d' connected components. One can also write

$$R_i(\rho, \beta) := \big\{x \in \varphi_i^{-1}\big(A(\rho, \beta)\big); \rho^{1/e_i} \leq \big|\xi_i(x)\big| \leq \beta^{1/e_i}\big\}.$$

For the slope of $b_i := \xi_i^{e_i} - \zeta\varepsilon_i$ we see by (4.4) that

$$|b_i|_\rho < \rho^{e_i} \quad \text{for } \rho \in [\varrho_0, 1]. \tag{4.7}$$

Now consider the finite extensions

$$K\langle\zeta/\beta, \rho/\zeta\rangle \subset K\big\langle\zeta/\beta, \rho/\zeta, \xi_i/\beta^{1/e_i}, \rho^{1/e_i}/\xi_i\big\rangle \subset \mathcal{O}_X\big(R_i(\rho,\beta)\big).$$

The degree of the first extension is at least e_i, as seen by considering the slope of $|\xi_i|$, and the extension from the first to the last is at most e_i, since the morphism $\varphi : R(\rho,\beta) \to A(\rho,\beta)$ is finite of degree n and $n = e_1 + \cdots + e_{d'}$. So ξ_i satisfies an integral equation $K\langle\zeta/\beta, \rho/\zeta\rangle$ over of degree e_i. Then we can represent every $f \in K\langle\zeta/\beta, \rho/\zeta, \xi_i/\beta^{1/e_i}, \rho^{1/e_i}/\xi_i\rangle$ as convergent series

$$f = \sum_{\mu\in\mathbb{Z}} \sum_{\nu=0}^{e_i-1} c_{\mu,\nu}\zeta^\mu \xi_i^\nu$$

Due to Lemma 4.1.4 the slope of $|f|_\rho$ is given by

$$|f|_\rho = \max\big\{|c_{\mu,\nu}|\rho^\mu \rho^{\nu/e_i}\big\} = \max\big\{|c_{\mu,\nu}|\rho^\mu \varrho^\nu\big\}$$

So we have $|f|_\rho \le 1$ if and only if $|c_{\mu,\nu}|\rho^\mu\varrho^\nu \le 1$ for all μ, ν. This implies

$$R\langle\zeta/\rho, \rho/\zeta, \xi_i/\varrho, \varrho/\xi_i\rangle = \mathring{\mathcal{O}}_X\big(R_i(\rho,\rho)\big) \cap K\langle\zeta/\rho, \rho/\zeta, \xi_i/\varrho, \varrho/\xi_i\rangle.$$

Then we see by Lemma 3.1.4 that the relation (4.7) yields

$$R\langle\zeta/\rho, \rho/\zeta, \xi_i/\varrho, \varrho/\xi_i\rangle = R\langle\xi_i/\varrho, \varrho/\xi_i\rangle.$$

Thus, the degree of the finite extension $K\langle\xi_i/\varrho, \varrho/\xi_i\rangle \subset \mathcal{O}_X(R_i(\rho,\beta))$ is equal to 1, and hence both rings coincide, because the first one is normal.

This settles all the assertions in the case, when $\tilde{f}$ gives rise to a finite morphism $\widetilde{\varphi} : \widetilde{X}' \to \widetilde{\mathbb{D}}$ with $\widetilde{\varphi}^{-1}(\tilde{0}) = \{\tilde{x}_1', \ldots, \tilde{x}_{d'}'\}$.

In the general case we can choose a $g \in A_K$ with $|g| \le 1$ such that $\tilde{x}$ is an isolated zero of $\tilde{g}$ and $\tilde{g}$ gives rise to a finite morphism $\widetilde{\varphi} : \widetilde{X} \to \widetilde{\mathbb{D}}$ as above. Then we can apply the result for $\tilde{g}$. For each $i = 1, \ldots, d'$ there exists a $\tau_i \in \mathbb{N}$ such that the order of $\tilde{f}$ at $\tilde{x}_i'$ is τ_i. Then $f|_{R_i(\rho,\beta)}$ behaves like $\xi_i^{\tau_i}$. Here τ_i corresponds to the number e_i of the assertion.

The assertion on h follows in a similar way, because due to Proposition 4.1.7 one can approximate h by an element $g \in A$ modulo a $(\tau_{i_0} + 1)$-power of $\tilde{x}_{i_0}'$. Then h and g have the same behavior on R_{i_0} up to a unit in $\mathring{\mathcal{O}}_X(R_{i_0})$. □

In simple cases one can determine the structure of the whole formal fiber.

Proposition 4.1.12. *Let X be a smooth affinoid space of pure dimension* 1. *Let $\tilde{x} \in \widetilde{X} = \operatorname{Spec} \widetilde{A}$ be a closed point.*

(a) *$\tilde{x}$ is a smooth point on $\widetilde{X}$ if and only if $X_+(\tilde{x}) \cong \mathbb{D}_+$.*
(b) *$\tilde{x}$ is an ordinary double point on $\widetilde{X}$ if and only if $X_+(\tilde{x}) \cong A(\varepsilon, 1)$ is isomorphic to an open annulus of a height $\varepsilon \in |K^{\times|}$ with $0 < \varepsilon < 1$.*

Proof. (a) If $X_+(\tilde{x}) \cong \mathbb{D}_+$, then $\widehat{\widetilde{A}}$ is a formal power series ring in one variable by Proposition 4.1.8, and hence $\widetilde{X}$ is smooth at $\tilde{x}$.

Now assume that $\widetilde{X}$ is smooth at $\tilde{x}$. Let $\tilde{\xi}$ be a uniformizer of $\widetilde{A}$ at $\tilde{x}$ and $\xi \in A$ a lifting of $\tilde{\xi}$. Then, the map $\xi : X \to \mathbb{D}$ is formally étale. Thus, the assertion follows from Corollary 4.1.10.

(b) Let $\tilde{x}$ be a double point, then $\widetilde{A} = k \oplus \widetilde{\mathfrak{m}}$, where $\widetilde{\mathfrak{m}} \subset \widetilde{A}$ is the maximal ideal associated to $\tilde{x}$. Since $\tilde{x}$ is a double point, $\widetilde{\mathfrak{m}} = \widetilde{A}\tilde{f} + \widetilde{A}\tilde{g}$ is generated by two elements after replacing $\widetilde{X}$ by a formal neighborhood of $\tilde{x}$. Assume first that $\tilde{x}$ lies on different components. Then we may assume that $\tilde{f} \cdot \tilde{g} = \tilde{0}$. There exist liftings $f \in A$ and $g \in A$ of $\tilde{f}$ and $\tilde{g}$. Then one shows $A = R \perp Af \perp Ag$. Moreover, $\tilde{f} \cdot \tilde{g} = \tilde{0}$ implies $f \cdot g = h_0$ with $\gamma := |h_0| < 1$. Using $A = R \perp Af \perp Ag$, we can write

$$h_0 := f \cdot g = c_1 + g_1 f + f_1 g \quad \text{with } c_1 \in R,\ f_1, g_1 \in A$$

satisfying $|c_1|, |f_1|, |g_1| \leq \gamma$. By induction there are elements $c_i \in R$ and $f_i, g_i, h_i \in A$ with

$$\left(f - \sum_{i=1}^{n} f_i\right) \cdot \left(g - \sum_{i=1}^{n} g_i\right) = \sum_{i=1}^{n} c_i + h_n$$

satisfying $|c_i|, |f_i|, |g_i| \leq \gamma^i$ and $|h_i| \leq \gamma^{i+1}$ for $i \in \mathbb{N}$. In fact, decompose

$$h_n = c_{n+1} + g_{n+1} f + f_{n+1} g$$

with $c_{n+1} \in R$, $f_{n+1}, g_{n+1} \in A$ and $|c_{n+1}|, |f_{n+1}|, |g_{n+1}| \leq \gamma^{n+1}$. Then we obtain $h_{n+1} \in A$ with $|h_{n+1}| \leq \gamma^{n+2}$. Therefore, the series

$$f' := f - \sum_{i=1}^{\infty} f_i, \qquad g' := g - \sum_{i=1}^{\infty} g_i \quad \text{and} \quad c := \sum_{i=1}^{\infty} c_i$$

converge and satisfies $f' \cdot g' = c' \in R$ with $|c'| < 1$. The case $c' = 0$ is excluded, since A is a domain. The reduction of f' and g' coincides with $\tilde{f}$ and $\tilde{g}$, respectively. Thus, we may assume that $f, g \in A$ satisfy $fg = c \in R - \{0\}$. We have the map $\varphi : K\langle \zeta, c/\zeta \rangle \to A_K, \zeta \mapsto f$. If $\rho \in K^\times$ with $|c| < |\rho| < 1$, then consider the mapping

$$\varphi_\rho : K\langle \zeta/\rho, c/\rho\zeta \rangle \longrightarrow A_K\langle f/\rho, g/\rho \rangle = \mathcal{O}_X\big(X_\rho(\tilde{x})\big),$$

where

$$X_\rho(\tilde{x}) := \big\{x \in X;\ |c/\rho| \leq \big|f(x)\big| \leq \rho\big\}.$$

By reason of dimensions it suffices to show that φ_ρ is surjective. Moreover, it suffices to show that the restriction of every element of $a \in A$ to X_ρ lies in the image of φ_ρ, since a generating system of the affinoid algebra $A_K\langle f/\rho, g/\rho\rangle$ is given by elements of A and the functions f/ρ and g/ρ. Indeed, by repeated application of the decomposition $A = R \perp Af \perp Ag$ and using $f \cdot g = c \in R$, for every $a \in A$ we can construct a series

$$a = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n f^n + \sum_{n=1}^{\infty} \beta_n g^n$$

with $\alpha_n, \beta_n \in R$ which converges on the domain $X_\rho(\tilde{x})$, as long as $|\rho| < 1$. Therefore, φ_ρ is surjective and hence bijective.

If the point $\tilde{x}$ lies only on one component, there exists a pointed étale morphism $(\widetilde{Y}, \tilde{y}) \to (\widetilde{X}, \tilde{x})$ from some affine scheme $\widetilde{Y}$ such that the point $\tilde{y}$ lies on two different components; cf. [78, Cor. 1, p. 99]. Now one can lift the morphism $\widetilde{Y} \to \widetilde{X}$ to a formal étale morphism $Y \to X$. Then by Lemma 4.1.9 the assertion follows from the special case discussed above.

The converse implication follows from Proposition 4.1.8(c), because the $\mathfrak{m}$-adic completion of $\mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ is isomorphic to $k[[\zeta, \eta]]/(\zeta\eta)$ if $X_+(\tilde{x})$ is an open annulus. Thus, the completion of $\mathcal{O}_{\widetilde{X},\tilde{x}}$ is isomorphic to $k[[\zeta, \eta]]/(\zeta\eta)$, and hence $\tilde{x}$ is an ordinary double point. □

4.2 Genus Formula

Proposition 4.1.11 enables us to define the genus of the formal fiber of a closed point of $\widetilde{X}$, where $\rho : X \to \widetilde{X}$ is a reduction of a connected smooth projective curve X. As a preparation we show the following

Proposition 4.2.1. *Let $B_1, \ldots, B_n$ be pairwise disjoint closed affinoid discs in X. Let $B_{\nu,+}$ be a formal fiber of B_ν, and set*

$$B_0 := X - (B_{1,+} \cup \cdots \cup B_{n,+}).$$

Then $\mathfrak{B} := \{B_0, \ldots, B_n\}$ is a formal covering of X. Its associated reduction $X_{\mathfrak{B}}$ consists of n rational curves $\widetilde{X}_1, \ldots, \widetilde{X}_n$ which meet at a common point q. The canonical reductions $\widetilde{B}_\nu$ are open subsets of $\widetilde{X}_\nu$ for each $\nu = 1, \ldots, n$. In particular,

$$\widetilde{X}_{\mathfrak{B}} = \{q\} \dot{\cup} \widetilde{B}_1 \dot{\cup} \cdots \dot{\cup} \widetilde{B}_n.$$

The point q is smooth on $\widetilde{X}_{\mathfrak{B}}$ if and only if $g(X) = 0$ and $n = 1$.

Proof. $\mathfrak{B} := \{B_0, B_1, \ldots, B_n\}$ is a formal covering of X by Proposition 3.1.10. The reduction $\widetilde{X}_{\mathfrak{B}}$ contains the canonical reduction $\widetilde{B}_\nu$ of B_ν as an open subset for $\nu = 1, \ldots, n$. So $\widetilde{X}_{\mathfrak{B}}$ is a union of rational curves which meet in a single point q. Obviously, q is smooth at $\widetilde{X}_{\mathfrak{B}}$ if and only if $\widetilde{X}_{\mathfrak{B}}$ is isomorphic to the projective line. This is equivalent to $n = 1$ and $g(\widetilde{X}_{\mathfrak{B}}) = 0$. □

Definition 4.2.2. Let X be a connected smooth projective curve equipped with a formal analytic structure covering $\rho : X \to \widetilde{X}$ in the sense of Definition 3.1.6. Let $\widetilde{X}_1, \dots, \widetilde{X}_n$ be the irreducible components of $\widetilde{X}$ and let $p : \widetilde{X}' \to \widetilde{X}$ be the normalization of $\widetilde{X}$ and $\widetilde{X}'_\nu \subset \widetilde{X}'$ be the normalization of $\widetilde{X}_\nu$. For a point $q \in \widetilde{X}$ we put

$$N(q) := \{\nu \in \{1, \dots, n\}; q \in \widetilde{X}_\nu\} \quad \text{and} \quad n(q) := \operatorname{card} N(q).$$

For $\nu \in N(q)$ we put

$$M(q,\nu) := \{q' \in \widetilde{X}'_\nu; p(q') = q\} \quad \text{and} \quad m(q,\nu) := \operatorname{card} M(q,\nu).$$

The *cyclomatic number* of the reduction $\widetilde{X}$ is defined by

$$z(\widetilde{X}) := \sum_{q \in \widetilde{X}} \sum_{\nu \in N(q)} \big(m(q,\nu) - 1\big) + \sum_{q \in \widetilde{X}} \big(n(q) - 1\big) - n + 1.$$

Remark 4.2.3. The cyclomatic number $z(\widetilde{X})$ can be interpreted as the cyclomatic number of a geometric graph G in the sense of Definition A.1.1. The vertices of G are the irreducible components $\widetilde{X}_1, \dots, \widetilde{X}_n$ of $\widetilde{X}$ and the singular points define edges. For each singular point $q \in \widetilde{X}$ attach $(m(q,\nu) - 1)$ loops to the vertex $\widetilde{X}_\nu$, then make a choice and connect the $n(q)$ components passing through q by $(n(q) - 1)$ the edges. The resulting geometric graph is connected.

If the singularities of $\widetilde{X}$ are at most ordinary double points, $G = G(\widetilde{X})$ is uniquely determined by $\widetilde{X}$, because each double point lies on at most two irreducible components.

If $z(\widetilde{X}) = 0$, then $m(q,\nu) = 1$ for all q and ν. The configuration of the components of $\widetilde{X}$ is tree-like; i.e., the associated graph $G(\widetilde{X})$ is a tree, and $\widetilde{X} - \{q\}$ consists of $n(q)$ components. □

Definition 4.2.4. In the situation of Definition 4.2.2 let $q \in \widetilde{X}$ be a closed point. Let $U \subset X$ be an open formal neighborhood of q. Let $f \in \mathcal{O}_X(U)$ with $|f|_U \le 1$ and assume that its reduction $\tilde{f}$ has a single zero at q. Consider constants $\rho, \rho_{\mu,\nu}$ with $0 < \rho, \rho_{\mu,\nu} < 1$ such that

$$X_+(q) \cap \{x \in X; \rho \le |f(x)|\} = \bigcup_{\nu \in N(q)} \bigcup_{\mu \in M(q,\nu)} A_{\mu,\nu}$$

is a disjoint union and such that there are isomorphisms

$$\varphi_{\mu,\nu} : A_{\mu,\nu} \xrightarrow{\sim} \{z \in \mathbb{D}; \rho_{\mu,\nu} \le |z| < 1\}.$$

These data identify the periphery of the formal fiber of q as in Proposition 4.1.11. Note that $\sum_{\nu \in N(q)} m(q,\nu) = d'$, where d' is the number introduced in Notation 4.1.3. Then we define a smooth proper curve $X(q)$ by pasting $X_+(q)$ with affinoid discs $D_{\mu,\nu}$ via $\varphi_{\mu,\nu}$; i.e.,

$$X(q) := X_+(q) \cup \bigcup_{\mu,\nu} D_{\mu,\nu},$$

where $D_{\mu,\nu} = \{z \in \mathbb{P}^1_K; \rho_{\mu,\nu} \le |z|\}$. Then $X(q)$ is a smooth projective curve by Theorem 1.8.1. Its genus is called the *genus of the formal fiber* $X_+(q)$.

In $X(q)$ there are pairwise disjoint affinoid subdomains

$$B_{\mu,\nu} := \{z \in D_{\mu,\nu}; 1 \le |z|\} \simeq \mathbb{D},$$

which are isomorphic to the unit disc. Furthermore,

$$B^-_{\mu,\nu} = \{z \in D_{\mu,\nu}; 1 < |z|\}$$

is a formal fiber of $B_{\mu,\nu}$. These discs give rise to a formal covering $\mathfrak{B}$ of $X(q)$ by Proposition 3.1.10. By Proposition 4.2.1 the reduction $\widetilde{X}(q)$ of $X(q)$ with respect to $\mathfrak{B}$ has a unique singular point; also denoted by q. The formal fibers of q on X with respect to $\rho : X \to \widetilde{X}$ and of the one of $X(q)$ with respect to $\widetilde{X}_{\mathfrak{B}}$ are isomorphic; i.e., $X_+(q)$ and $X(q)_+(q)$ are isomorphic as rigid analytic spaces.

Lemma 4.2.5. *In the above situation let* $\alpha : \widetilde{X}' \to \widetilde{X}$ *and* $\beta : \widetilde{X}(q)' \to \widetilde{X}(q)$ *be the normalization of* $\widetilde{X}$ *and* $\widetilde{X}(q)$, *respectively. Then there exists a canonical isomorphism*

$$(\alpha_*\mathcal{O}_{\widetilde{X}'}/\mathcal{O}_{\widetilde{X}})_q \xrightarrow{\sim} (\beta_*\mathcal{O}_{\widetilde{X}(q)'}/\mathcal{O}_{\widetilde{X}(q)})_q.$$

Proof. The modules in question are of finite length, so it is enough to show the isomorphism for their q-adic completions. From Proposition 4.1.8 we can conclude $\widehat{\mathcal{O}}_{\widetilde{X},q} \simeq \widehat{\mathcal{O}}_{\widetilde{X}(q),q}$ as $X_+(q) \simeq X(q)_+(q)$. Since normalization and completion processes are compatible in the case of affine algebras, the q-adic completion of $(\alpha_*\mathcal{O}_{\widetilde{X}'})_q$ and $(\beta_*\mathcal{O}_{\widetilde{X}(q)'})_q$ are the normalization of $\widehat{\mathcal{O}}_{\widetilde{X},q}$ and $\widehat{\mathcal{O}}_{\widetilde{X}(q),q}$, respectively. This implies the assertion. □

Proposition 4.2.6. *In the situation of Definition 4.2.2 we have the following formula*

$$g(X) = \sum_{\nu=1}^{n} g(\widetilde{X}_\nu) + \sum_{q\in\widetilde{X}} g\big(X(q)\big) + z(\widetilde{X}),$$

where $g(\widetilde{X}_\nu)$ *is the genus of* $\widetilde{X}_\nu$. *In particular, if the singularities of* $\widetilde{X}$ *are at most ordinary double points, then*

$$g(X_K) = z(\widetilde{X}) + \sum_{\nu=1}^{n} g(\widetilde{X}_\nu).$$

Proof. By Remark 3.2.5 there is a formal R-model with special fiber $\widetilde{X}$. Since the Euler-Poincaré characteristic is constant in a flat family, the genus of X can be computed on the special fiber $\widetilde{X}$. We have the exact sequence

$$\begin{aligned} 0 &\to H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to H^0\big(\widetilde{X}', \mathcal{O}_{\widetilde{X}'}\big) \to H^0(\widetilde{X}, p_*\mathcal{O}_{\widetilde{X}'}/\mathcal{O}_{\widetilde{X}}) \\ &\to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to H^1\big(\widetilde{X}', \mathcal{O}_{\widetilde{X}'}\big) \to 0, \end{aligned}$$

where $p : \widetilde{X}' \to \widetilde{X}$ is the normalization of $\widetilde{X}$. Thus, we obtain that

$$g(X) = \dim H^1\big(\widetilde{X}', \mathcal{O}_{\widetilde{X}'}\big) + \Bigg[\sum_{q\in\widetilde{X}} \dim(p_*\mathcal{O}_{\widetilde{X}'}/\mathcal{O}_{\widetilde{X}})_q\Bigg] - n + 1.$$

By applying this formula to the curve $X(q)$, we obtain that

$$g\big(X(q)\big) = \dim\big(p_*(\mathcal{O}_{\widetilde{X}(q)'})/\mathcal{O}_{\widetilde{X}(q)}\big)_q - \Bigg[\sum_{\nu\in N(q)} m(q,\nu)\Bigg] + 1 \qquad (*)$$

because $H^1(\widetilde{X}(q)', \mathcal{O}_{\widetilde{X}(q)'}) = 0$, as $\widetilde{X}(q)'$ is rational. By Lemma 4.2.5 we have that

$$\begin{aligned}\dim(p_*\mathcal{O}_{\widetilde{X}'}/\mathcal{O}_{\widetilde{X}})_q &= \dim(p_*\mathcal{O}_{\widetilde{X}(q)'}/\mathcal{O}_{\widetilde{X}(q)})_q \\ &= g\big(X(q)\big) + \Bigg[\sum_{\nu\in N(q)} \big(m(q,\nu)-1\big)\Bigg] + n(q) - 1.\end{aligned}$$

Since $\dim H^1(\widetilde{X}', \mathcal{O}_{\widetilde{X}'}) = \sum_{\nu=1}^{n} g(\widetilde{X}_\nu)$, we see that

$$g(X) = \sum_{\nu=1}^{n} g(\widetilde{X}_\nu) + \sum_{q\in\widetilde{X}} g\big(X(q)\big) + b,$$

where

$$b := \sum_{q\in\widetilde{X}} \Bigg[\sum_{\nu\in N(q)} \big(m(q,\nu)-1\big) + \big(n(q)-1\big)\Bigg] - n + 1 = z(\widetilde{X}).$$

If the singularities of $\widetilde{X}$ are at most ordinary double points, then $g(X(q)) = 0$ by Proposition 4.1.12 for all $q \in \widetilde{X}$. Thus, the asserted formulas are true. □

Corollary 4.2.7. *If there exists a point $q \in \widetilde{X}$ with $g(X(q)) = g(X)$, then*

$$X - X_+(q) = B_1 \dot{\cup} \cdots \dot{\cup} B_{n(q)}$$

is a union of pairwise disjoint closed affinoid discs $B_1, \ldots, B_{n(q)}$.

Proof. If $g(X(q)) = g(X)$, then all irreducible components $\widetilde{X}_\nu$ are rational, the curves $X(p)$ have genus $g(X(p)) = 0$ for all $p \in \widetilde{X} - \{q\}$, and its cyclomatic number $z(\widetilde{X})$ is zero, as seen by the genus formula in Proposition 4.2.6. Then $\widetilde{X} - \{q\}$ decomposes into $n(q)$ connected components

$$\widetilde{X} - \{q\} = \widetilde{B}_1 \dot{\cup} \cdots \dot{\cup} \widetilde{B}_{n(q)};$$

cf. Remark 4.2.3. Let $B_\nu \subset X$ be the formal open part which induces $\widetilde{B}_\nu$. Using Proposition 4.1.11 we can add an open disc D_ν^- to B_ν in order to compactify B_ν to a proper smooth curve Y_ν with a formal structure $\widetilde{Y}_\nu$ such that $\widetilde{Y}_\nu = \{q\} \cup \widetilde{B}_\nu$. Then q is a smooth point on $\widetilde{Y}_\nu$. The genus formula in Proposition 4.2.6 yields $g(Y_\nu) = 0$. Thus, each B_ν is isomorphic to a closed affinoid disc. □

The other extreme case $g(X(q)) = 0$ can be characterized by our method as well. A point q on a reduced algebraic curve $\widetilde{X}$ is called an *ordinary n-fold point* if the q-adic completion of $\mathcal{O}_{\widetilde{X},q}$ is isomorphic to $k[[\zeta_1, \dots, \zeta_n]]/(\zeta_i\zeta_j;\, 1 \le i < j \le n)$.

Corollary 4.2.8. *For a point $q \in \widetilde{X}$ the following conditions are equivalent*:

(a) *$g(X(q)) = 0$ and $\widetilde{X}(q)$ has n components.*
(b) *q is an ordinary n-fold point on $\widetilde{X}$.*
(c) *$X_+(q) \simeq \mathbb{P}^1_K - (B_1 \dot\cup \cdots \dot\cup B_n)$, where $B_1, \dots, B_n$ are pairwise disjoint closed affinoid discs in the projective line.*
(d) *For every subdomain $D \subset X_+(q)$ the ring $\mathcal{O}_X(D)$ is factorial and $\widetilde{X}(q)$ has n components.*

Proof. (a) ↔ (b): Let $\alpha : \widetilde{X}' \to \widetilde{X}$ and $\beta : \widetilde{X}(q)' \to \widetilde{X}(q)$ be the normalization. Put

$$A := (\alpha_*\mathcal{O}_{\widetilde{X}'}/\mathcal{O}_{\widetilde{X}})_q \quad \text{and} \quad B := (\beta_*\mathcal{O}_{\widetilde{X}(q)'}/\mathcal{O}_{\widetilde{X}(q)})_q.$$

The formula ($*$) in the proof of Proposition 4.2.6 shows $g(X(q)) = \dim B - n + 1$, where n is the number of components of $\widetilde{X}(q)$. Thus, (a) is equivalent to $\dim B = n - 1$. The condition (b) is equivalent to $\dim A = n - 1$. Since $\dim A = \dim B$ by Lemma 4.2.5, the equivalence of (a) and (b) is clear.

(a) → (c): This is obvious as $X_+(q) \simeq X(q)_+(q)$ and $X(q) \simeq \mathbb{P}^1_K$.

(c) → (b): This follows by a direct computation; cf. Example 3.1.11.

(c) → (d): This is clear, because the polynomial ring is factorial.

(d) → (a): It suffices to show $g(X(q)) = 0$. Let $\mathfrak{B} = (B_0, \dots, B_n)$ be the formal covering which defines the formal structure of $X(q)$; cf. Definition 4.2.4, where $B_1, \dots, B_n$ are pairwise disjoint closed discs and B_0 is the complement of the union of the open discs $B_1^-, \dots, B_n^-$, as in Proposition 3.1.10.

Next we make use of some facts we will show below. Due to Lemma 4.2.9 we can enlarge the discs $B_1, \dots, B_n$ so that we can assume by (d) that $\mathcal{O}_X(B_0)$ is factorial. Since the meromorphic functions on $X(q)$ are dense in $\mathcal{O}_X(B_0)$, the module $\Gamma(B_0, \mathcal{O}_{X(q)}(D))$ can be generated by meromorphic functions on $X(q)$ for every divisor D on B_0.

Thus, we obtain that every divisor D is linearly equivalent to a divisor E with support in $B_1 \cup \dots \cup B_n$. Then Lemma 4.2.10 implies that the subgroup of the Picard group $\operatorname{Pic}(X(q))$ consisting of isomorphism classes of invertible sheaves of finite order prime to $\operatorname{char} k$ is finitely generated. So we see that $g(X(q)) = 0$, because this subgroup would not be of finite type if the genus of $X(q)$ would be positive. □

To verify the last implication we needed some preparations, which are also used in later contexts. In the following let X be a connected smooth projective curve over an algebraically closed field K.

Lemma 4.2.9. *Let $B_1, \dots, B_n \subset X$ be pairwise disjoint closed discs in X. Then there are pairwise disjoint closed discs $B'_1, \dots, B'_n \subset X$ and formal fibers $B'_{1,+}, \dots, B'_{n,+}$ of $B'_1, \dots, B'_n$ such that $B_i \subset B'_{i,+}$ for $i = 1, \dots, n$.*

Proof. First we enlarge every B_i to a subdomain B'_i with $B_i \subset B'_{i,+}$. By Lemma 3.1.9, there exists a meromorphic function ζ_i on X which serves as a coordinate function on B_i. Then, for all $c \in K^\times$ with $|c| > 1$ and small enough, the subset $\{x \in X; |\zeta_i(x)| \le c\}$ decomposes again in a disjoint union $V(c) \dot\cup B_i(c)$ with $B_i \subset B_i(c)$ such that ζ_i/c is a coordinate function on $B_i(c)$ as well. For $c \in K^\times$ with $|c| > 1$ and close to 1, the subsets $B_1(c), \dots, B_n(c)$ satisfy the claim. □

Lemma 4.2.10. *Let $B_1, \dots, B_n \subset X$ be pairwise disjoint closed discs in X and let ℓ be an integer prime to the residue characteristic* $\mathrm{char}(k)$. *If E is a divisor on X such that*

(i) $\mathrm{Supp}(E) \subset B_1 \cup \dots \cup B_n$,
(ii) $\deg(E|_{B_i}) = 0$ *for* $i = 1, \dots, n$,
(iii) $\ell \cdot E = \mathrm{div}(f)$ *for some* $f \in \mathcal{M}(X)$,

then E is principal.

Proof. By Lemma 4.2.9 we may assume that $\mathrm{Supp}(E|_{B_i}) \subset B_{i,+}$ for $i = 1, \dots, n$. Then we denote by $\mathfrak{B} := (B_0, B_1, \dots, B_n)$ the formal covering of X as defined in Proposition 3.1.10. We view f as a function on B_0 and adjust its supremum norm to 1. Then we have that $|f|_{B_i} := |f|_{B_i - B_{i,+}}| \le 1$ for $i = 1, \dots, n$.

Since $\deg(f|_{B_i}) = 0$, we see that f induces a constant function on the reduction $\widetilde{B}_i$ for $i = 1, \dots, n$. In particular, there exists an index j in $\{1, \dots, n\}$ such that $|f|_{B_j} = 1$, and hence $\tilde{f}(q) \ne 0$ for the singular point $q \in \widetilde{X}_{\mathfrak{B}}$ and so $|f|_{B_i} = 1$ for all $i = 1, \dots, n$. Therefore, f reduces to a constant function on $\widetilde{X}_{\mathfrak{B}}$.

Using the binomial series for ℓ-th roots, f has an ℓ-th root g_0 over B_0, because ℓ is prime to $\mathrm{char}(k)$. Since $\mathcal{O}_X(B_i)$ is factorial for $i = 1, \dots, n$, there are elements $g_i \in \mathcal{M}(B_i)$ with $\mathrm{div}(g_i) = E|_{B_i}$. Thus, we have that $g_i^\ell f^{-1}|_{B_i} =: h_i \in \mathcal{O}_X(B_i)^\times$ for $i = 1, \dots, n$. Since B_i is a disc, there exist elements $k_i \in \mathcal{O}_X(B_i)^\times$ with $h_i = k_i^\ell$. So we can assume that $f|_{B_i} = g_i^\ell$ and that $g_0 g_i^{-1}|_{B_0 \cap B_i} = c_i$ is constant. Then g_0 extends to a meromorphic function on C satisfying $f = g_0^\ell$, and hence $E = \mathrm{div}(g_0)$. □

Proposition 4.2.11. *Let $g(X)$ be the genus of X. Then we have the following*:

(a) *Let $B_1, B_2 \subset X$ be closed discs with $B_1 \cap B_2 \ne \emptyset$. If $g(X) \ge 1$, then $B_1 \subset B_2$ or $B_2 \subset B_1$.*
(b) *Let $A \subset X$ be an annulus of height* 1 *and let $B \subset X$ be a disc with $A \cap B \ne \emptyset$. If $g(X) \ge 1$, then $B \subset A$ or $A \subset B$.*
(c) *If X is a union of finitely many subsets $B'_1, \dots, B'_n$ which are unions of increasing sequences of discs, then $g(X) = 0$.*

Proof. (a) If $B_1 \cup B_2 = X$, every divisor on X is linearly equivalent to a divisor with support in B_1 as easily follows from Lemma 3.1.9. Then it would follow $g(X) = 0$

by Lemma 4.2.10 in contradiction to the assumption. Thus, there exists a point a in $X-(B_1\cup B_2)$. Let $b\in B_1\cap B_2$. Due to Lemma 3.1.9 there exists a meromorphic function ζ which serves as a coordinate function on B_1 with $|\zeta|_{B_1}=1$, $\zeta(b)=0$ and $\mathrm{Pol}(\zeta)=\{a\}$. If $|\zeta|_{B_2}\le 1$, then $B_2\subset B_1$, because B_1 is a connected component of $\{x\in X; |\zeta(x)|\le 1\}$. If $c:=|\zeta|_{B_2}\ge 1$, then $B_1\subset B_2$, since B_2 is a connected component of $\{x\in X; |\zeta(x)|\le c\}$.

(b) At first, it is clear that $A\cup B\neq X$. In fact, assume the contrary; i.e., $A\cup B=X$. If A is not contained in B and B not contained in A, then $A\cap B$ is formal open in A and B with respect to their canonical reductions. Indeed, by (a) the intersection is a union of formal fibers. Therefore, the covering (A,B) gives rise to a formal analytic structure $\rho: X\to\widetilde{X}$. One easily shows $g(\widetilde{X})=0$. Thus, we arrive at a contradiction as $g(X)=g(\widetilde{X})$. Therefore, we may assume that there exists points b in $B-A$ and a in $X-(A\cup B)$.

By Lemma 3.1.9 there exists a meromorphic function ζ which serves as a coordinate function on B with $|\zeta|_B=1$, $\zeta(b)=0$ and $\mathrm{Pol}(\zeta)=\{a\}$. We may assume $B\not\subset A$; otherwise we are done. Since $A\cap B\neq\emptyset$, the set B contains a dense open formal part of A as before. Thus, it follows $|\zeta|_A\le 1$ and hence $A\subset B$ as ζ is coordinate function on B.

(c) For a divisor $D\in\mathrm{Div}(X)$ on X set $d_i(D):=\deg(D|_{B_i'})$ and then put $d(D):=(d_1(D),\ldots,d_n(D))\in\mathbb{Z}^n$. Then consider the subgroup

$$T:=\big\{[D]\in\mathrm{Pic}\,X; [D]\text{ is of finite order prime to }\mathrm{char}\,k\text{ and }d(D)=0\big\}$$

of the Picard group $\mathrm{Pic}\,X:=\mathrm{Pic}_{X/K}(K)$; cf. Proposition 5.1.1. Since $g(X)\ge 1$, the prime to $\mathrm{char}\,k$ torsion of $\mathrm{Pic}\,X$ is not finitely generated, and hence T is not finitely generated. Using the assertion (a), one easily deduces from Lemma 4.2.10 that $T=0$. Thus, we arrive at a contradiction, and hence $g(X)=0$. □

4.3 Meromorphic Functions

Let X be a reduced rigid analytic variety. If one associates with each open affinoid subvariety $U\subset X$ the total ring of fractions $Q(\mathcal{O}_X(U))$ of the ring $\mathcal{O}_X(U)$ of holomorphic functions on U, one obtains a sheaf on the family of all open affinoid subdomains of X. This extends canonically to all admissible open subsets of X. The resulting sheaf $\mathcal{M}$ is called the *sheaf of meromorphic functions on* X. If X is the analytification of a projective algebraic variety, one knows from the rigid analytic GAGA theorems that $\mathcal{M}(X)$ equals the field of rational functions $K(X)$; cf. Theorem 1.8.1.

In the following we assume that X is the analytification of a connected smooth projective curve over an *algebraically closed field* K. Then each non-constant function $f\in\mathcal{M}(X)$ gives rise to a finite morphism $X\to\mathbb{P}^1_K$. All the subsets of type

$$U_\alpha^0:=\big\{x\in X; \big|f(x)\big|\le\alpha\big\}\quad\text{and}\quad U_\alpha^\infty:=\big\{x\in X; \big|f(x)\big|\ge\alpha\big\}$$

are affinoid for each $\alpha \in |K^\times|$. If $U \subset X$ is an affinoid subdomain of X, then each $m \in \mathcal{M}(U)$, $m \neq 0$, has a well-defined divisor $\operatorname{div}(m)$.

Furthermore, if the canonical reduction $\widetilde{U}$ of U is irreducible, the sup-norm is multiplicative on $\mathcal{O}_X(U)$ due to Remark 1.4.6. Thus the sup-norm extends to a valuation on the field $\mathcal{M}(U) = Q(\mathcal{O}_X(U))$. One has $|m| = |f| \cdot |g|^{-1}$ if $m = f/g$ with $f, g \in \mathcal{O}_X(U)$. It is clear that a restriction $\mathcal{M}(U) \to \mathcal{M}(U')$, where $U' \subset U$ is a non-empty formal subdomain of U, leaves such norms of meromorphic functions invariant.

Now let us fix a formal affinoid covering $\mathfrak{U}$ of X and consider the associated reduction $\widetilde{X} := \widetilde{X}_{\mathfrak{U}}$. Let $\widetilde{X}_1, \ldots, \widetilde{X}_n$ be the irreducible components of $\widetilde{X}$ and choose open dense affine subsets $\widetilde{U}_\nu \subset \widetilde{X}_\nu$, $\nu = 1, \ldots, n$, which are open in $\widetilde{X}$. Then let $U_\nu \subset X$ be the associated open affinoid subdomain. Thus, for each $m \in \mathcal{M}(X) - \{0\}$ we can define the *sup-norm of m at the component $\widetilde{X}_\nu$* by

$$|m|_\nu := |m|_{U_\nu}|.$$

Assuming $m \neq 0$, there are constants $c_\nu \in K^\times$ such that $|m|_\nu = |c_\nu|$. Then m/c_ν has sup-norm 1 on U_ν and, hence, reduces to a rational function $\widetilde{m/c_\nu}$ on $\widetilde{X}_\nu$. Its divisor on the normalization $\widetilde{X}'$ of $\widetilde{X}$ depends only on m but not on the choice of the constant c_ν. Therefore, we can define the *order of m at a point $\tilde{y} \in \widetilde{X}'_\nu$* of the normalization $\widetilde{X}'_\nu$ by

$$\operatorname{ord}_{\tilde{y}}(m) := \operatorname{ord}_{\tilde{y}}(\widetilde{m/c_\nu}).$$

Proposition 4.3.1. *Let $m \neq 0$ be a meromorphic function on X, let $\tilde{x}$ be a point of $\widetilde{X}$ and denote by $\tilde{y}_1, \ldots, \tilde{y}_r$ the points in the normalization $\widetilde{X}'$ of $\widetilde{X}$ lying over $\tilde{x}$. If $\tilde{x}$ is smooth or a double point, then the divisor of m has the degree*

$$\deg\bigl(\operatorname{div}(m|_{X_+(\tilde{x})})\bigr) = \sum_{i=1}^{r} \operatorname{ord}_{\tilde{y}_i}(m)$$

on the formal fiber $X_+(\tilde{x})$.

Remark 4.3.2. Actually, the assumption on the type of singularity of $\tilde{x}$ is superfluous. However, in order to verify this, one needs the existence of semi-stable reductions for X; cf. Theorem 4.4.3.

Proof of Proposition 4.3.1. At first, let us consider the case, where $\tilde{x}$ is smooth. Then $X_+(\tilde{x})$ is an open disc by Proposition 4.1.12. Let ξ be a coordinate function on $X_+(\tilde{x})$. Since m is a quotient of analytic functions, we may assume that m has no poles in $X_+(\tilde{x})$. Then m is a power series on $X_+(\tilde{x})$, say

$$m = \sum_{\nu=0}^{\infty} d_\nu \xi^\nu \in R[[\xi]],$$

up to a multiplicative scalar. The term of degree $\nu = \operatorname{ord}_{\tilde{x}}(m)$ in this expansion is dominant at the periphery of $X_+(\tilde{x})$. Hence, by the Weierstraß preparation theorem, m has precisely $\operatorname{ord}_{\tilde{x}}(m)$ zeros in $X_+(\tilde{x})$.

Next we consider the case, where $\tilde{x}$ is an ordinary double point. Then $X_+(\tilde{x})$ is an open annulus, say of height ε; cf. Proposition 4.1.12, and there are two points $\tilde{y}_1, \tilde{y}_2$ in the normalization $\widetilde{X}'$ lying over $\tilde{x}$. For $i = 1, 2$, let $\widetilde{X}_i$ be the component of $\widetilde{X}'$ whose normalization $\widetilde{X}_i$ contains $\tilde{y}_i$; in particular, $\widetilde{X}_1 = \widetilde{X}_2$ if and only if $\tilde{x}$ belongs to a single component of $\widetilde{X}$. Similarly as before, let $\xi \in \mathcal{O}_X(X_+(\tilde{x}))$ be a coordinate function on $X_+(\tilde{x})$ with $\operatorname{ord}_{\tilde{y}_1}(\tilde{\xi}) = 1$. If $c \in K^\times$ satisfies $|c| = \varepsilon$, then $\eta := c/\xi \in \mathring{\mathcal{O}}_X(X_+(\tilde{x}))$ is a coordinate function on $X_+(\tilde{x})$ with $\operatorname{ord}_{\tilde{y}_2}(\tilde{\eta}) = 1$. We may assume that m has no poles on $X_+(\tilde{x})$. Then m admits a Laurent series expansion

$$m = \sum_{\nu=-\infty}^{\infty} d_\nu \xi^\nu = \sum_{\nu=-\infty}^{\infty} d_{-\nu} c^{-\nu} \eta^\nu \in R\big[\big[\xi, \xi^{-1}\big]\big]$$

on $X_+(\tilde{x})$. By Proposition 1.3.4 the term $d_\nu \xi^\nu$ of degree $\nu_1 := \operatorname{ord}_{\tilde{y}_1}(m)$ satisfies $|d_{\nu_1}| = |m|_1$ and is dominant on the component of the periphery of $X_+(\tilde{x})$ corresponding to $\tilde{y}_1$. Likewise, the term $d_{-\nu_2} c^{-\nu_2} \eta^{\nu_2}$ of degree $\nu_2 := \operatorname{ord}_{\tilde{y}_2}(m)$ satisfies $|d_{-\nu_2} c^{-\nu_2}| = |m|_2$ and is dominant on the component of the periphery of $X_+(\tilde{x})$ corresponding to $\tilde{y}_2$. Similarly as in the power series case, one concludes that m has $\nu_1 + \nu_2$ zeros on $X_+(\tilde{x})$. □

If m has neither zeros nor poles on the formal fiber $X_+(\tilde{x})$ over the double point $\tilde{x}$, then $\nu_1 + \nu_2 = 0$ and hence $|m|_2 = |d_{\nu_1} c^{\nu_1}| = |c|^{\nu_1} |m|_1$. Thus we verified the following result.

Corollary 4.3.3. *Let $m \neq 0$ be a meromorphic function on X without zeros and poles on the formal fiber $X_+(\tilde{x})$ above an ordinary double point $\tilde{x} \in \widetilde{X}$. Let $\tilde{y}_1, \tilde{y}_2$ be the points in the normalization of $\widetilde{X}$ lying over $\tilde{x}$. Let $\widetilde{X}_1, \widetilde{X}_2$ be the components of $\widetilde{X}$ passing through $\tilde{x}$, where $\widetilde{X}_1 = \widetilde{X}_2$ if $\tilde{x}$ lies on a single component. Assume $\tilde{y}_i \in \widetilde{X}_i$ for $i = 1, 2$. Then*

$$|m|_2 = \varepsilon^{\nu_1} |m|_1,$$

where ε is the height of the annulus $X_+(\tilde{x})$ and $\nu_1 := \operatorname{ord}_{\tilde{y}_1}(m)$.

Corollary 4.3.4. *Let X be a connected smooth projective curve equipped with a formal structure $\rho : X \to \widetilde{X}$, which is semi-stable; cf. Definition 4.4.1. Let $\widetilde{X}_1, \ldots, \widetilde{X}_n$ be the irreducible components of $\widetilde{X}$ and let $\widetilde{X}'_1, \ldots, \widetilde{X}'_n$ be their corresponding normalizations. Put $X_\nu := \rho^{-1}(\widetilde{X}_\nu - \operatorname{Sing}(\widetilde{X}))$.*

If m is a meromorphic function on X without zeros and poles in the singular formal fibers, then the following conditions are equivalent:

(a) $\deg(\operatorname{div}(m|_{X_\nu})) = 0$ *for* $\nu = 1, \ldots, n$.
(b) $|m|_\mu = |m|_\nu$ *for all* $\mu, \nu = 1, \ldots, n$.
(c) $\operatorname{ord}_{\tilde{y}}(m) = 0$ *for all points* $\tilde{y} \in \widetilde{X}'$ *lying over singular points of* $\widetilde{X}$.

Proof. Let $c_\nu \in K^\times$ be an element with $|c_\nu| = |m|_\nu$ for $\nu = 1, \ldots, n$.

(a) → (b): If $|m|_\mu \neq |m|_\nu$ for some pair (μ, ν), then there exists a point $\tilde{y}_1 \in \widetilde{X}'$, say $\tilde{y}_1 \in \widetilde{X}'_{\nu_1}$ lying over $\tilde{x}_1 \in \widetilde{X}_{\nu_1} \cap \operatorname{Sing}(\widetilde{X})$ such that $\operatorname{ord}_{\tilde{y}_1}(m) > 0$. From the assumption (a) we know by Proposition 4.3.1 that

$$\deg\bigl(\operatorname{div}(\widetilde{m/c_\nu}|_{\widetilde{X}_{\nu_1}})\bigr) = 0.$$

Since the degree of a principal divisor is 0, there exists a point $\tilde{y}'_2 \in \widetilde{X}'_{\nu_2}$ lying over a point $\tilde{x}_2 \in \widetilde{X}_{\nu_1} \cap \operatorname{Sing}(\widetilde{X})$ such that $\operatorname{ord}_{\tilde{y}'_2}(m) < 0$. Let $\tilde{y}_2 \in \widetilde{X}'_{\nu_2}$ be the second point in $\widetilde{X}'$ lying over $\tilde{x}_2$. Since m has no zeros and no poles in $X_+(\tilde{x}_2)$, we obtain from Proposition 4.3.1 and Corollary 4.3.3

$$\operatorname{ord}_{\tilde{y}_2}(m) = -\operatorname{ord}_{\tilde{y}'_2}(m) \quad \text{and} \quad |c_{\nu_2}| > |c_{\nu_1}|.$$

Continuing this way, we obtain an infinite ascending chain $|c_{\nu_1}| < |c_{\nu_2}| < \cdots$. This is not possible, because $\widetilde{X}$ has only finitely many components.

(b) → (c): This follows from Corollary 4.3.3.

(c) → (a): The assumption (c) yields $\deg(\operatorname{div}(\widetilde{m/c_\nu}|_{\widetilde{X}_\nu})) = 0$ for $\nu = 1, \ldots, n$. By Corollary 4.3.3 this is equivalent to (a). □

Proposition 4.3.5. *Let X be a connected smooth projective curve equipped with a formal structure $\rho : X \to \widetilde{X}$. Let $\widetilde{Q} \subset \widetilde{X}$ be a non-empty finite set of smooth closed points and let $\widetilde{A} \subset \widetilde{X}$ be the union of all components $\widetilde{X}_\nu$ of $\widetilde{X}$ with $\widetilde{X}_\nu \cap \widetilde{Q} = \emptyset$. Denote by $X_{\widetilde{Q}} \subset X$ the open subset $\rho^{-1}(\widetilde{X} - \widetilde{Q})$. Then we have the following results*:

(a) *$X_{\widetilde{Q}}$ is an affinoid subdomain of X.*
(b) *The rational functions $\mathcal{M}(X)$ are dense in $\mathcal{O}_X(X_{\widetilde{Q}})$.*
(c) *If $\widetilde{X}_\nu$ is a component of $\widetilde{X}$ with $\widetilde{X}_\nu \cap \widetilde{Q} \neq \emptyset$, then there exists a non-empty open subset $\widetilde{U}$ of $\widetilde{X} - \widetilde{Q}$, contained in $\widetilde{X}_\nu$ such that $U := \rho^{-1}(\widetilde{U})$ is a formal open subdomain of $X_{\widetilde{Q}}$ with canonical reduction $\widetilde{U}$.*
(d) *Let $\tilde{x}$ be a point in $\widetilde{X} - \widetilde{Q}$. The formal fiber $X_{\widetilde{Q},+}(\tilde{x})$ of $X_{\widetilde{Q}}$ containing x with respect to the canonical reduction $\widetilde{X_{\widetilde{Q}}}$ of the affinoid space $X_{\widetilde{Q}}$ is the open subset $\rho^{-1}(\widetilde{Z}_{\tilde{x}})$, where $\widetilde{Z}_{\tilde{x}}$ is the connected component of $\widetilde{A} \cup \{\tilde{x}\}$ which contains $\tilde{x}$.*
(e) *If $\widetilde{Q}$ meets every component of $\widetilde{X}$, the canonical reduction $\widetilde{X_{\widetilde{Q}}}$ of $X_{\widetilde{Q}}$ is canonically isomorphic to $\widetilde{X} - \widetilde{Q}$.*

Proof. Let $\widetilde{Q} := \{\tilde{q}_1, \ldots, \tilde{q}_r\}$ and denote by $\widetilde{X}_{\nu(i)}$ the component of $\widetilde{X}$ containing $\tilde{q}_i$. Moreover, choose a finite set $\widetilde{P} = \{\tilde{p}_1, \ldots, \tilde{p}_n\}$ of smooth points of $\widetilde{X}$ which is disjoint from $\widetilde{Q}$ such that $\widetilde{P}$ meets every irreducible component of $\widetilde{X}$. For each $\tilde{q} \in \widetilde{Q}$ and $\tilde{p} \in \widetilde{P}$ choose a lifting $q \in X$ of $\tilde{q}$ and $p \in X$ of $\tilde{p} \in \widetilde{P}$, respectively.

By the Riemann-Roch Theorem 1.7.6 there exist meromorphic functions f_i for $i = 1, \ldots, r$ with $\operatorname{Pol}(f_i) = \{q_i\}$ and $|f_i|_{\nu(i)} = 1$ such that f_i vanishes at all points of P. Then we obtain that $|f_i(x)| \leq 1$ for all points $x \in X - X_+(\tilde{q}_i)$, and hence by Corollary 4.3.3 that $|f_i(x)| < 1$ for all x, which reduce to points $\tilde{x}$ lying in $\bigcup_{\nu \neq \nu(i)} \widetilde{X}_\nu$. In particular, we have $f_1, \ldots, f_r \in \mathring{\mathcal{O}}_X(X_{\widetilde{Q}})$.

Now put $f := f_1 + \cdots + f_r \in \mathcal{O}_X(X_{\widetilde{Q}})$. Then $\tilde{f}$ has poles at the points $\tilde{q}_1, \ldots, \tilde{q}_r$. The proof of Lemma 3.1.9 shows that $X_{\widetilde{Q}}$ is a connected component of $\{x \in X; |f(x)| \leq 1\}$. Thus, the assertions (a) and (b) are clear.

(c) Put $\widetilde{U}_i := \{\tilde{x} \in \widetilde{X} - \{\tilde{q}_i\}; \tilde{f}_i(\tilde{x}) \neq 0\}$. Then $\widetilde{U}_i \subset \widetilde{X}_{\nu(i)}$ is affine open in $\widetilde{X} - \widetilde{Q}$ because of the pole of $\tilde{f}_i$ at $\tilde{q}_i$ and the zeros on all other components. Moreover, $U_i := \rho^{-1}(\widetilde{U}_i)$ is an affinoid formal open subdomain of $X_{\widetilde{Q}}$ with the canonical reduction $\widetilde{U}_i$.

(d) Consider the natural map $\sigma : \widetilde{X} - \widetilde{Q} \longrightarrow \widetilde{X_{\widetilde{Q}}}$. Since $\widetilde{X_{\widetilde{Q}}}$ is affine, σ maps each $\rho^{-1}(\widetilde{Z}_{\tilde{x}})$ onto a point in $\widetilde{X_{\widetilde{Q}}}$ for $\tilde{x} \in \widetilde{X} - \widetilde{Q}$. Since formal fibers are connected by Proposition 4.1.8(e), we have $\sigma^{-1}(\sigma(\tilde{x})) = \widetilde{Z}_{\tilde{x}}$ for all $\tilde{x} \in \widetilde{X} - \widetilde{Q}$. This verifies the assertion (d).

(e) In this case $\widetilde{X_{\widetilde{Q}}}$ and $\widetilde{X} - \widetilde{Q}$ are affine and, due to (d), the morphism $\sigma : \widetilde{X} - \widetilde{Q} \longrightarrow \widetilde{X_{\widetilde{Q}}}$ is bijective. Then σ is an isomorphism. □

As a first application of the proposition we show:

Corollary 4.3.6. *Let X be a connected smooth projective curve with semi-stable reduction $\widetilde{X}$ and let $\widetilde{X}_1, \ldots, \widetilde{X}_n$ be the irreducible components of $\widetilde{X}$. If $\widetilde{m}_1, \ldots, \widetilde{m}_n$ are rational functions on $\widetilde{X}_1, \ldots, \widetilde{X}_n$ such that each $\widetilde{m}_\nu$ is regular at every singular point of $\widetilde{X}$ which lies on $\widetilde{X}_\nu$ and such that*

$$\widetilde{m}_\mu(\tilde{x}) = \widetilde{m}_\nu(\tilde{x}) \quad \textit{for all } \tilde{x} \in \widetilde{X}_\mu \cap \widetilde{X}_\nu,$$

then there exists a meromorphic function m on X with poles only in smooth formal fibers which induces a rational function $\widetilde{m}$ on $\widetilde{X}$ with

$$\widetilde{m}|_{\widetilde{X}_\nu} = \widetilde{m}_\nu \quad \textit{for } \nu = 1, \ldots, n.$$

Proof. Let $\widetilde{Q} \subset \widetilde{X}$ be a finite set of smooth points such that $\widetilde{Q}$ meets every component $\widetilde{X}_\nu$ and contains the set of poles of every $\widetilde{m}_\nu$. Since $\widetilde{X}$ is semi-stable, the n-tuple $(\widetilde{m}_1, \ldots, \widetilde{m}_n)$ yields a regular function on $\widetilde{X} - \widetilde{Q}$. Since $\widetilde{\mathcal{O}_X(X_{\widetilde{Q}})} = \mathcal{O}_{\widetilde{X}}(\widetilde{X} - \widetilde{Q})$, there exists a function $h \in \mathring{\mathcal{O}}_X(X_{\widetilde{Q}})$ inducing $\widetilde{m}_1, \ldots, \widetilde{m}_n$. Due Proposition 4.3.5(b) we can assume that $h = m$ is meromorphic on X. □

As a second application of Proposition 4.3.5 we have the technique of *blowing-down components of the reduction* $\widetilde{X}$ of a formal analytic structure $\rho : X \to \widetilde{X}$.

Corollary 4.3.7. *Let X be a connected smooth projective curve equipped with a formal analytic structure $\rho : X \to \widetilde{X}$. Let $\widetilde{A} \subset \widetilde{X}$ be a union of irreducible components of $\widetilde{X}$ and let $\widetilde{Q}, \widetilde{Q}' \subset \widetilde{X} - \widetilde{A}$ be non-empty finite sets of smooth points such that each component of $\widetilde{X}$, which is not contained in $\widetilde{A}$, meets $\widetilde{Q}$ and $\widetilde{Q}'$. Assume that $\widetilde{Q}$ and $\widetilde{Q}'$ are disjoint.*

Then $\mathfrak{V} := \{X_{\widetilde{Q}}, X_{\widetilde{Q}'}\}$ *is a formal analytic covering of* X *by affinoid subsets. The canonical map* $\widetilde{X} \to \widetilde{X}_{\mathfrak{V}}$ *is surjective by Proposition 4.3.5 and injective on* $\widetilde{X} - \widetilde{A}$*. It maps the connected components of* $\widetilde{A}$ *to points in* $\widetilde{X}_{\mathfrak{V}}$*.*

Definition 4.3.8. In the above situation we say that $X_{\mathfrak{V}}$ is *constructed from the formal structure* $\rho : X \to \widetilde{X}$ *by blowing down* $\widetilde{A} \subset \widetilde{X}$*.*

Moreover, we obtain a criterion to compare formal analytic structures.

Corollary 4.3.9. *Let* X *be a smooth connected projective curve equipped with two formal analytic structures* $\rho : X \to \widetilde{X}$ *and* $\rho' : X \to \widetilde{X}'$*. Let* $\widetilde{X}_1, \ldots, \widetilde{X}_n$ *be the irreducible components of* $\widetilde{X}$ *and let* $p_1, \ldots, p_n$ *be points of* X *such that* $\rho(p_i)$ *is a smooth point of* $\widetilde{X}$ *and belongs to* $\widetilde{X}_i$ *for* $i = 1, \ldots, n$*. If the formal fiber* $X_+(\rho(p_i))$ *of each* p_i *is also a formal fiber* $X'(\rho'(p_i))$ *with respect to* ρ'*, then the identity morphism* $(X, \rho') \to (X, \rho)$ *is formal analytic.*

Proof. Put $\widetilde{P} := \{\rho(p_1), \ldots, \rho(p_n)\}$ and $\widetilde{P}' := \{\rho'(p_1), \ldots, \rho'(p_n)\}$. By Proposition 4.3.5 we know that $X_{\widetilde{P}}$ is affinoid with canonical reduction $\widetilde{X} - \widetilde{P}$. From the assumption we have $X_{\widetilde{P}} = X_{\widetilde{P}'}$, where the last one is defined with the respect to ρ'. Also by Proposition 4.3.5 we have that the identity map id_X gives rise to a surjective morphism $\widetilde{X}' \to \widetilde{X}$ which contracts the irreducible components of $\widetilde{X}'$, which are not met by $\widetilde{P}'$, to points of $\widetilde{X}$. □

It is useful to know that the formal analytic structure of $\rho : X \to \widetilde{X}$ depends only on the formal fibers.

Corollary 4.3.10. *Let* $\rho : X \to \widetilde{X}$ *and* $\rho' : X \to \widetilde{X}'$ *be reductions. Assume that they have the same formal fibers. Then* ρ *and* ρ' *are equivalent*; *i.e., the identity map* $(X, \rho') \to (X, \rho)$ *is a formal analytic isomorphism.*

4.4 Formal Stable Reduction

Definition 4.4.1. Let S be a scheme. A *semi-stable curve* over S is a quasi-compact, separated flat morphism $X \to S$ such that its fibers $X_{\overline{s}}$ over geometric points $\overline{s}$ of S are reduced, connected, 1-dimensional and its singularities are at most ordinary double points.

A semi-stable curve is called *stable* if for every geometric fiber $X_{\overline{s}}$ the following conditions are satisfied:

(i) every smooth rational component of $X_{\overline{s}}$ meets the remaining components in at least 3 points,
(ii) every singular rational component of $X_{\overline{s}}$ has a least 2 singular point or it has 1 singular point and meets at least one remaining component.

Usually one also asks $X \to S$ to be proper, but for technical reasons we consider the more general definition. One can also introduce n-marked stable curves.

Definition 4.4.2. An *n-marked stable curve* over a base scheme S is a semi-stable curve $X \to S$ as above together with S-sections $\sigma_1, \dots, \sigma_n$, which are pairwise disjoint and factorize through the smooth locus of X/S, such that the following is satisfied. If C is a *rational* component of a geometric fiber $X_{\overline{s}}$ of X/S, then

(i) $m(C) + r(C) \geq 2$ if $m(C) \geq 1$, or
(ii) $r(C) \geq 3$.

Here the integer $m(C)$ is the number of singular points of C and the integer $r(C)$ the sum of the number of points among $\sigma_1(s), \dots, \sigma_n(s)$, which meet C, and of the number of points, where C meets the rest of $X_{\overline{s}}$.

The genus of C is the genus of the normalization of its projective closure. In the absence of marked points a 0-marked stable curve is a stable curve. If, in addition, $X \to S$ is proper and $2g - 2 + n \geq 1$, then the condition (i) and (ii) can be replaced by asking $\dim H^1(X_{\overline{s}}, \mathcal{O}_{X_{\overline{s}}}) = g$ and $r(C) \geq 3$ for every smooth rational component of $X_{\overline{s}}$.

The main subject in this section is the proof of the following theorem.

Theorem 4.4.3. *Let K be a non-Archimedean field and let X_K be the analytification of a smooth geometrically connected projective curve with pairwise distinct K-rational points $x_1, \dots, x_n$. Then there exists a finite separable field extension K'/K such that $X_K \otimes_K K'$ admits an admissible formal R'-model X' with a projective semi-stable special fiber $\widetilde{X}'$, where R' is the valuation ring of K'. Moreover, the points $x_1, \dots, x_n$ specialize to pairwise distinct smooth points $\tilde{x}_1, \dots, \tilde{x}_n$ on $\widetilde{X}'$.*

If $2g(X) + n \geq 3$, then X can be chosen to be n-marked stable. An n-marked stable formal R-model of X_K is uniquely determined up to a canonical isomorphism.

We remind the reader that a rigid analytic curve X_K is proper if and only if the reduction is proper; cf. Theorem 3.3.12. In the case of curves the equivalence is easy to show; cf. Remark 3.3.13. Moreover, a proper rigid analytic curve is an analytification of a projective curve, cf. Theorem 1.8.1.

In the preceding sections we always worked in the context of canonical reductions of affinoid K-algebras over an algebraic closed field K. They have automatically formal R-models by Remark 3.2.5, because the R-algebras $\mathring{A}_K$ are of topologically finite presentation due to Theorems 3.1.17 and 3.2.1. Likewise formal analytic spaces have a canonical formal R-model. Thus, to verify Theorem 4.4.3 it suffices to show that X_K admits a formal analytic structure whose reduction is semi-stable or stable, respectively.

If K is not necessarily algebraically closed, then we will perform a base change to a complete algebraic closure $\overline{K}$ of K. By Corollary 3.4.3 we can descend the result of Theorem 4.4.3 for algebraically closed fields to a finite separable field extension K'/K.

In the following we assume that K is algebraically closed and we write X instead of X_K. We start with some preparations for the proof.

Lemma 4.4.4. *Assume that the genus of X satisfies $g(X) \geq 1$. Let $\ell \geq 2$ be an integer prime to* $\operatorname{char} k$ *and let E be a divisor on X such that $\ell \cdot E = \operatorname{div}(f)$ is a principal divisor, whereas E is not principal.*

Then there exists an element $\alpha \in |K^\times|$ such that, with the notations of Definition 4.2.4, we have that $g(X_\alpha(q)) < g(X)$ for all $q \in \widetilde{X}_\alpha$, where X_α is the formal analytic curve with respect to the covering

$$V_\alpha := \{x \in X; |f(x)| \leq \alpha\}, \qquad W_\alpha := \{x \in X; |f(x)| \geq \alpha\}.$$

Proof. For $\alpha \in |K^\times|$ we define the following property:

$$A(\alpha): \quad \text{there exists a point } q_\alpha \in \widetilde{X}_\alpha \text{ such that } g\big(X_\alpha(q_\alpha)\big) = g(X).$$

Since $g(X) \geq 1$, the point $q_\alpha \in \widetilde{X}_\alpha$ is uniquely determined due to Proposition 4.2.6. Furthermore, we introduce the following sets

$$\begin{aligned} A &:= \{\alpha \in |K^\times|; A(\alpha) \text{ is true}\}, \\ A_0 &:= \{\alpha \in A; X_{\alpha,+}(q_\alpha) \subset \{x \in X; |f(x)| > \alpha\}\}, \\ A_\infty &:= \{\alpha \in A; X_{\alpha,+}(q_\alpha) \subset \{x \in X; |f(x)| < \alpha\}\}. \end{aligned}$$

Then the following assertions are true:

(a) $A = A_0 \cup A_\infty$.
(b) There exist subsets $B_1^0, \ldots, B_n^0, B_1^\infty, \ldots, B_n^\infty$ of X which are unions of increasing sequences of discs such that

$$\bigcup_{\alpha \in A_0} V_\alpha = B_1^0 \cup \cdots \cup B_n^0; \qquad \bigcup_{\alpha \in A_\infty} W_\alpha = B_1^\infty \cup \cdots \cup B_n^\infty.$$

(a) If $\alpha \in A$, then we obtain by Corollary 4.2.7 a decomposition

$$X - X_{\alpha,+}(q_\alpha) = B_1 \dot\cup \cdots \dot\cup B_{m_\alpha}$$

with pairwise disjoint closed discs B_i in X. If we define the reduction $\mathbb{P}^1_\alpha$ of $\mathbb{P}^1_K$ with respect to the formal covering $\{\{|z| \leq \alpha\}, \{|z| \geq \alpha\}\}$, then f induces a finite morphism $\tilde{f}_\alpha : \widetilde{X}_\alpha \to \mathbb{P}^1_\alpha$. Thus, we obtain a decomposition

$$X - X_{\alpha,+}\big(\tilde{f}_\alpha^{-1}\big(\tilde{f}_\alpha(q_\alpha)\big)\big) = U_1 \dot\cup \cdots \dot\cup U_{n_\alpha}$$

with pairwise disjoint $U_1, \ldots, U_{n_\alpha}$, where each $U_i \subset B_{j(i)}$ with some $j(i)$ in $\{1, \ldots, m_\alpha\}$ for $i = 1, \ldots, n_\alpha$. Furthermore, f induces finite morphisms

$$f : U_i \longrightarrow D := \mathbb{P}^1_K - \mathbb{P}^1_{K,+}\big(\tilde{f}_\alpha(q_\alpha)\big) \quad \text{for } i = 1, \ldots, n_\alpha.$$

If $X_{\alpha,+}(q_\alpha) \subset \{x \in X; |f(x)| = \alpha\}$, then D contains 0 and ∞. Since D is connected, we see that the divisors $f^*(0)$ and $f^*(\infty)$ have the same degree on each U_i. Thus, we obtain that $\deg(E|_{B_i}) = 0$ for $i = 1, \dots, n$, and hence $E \sim 0$ by Lemma 4.2.10. However, this is a contradiction to our assumption. Thus, we see that $\alpha \in A_0 \cup A_\infty$.

(b) Let $\operatorname{div}(f) = \sum_{i=1}^n (x_i - y_i)$ and $\alpha \in A_0$. From Corollary 4.2.7 we obtain a decomposition

$$V_\alpha = X - X_{\alpha,+}(q_\alpha) = B_1^\alpha \dot{\cup} \cdots \dot{\cup} B_{n_\alpha}^\alpha$$

with closed discs B_i^α in X, and hence a decomposition

$$V_\alpha = V_1^\alpha \dot{\cup} \cdots \dot{\cup} V_{n_\alpha}^\alpha,$$

where $V_i^\alpha := V_\alpha \cap B_i^\alpha$. Since f gives rise to a rational function on $\widetilde{X}_\alpha$ which is not constant on each component of $\widetilde{X}_\alpha$, we see that each V_i^α is not empty. In particular, f induces finite surjective maps

$$V_i^\alpha \longrightarrow \left\{z \in \mathbb{P}_K^1; |z| \le \alpha\right\}$$

and each V_i^α must contain a zero of f. Thus, allowing repetitions, we may assume $n_\alpha = n$ and $x_i \in B_i^\alpha$. For $\alpha, \beta \in A_0$ with $\alpha < \beta$ we have that $B_i^\alpha \subset B_i^\beta$ by Proposition 4.2.11(a). Thus, writing

$$B_i^0 := \bigcup_{\alpha \in A_0} B_i^\alpha \quad \text{for } i = 1, \dots, n,$$

it follows $\bigcup_{\alpha \in A_0} V_\alpha \subset B_1^0 \cup \cdots \cup B_n^0$ as asserted. The case A_∞ is treated similarly.

By (a) we have $A = A_0 \cup A_\infty$. Furthermore, $A_0 \cup A_\infty \ne |K^\times|$ by (b) and Proposition 4.2.11(c) because $g(X) \ge 1$. Thus, $A \ne |K^\times|$ due to (a), and hence the assertion follows. □

Lemma 4.4.5. *Let $B_1, \dots, B_n$ be pairwise disjoint closed discs in X. Then every divisor D on X is linearly equivalent to a divisor E on X with* $\operatorname{Supp}(E) \cap B_i = \emptyset$ *for $i = 1, \dots, n$.*

Proof. It suffices to show that, for each $x \in B_j$, there exists a meromorphic function m on X with zero x and no other zeros or poles in $B_1, \dots, B_n$. We see by Lemma 4.2.9 that it is enough to verify this for $B_{1,+}, \dots, B_{n,+}$ instead of $B_1, \dots, B_n$, where each $B_{j,+}$ is a maximal open disc contained in B_j.

Let $\mathfrak{B}$ be the formal covering of X introduced in Proposition 3.1.10, and let $\tilde{b}_1, \dots, \tilde{b}_n$ be the points in $\widetilde{X}_\mathfrak{B}$ which correspond to the formal fibers $B_{1,+}, \dots, B_{n,+}$. Let $\widetilde{Q} \subset \widetilde{X}_\mathfrak{B}$ be a finite subset of smooth points which meet each component of $\widetilde{X}_\mathfrak{B}$ and which do not contain any of the points $\tilde{b}_1, \dots, \tilde{b}_n$. Then there exists a rational function $\widetilde{m}$ on $\widetilde{X}_\mathfrak{B}$ which has a simple zero in $\tilde{b}_j$ and does not vanish at $\tilde{b}_i$ for i with $i \ne j$. Due to Proposition 4.3.5 the function $\widetilde{m}$ is induced by a meromorphic function

$m \in \mathcal{M}(X) \cap \mathring{\mathcal{O}}_X(X_{\widetilde{Q}})$, where $X_{\widetilde{Q}}$ is the affinoid subdomain induced by $\widetilde{X}_{\mathfrak{B}} - \widetilde{Q}$. Then m restricts to a coordinate function on $B_{j,+}$ which satisfies $|m(y)| < 1$ for all $y \in B_{j,+}$ and $|m(y)| = 1$ for all $y \in B_{i,+}$. Thus, the function $m - m(x)$ is a desired one. □

Lemma 4.4.6. *Let X be a smooth projective curve of genus g over an algebraically closed field K. Then there exists a reduction $\rho : X \to \widetilde{X}$ in the sense of Definition 3.1.6 such that $\widetilde{X}$ is semi-stable.*

Proof. Let us start with some reduction $\rho : X \to \widetilde{X}$, which is given by a meromorphic function as in Example 3.1.7(b). Using the notation of Definition 4.2.4, consider

$$\gamma(\widetilde{X}) := \sum_{q \in \widetilde{X}} g\big(X(q)\big).$$

Due to Proposition 4.2.6 one knows $\gamma(\widetilde{X}) \le g(X)$. At first we assert that one can refine the reduction in order to achieve $\gamma(\widetilde{X}) = 0$. Thus, assume that the given reduction has $\gamma(\widetilde{X}) \ge 1$. Then we pick a point $q \in \widetilde{X}$ with $g(X(q)) \ge 1$. Then it suffices to refine the formal structure on the formal fiber $X_+(q)$. We choose an affine formal open neighborhood U of q in X. After shrinking U, we may assume that there is a function $h \in \mathring{\mathcal{O}}_X(U)$ such that q is the only zero of $\tilde{h}$. By Proposition 4.1.11 we have a representation of the periphery of the formal fiber of $X_+(q)$ as in Definition 4.2.4

$$\big\{x \in U;\, \varepsilon \le \big|h(x)\big| < 1\big\} = \dot{\bigcup}_{\mu,\nu} A_{\mu,\nu},$$
$$\varphi_{\mu,\nu} : A_{\mu,\nu} \xrightarrow{\sim} \big\{z \in \mathbb{D}_K;\, \varepsilon_{\mu,\nu} \le |z| < 1\big\}.$$

Then put

$$U_1 := \big\{x \in U;\, \big|h(x)\big| = 1\big\},$$
$$U_2 := \big\{x \in U;\, \big|h(x)\big| \ge \varepsilon\big\},$$
$$U_3 := \big\{x \in U;\, \big|h(x)\big| \le \varepsilon\big\}.$$

Denote by $X(q)$ the curve constructed by pasting the periphery of $X_+(q)$ with discs $B_{\mu,\nu}$ via the isomorphisms $\varphi_{\mu,\nu}$. Since $g(X(q)) \ge 1$, there exists a non-trivial divisor E on $X(q)$ of order $\ell \ge 2$, prime to $\operatorname{char} k$, with $\ell \cdot E = \operatorname{div}(f)$ for some meromorphic function $f \in \mathcal{M}(X(q))$. Due to Lemma 4.4.5 we can assume

$$\operatorname{Supp}(E) \subset U_{3,+} := \big\{x \in U_3;\, \big|h(x)\big| < \varepsilon\big\}$$

and we have the following disjoint decomposition of $X(q)$

$$X(q) = U_{3,+} \,\dot{\cup}\, \dot{\bigcup}_{\mu,\nu} B_{\mu,\nu},$$
$$\varphi_{\mu,\nu} : B_{\mu,\nu} \xrightarrow{\sim} \big\{z \in \mathbb{P}^1_K;\, \varepsilon_{\mu,\nu} \le |z|\big\}.$$

Now apply Lemma 4.4.4 to the curve $X(q)$ and the divisor E. Thus, there exists an element $\alpha \in |K^\times|$ such that $g(X(q)_\alpha(p)) < g(X(q))$ for all $p \in \widetilde{X}(q)_\alpha$. Then we introduce a new reduction $\rho' : X \to \widetilde{X}'$ as a refinement of the old reduction $\rho : X \to \widetilde{X}$ above the point q by the formal covering

$$\begin{aligned} U_{|h|\geq\varepsilon} &:= \{x \in U;\ |h(x)| \geq \varepsilon\}, \\ U_{|h|\leq\varepsilon} &:= \{x \in U;\ |h(x)| \leq \varepsilon\}, \\ U'_{f,\alpha} &:= \{x \in U_{|h|\leq\varepsilon};\ |f(x)| \geq \alpha\}, \\ U''_{f,\alpha} &:= \{x \in U_{|h|\leq\varepsilon};\ |f(x)| \leq \alpha\}. \end{aligned}$$

One easily verifies that $\rho' : X \to \widetilde{X}'$ satisfies $\gamma(\widetilde{X}') < \gamma(\widetilde{X})$, since every formal fiber contained in $U_{|h|\leq\varepsilon}$ with respect to ρ' is part of a formal fiber of the reduction with respect to $\widetilde{X}(q)_\alpha$.

Next we show that one can refine the reduction $\rho : X \to \widetilde{X}$ such that $\widetilde{X}$ is semi-stable. By first step we may assume that $g(X(q)) = 0$ for all points $q \in \widetilde{X}$. Due to Corollary 4.2.8 we have an explicit representation

$$X(q) = \mathbb{P}^1_K - (B_1 \dot\cup \cdots \dot\cup B_{n(q)})$$

for every $q \in \widetilde{X}$. We can proceed as in the first step.

We have to construct a formal covering of the affinoid subdomain U_3 such that the corresponding reduction of U_3 has only ordinary double points as singularities. Since U_3 is isomorphic to an affinoid subdomain of $\mathbb{P}^1_K$, this problem was solved in Proposition 2.4.6 and Lemma 2.4.5. □

Lemma 4.4.7. *Let X be given with points as in Theorem 4.4.3 and let $\rho : X \to \widetilde{X}$ be a semi-stable reduction. Then there exists a refinement $X \xrightarrow{\rho'} \widetilde{X}' \to \widetilde{X}$ by a semi-stable curve $\widetilde{X}'$ such that $x_1, \ldots, x_n$ specialize to distinct points $\rho'(x_1), \ldots, \rho'(x_n)$ of the smooth locus of $\widetilde{X}'$.*

Proof. We have only to treat the cases, where two points $\rho(x_i) = \rho(x_j)$ are reduced to the same point or where $\rho(x_i)$ is mapped to a singular point of $\widetilde{X}$. The addressed formal fibers are open discs or open annuli; cf. Proposition 4.1.12. Then the assertion follows by a similar construction as in the proof of Lemma 2.4.5. □

Lemma 4.4.8. *Let X be a smooth rigid analytic curve which is quasi-compact, separated and geometrically connected. Assume that X admits a reduction $\rho : X \to \widetilde{X}$ in the formal analytic sense, where $\widetilde{X}$ is smooth. Consider an (arbitrary) reduction $\rho' : X \to \widetilde{X}'$ with geometrically reduced special fiber.*

If $\widetilde{X}$ is not proper, then there exists a non-empty admissible open subdomain U of X which is formally open with respect to ρ and ρ'. Moreover, there exists a component C' of $\widetilde{X}'$ which is birational to $\widetilde{X}$ and all the other irreducible components of $\widetilde{X}'$ are smooth and rational.

If $\widetilde{X}'$ has more than 1 *component, then the configuration of the irreducible components of $\widetilde{X}'$ is tree-like and there is at least one proper smooth rational component C'' of $\widetilde{X}'$ which meets the remaining components in a single point.*

Proof. Since $\tilde{X}$ is not proper, $\widetilde{X}$ is affine and hence X is affinoid. By Proposition 3.1.12 we see that there exists an open subdomain $U \subset X$, which is formal open with respect to ρ, such that U is contained in an affine formal open subset U' with respect to ρ'. Since X is affinoid and $\widetilde{X}$ is affine and irreducible, there exists an $f \in \mathcal{O}_X(X)$ with $|f|_X = 1$ such that $\widetilde{X}_{\tilde{f}} \subset \widetilde{U}$. Since $U'(1/f)$ is formal open in U' with respect to ρ', we see that there is an irreducible component C' of $\widetilde{X}'$ which is birational to $\widetilde{X}$.

We can glue X along an open formal part with a smooth formal curve in order to complete it to a formal proper curve Y; cf. Proposition 3.7.1. Note that the complement $X - Y$ consists of finitely many open discs which correspond to the missing points $\overline{\widetilde{X}} - \widetilde{X}$, where $\overline{\widetilde{X}}$ is the normal projective closure of $\widetilde{X}$. Obviously, the reductions ρ and ρ' extend to Y. Thus, we may assume $X = Y$. Then the genus formula in Proposition 4.2.6 implies that all the other irreducible components of $\widetilde{X}'$ are smooth rational curves and that there is no circuit in the configuration of the irreducible components.

If all these irreducible components meet the remaining components in at least two points, we obtain a contradiction. In fact, let $\tilde{x}_1 \in C' \cap \widetilde{X}'_1$ where $\widetilde{X}'_1$ is an irreducible component of $\widetilde{X}'$ different from C'. Then there exists a further irreducible component $\widetilde{X}'_2$ and an intersection point $\tilde{x}_2 \in \widetilde{X}'_1 \cap \widetilde{X}'_2$. Continuing this way, one can construct an infinite sequence of distinct irreducible components of $\widetilde{X}'$, because there are no circuits. □

Corollary 4.4.9. *Let $\rho : X \to \widetilde{X}$ be a reduction of a smooth quasi-compact separated rigid analytic curve X. Then the following conditions are equivalent*:

(a) *X is isomorphic to a closed disc.*
(b.1) *All irreducible components of $\widetilde{X}$ are smooth and rational.*
(b.2) *The singularities are at most ordinary multiple points*; *cf. Corollary* 4.2.8.
(b.3) *$\widetilde{X}$ is connected and can be completed by adding exactly one smooth point.*
(b.4) *The configuration of the irreducible components of $\widetilde{X}$ is tree-like.*

If $\widetilde{X}$ has more than 1 *component, there exists at least one proper smooth rational component of $\widetilde{X}$ which meets the remaining components in at most one point.*

Proof. (b) → (a): Let C be the component of $\widetilde{X}$ which is not complete. Then we can glue X over a formal open part which reduces to an open subset of C with a formal open part of a disc to obtain a new smooth proper curve Y with a reduction $Y \to \widetilde{Y}$, which extends the given reduction ρ; cf. Proposition 3.7.1. The genus formula in Proposition 4.2.6 tells us that Y has genus 0, because $g(X(q)) = 0$ for every point $q \in \widetilde{X}$ by Corollary 4.2.8. Thus, Y is the projective line. Removing the formal fiber of the inserted smooth point shows that X is isomorphic to $\mathbb{P}^1_K$ minus an open disc. Thus, X is a closed disc by Example 3.1.11.

(a) → (b): Due to Lemma 4.4.8 there exists a formal open subdomain V with respect to ρ which is formally dense open in X with respect to the canonical reduction. We can glue the disc X with a formal open part of a disc to obtain a smooth proper rigid analytic curve of genus 0 with a reduction extending ρ. Then the assertion follows from the genus formula in Proposition 4.2.6 and Corollary 4.2.8.

The additional assertion follows from Lemma 4.4.8. □

Corollary 4.4.10. *In the situation of Corollary 4.4.9 the following conditions are equivalent*:

(a) *X is isomorphic to a closed annulus.*
(b.1) *All irreducible components are smooth and rational.*
(b.2) *The singularities are at most ordinary multiple points*; *cf. Corollary* 4.2.8.
(b.3) *$\widetilde{X}$ is connected and can be completed by adding exactly two smooth points.*
(b.4) *The configuration of the irreducible components of $\widetilde{X}$ is tree-like.*

In particular, there exists at least one proper smooth rational component which meets the remaining components in at most one point or $\widetilde{X}$ is a chain of smooth rational components.

Proof. (b) → (a): follows as in the proof of Corollary 4.4.9, except for the fact that one has to remove two disjoint open discs from $\mathbb{P}^1_K$. Thus, X isomorphic to a closed annulus due to Example 3.1.11.

(a) → (b): Obviously, a boundary component of the annulus cannot be contained in a formal fiber of ρ as follows from Lemma 4.4.8. Thus, there are precisely two points missing on $\widetilde{X}$ in a compactification of $\widetilde{X}$ to a proper curve. We can glue X by two formal open parts of a disc in order to complete X to a smooth proper rigid analytic curve of genus 0 with a reduction extending ρ. Then the assertion follows from Proposition 4.2.6 and Corollary 4.2.8.

The additional assertion follows by a similar reasoning as in the proof of Lemma 4.4.8. Due to the maximum principal the reduction of both boundary components of the annulus have to show up in every reduction in the birational sense. Then one can compactify both components as in the proof of Corollary 4.4.10 and proceed as in the proof of Lemma 4.4.8. □

Lemma 4.4.11. *Let $\rho : X \to \widetilde{X}$ be a reduction of a smooth projective curve X. Let X be equipped with K-rational points $x_1, \dots, x_n$ as in Theorem 4.4.3 which specialize to pairwise distinct smooth points on $\widetilde{X}$. Assume that $\rho : X \to \widetilde{X}$ is n-marked stable with respect to these points.*

(a) *If B is a closed disc contained in X and contains at most one of the marked points*, *then $\rho(B)$ reduces to one point of $\widetilde{X}$.*
(b) *If A is a closed annulus contained in X which does not contain any of the marked points*, *then $\rho(A)$ reduces to one point of $\widetilde{X}$.*

Proof. (a) Let $a \in X - B$ be a closed point which reduces to a smooth point $\rho(a)$ of $\widetilde{X}$. We may assume $X_+(a) \cap B = \emptyset$. Let $b \in B$ be a closed point. Due to Lemma 3.1.9 there exists a meromorphic function on X which gives rise to a coordinate function ζ of B with $\zeta(b) = 0$ and $\mathrm{Pol}(\zeta) = \{a\}$. Thus, B is a connected component of the subset $\{x \in X; |\zeta(x)| \leq 1\}$. Assume that B is not contained in a formal fiber with respect to ρ. By subdividing the annuli which correspond to the double points of $\widetilde{X}$ at $\{|\zeta| = 1\}$ we obtain a refinement $\rho' : X \to \widetilde{X}'$ of ρ such that B is formal open with respect to ρ'. The induced map $\widetilde{X}' \to \widetilde{X}$ is just the contraction of the smooth rational components corresponding to the subdivision. Due to Corollary 4.4.9 there exists a proper smooth rational component C' of $\widetilde{X}'$ which meets the remaining components in at most one point such that the lifting of its smooth part of C' is contained in B. This component cannot coincide with the newly introduced components, since they are connected to two irreducible components. Thus, we see that C' appears already in $\widetilde{X}$ and meets the other irreducible components of $\widetilde{X}$ in at most one point. Since $r(C) \geq 3$ for an n-marked stable curve in Definition 4.4.2, we arrive at a contradiction.

(b) Assume first that A is of height 1. Thus, the formal fibers of A with respect to the canonical reduction are open discs. Due to (a) every formal fiber of A with respect to its canonical reduction is contained in a formal fiber of X with respect to ρ. If there exists a point $p \in A$ with $A_+(p) \neq X_+(p)$, then $A \subset X_+(p)$ as follows from the maximum principle. So we may assume that A is a formal open part of X with respect to ρ. Since the reduction $\widetilde{A}$ is isomorphic to the projective line minus two points, there is a rational component C of $\widetilde{X}$ which has at most two points in common with the remaining components. If C is singular, then $m(C) = 1$ and $r(C) = 0$, we arrive at a contradiction. If C is smooth, then $m(C) = 0$ and $r(C) \leq 2$ and we obtain a contradiction as well. Thus, we see that A is contained in a formal fiber of X with respect to ρ.

Now consider the case of an annulus of height $\varepsilon < 1$. By what we have shown already every concentric subannulus of height 1 is contained in a formal fiber of ρ. Then it is clear that A is contained in a formal fiber. □

Proof of Theorem 4.4.3. Due to Lemma 4.4.6 there exists a formal covering $\mathfrak{U}$ of X such that $X_{\mathfrak{U}}$ has a semi-stable reduction. Due to Lemma 4.4.7 we can refine the covering to separate the given points $x_1, \ldots, x_n$. Thus, we may assume that there exists a reduction $\rho : X \to \widetilde{X}$ such that $\widetilde{X}$ is semi-stable and the points $\rho(x_1), \ldots, \rho(x_n)$ are smooth on $\widetilde{X}$ and pairwise distinct. Thus, it remains to blow down superfluous components. This can be done by using Corollary 4.3.7. Indeed, if C is an irreducible component which is a smooth rational line with $r(C) \leq 2$, then we contract this component by the method of Corollary 4.3.7. Hereby we obtain a map $\varphi : \widetilde{X} \to \widetilde{X}'$ which yields a new reduction $\varphi \circ \rho$. The component C collapses to a point $\tilde{x}' \in \widetilde{X}'$. The formal fiber of $\tilde{x}'$ is an open disc or an open annulus. Then $\tilde{x}'$ is a smooth point or a double point of $\widetilde{X}'$ due to Proposition 4.1.12. If one of the marked points meets C, the point $\tilde{x}'$ is smooth. The condition $r(C) \leq 2$ means that C is connected to the remaining components by one or two intersection points. If it is connected by two points, then none of the points $\rho(x_1), \ldots, \rho(x_n)$ meets C. If it is connected by one point, then at most one of the points meets C. In particular, $\tilde{x}'$ is a smooth point of $\widetilde{X}'$ and the marked point specializes to $\tilde{x}'$. By repeating this process finitely many times, we get rid of all smooth rational components C with $r(C) \leq 2$. Then the reduction $\rho : X \to \widetilde{X}$ is n-marked stable.

It remains to show the uniqueness of n-marked stable reductions. Let $\rho_i : X \to \widetilde{X}_i$ for $i = 1, 2$ be n-marked stable reductions. It follows from Lemma 4.4.11 and Corollary 4.3.10 that the identity map extends to a formal analytic isomorphism $(X, \rho_1) \to (X, \rho_2)$. Therefore, the formal structures given by ρ_1 and ρ_2 are equal.

As mentioned just after the statement in Theorem 4.4.3 the formal analytic structures are induced by proper formal R-models. Any morphism between formal analytic R-spaces is a morphism of their formal R-models. □

4.5 Stable Reduction

The stable reduction theorem for algebraic curves can be deduced from Theorem 4.4.3 in the formal case. We have to see that a formal semi-stable R-model of a curve is induced by a flat projective R-curve. In order to distinguish between the curve over K and its R-model, we will write "X_K" for the curve over K and "X" for an R-model of X_K.

Proposition 4.5.1. *Let X_K be a smooth rigid space over a non-Archimedean field K. Let $\overline{K}$ be an complete algebraic closure and $K_{\text{sep}} \subset \overline{K}$ be a separable algebraic closure of K. Then the set of K_{sep}-valued points of X_K is dense in $X_K(\overline{K})$ with respect to its canonical topology given by the absolute value.*

Proof. Let $U_K \subset X_K$ be a non-empty open affinoid subset. Since X_K is smooth over K, there exists a non-empty open subdomain $V_K \subset U_K$ such that V_K admits an étale morphism $V_K \to \mathbb{B}^n_K$ of V_K to an n-dimensional polydisc $\mathbb{B}^n$. Thus, it suffices to see that one can approximate an element $\alpha \in \overline{K}$, which is algebraic over K, by an element $\beta \in K_{\text{sep}}$. Consider the minimal polynomial $p \in K[T]$ of α. Then every approximation of p (with respect to the Gauss norm) by a monic polynomial $q \in K[T]$ has a zero $\beta \in \overline{K}$ which is close to α. Since there are separable polynomials which approximate p, the zero β lies in K_{sep}. □

Corollary 4.5.2. *Let K be a field equipped with a non-Archimedean absolute value, not necessarily complete, and let $\widehat{K}$ be its completion. Consider a smooth K-scheme X_K of finite type. Let $X^{\text{an}}_{\widehat{K}}$ be the associated rigid analytic space over $\widehat{K}$. Then the K_{sep}-valued points of X_K are dense in $X^{\text{an}}_{\widehat{K}}$ with respect to its canonical topology given by the absolute value.*

Theorem 4.5.3 (Stable reduction theorem). *Let K be a field equipped with a non-Archimedean absolute value, not necessarily complete, and R its valuation ring. Let X_K be a geometrically connected smooth projective curve over K of genus g and $x_1, \ldots, x_n$ be pairwise distinct K-valued points.*

Then there exists a finite separable field extension L of K and a flat projective R_L-model X of $X \otimes_K L$ with semi-stable fibers such that the points $x_1, \ldots, x_n$ extend to pairwise disjoint R_L-valued points of the smooth locus of X/R_L.

If $2g + n \geq 3$, the curve X can be chosen to be n-marked stable. An n-marked stable R-model of X_K is uniquely determined up to a canonical isomorphism.

This is the famous Stable Reduction Theorem usually presented for discrete valuation rings; cf. [5] or [21]. If R is not Noetherian, the classical proofs fail, since they essentially make use of the fact that R is Noetherian as they use the resolution of singularities in dimension 2. However, the result in the Noetherian case is sufficient for the construction of the moduli space $\overline{\mathcal{M}}_{g,n}$ of n-marked stable curves and to show that $\overline{\mathcal{M}}_{g,n}$ is proper. Then the case of a general valuation ring follows from the properness of $\overline{\mathcal{M}}_{g,n}$; cf. Theorem 7.5.2. By our concept we can deduce Theorem 4.5.3 from Theorem 4.4.3.

Proof. Let $\widehat{K}$ be the completion of K. Let $\widehat{K}'/\widehat{K}$ be a finite separable extension such that $X_K \otimes_K \widehat{K}'$ admits an admissible formal $\widehat{R}'$-model X' with semi-stable reduction; cf. Theorem 4.4.3. Let $\widetilde{X}'_1, \ldots, \widetilde{X}'_n$ be the irreducible components of the special fiber $\widetilde{X}' := X' \otimes_{\widehat{R}'} k'$. Due to the lemma of Krasner [10, 3.4.2/5] the field $\widehat{K}'$ is the completion of a finite separable field extension K' of K. Due to Corollary 4.5.2 there exists a finite Galois field extension K''/K such that there exist K''-valued points $z_1, \ldots, z_n$ of X_K such that z_i specializes into the smooth part of $\widetilde{X}'$ meeting $\widetilde{X}'_i$ for $i = 1, \ldots, n$. Then consider the divisor

$$D_K := N \cdot z_1 + \cdots + N \cdot z_n$$

on $X_{K''} := X_K \otimes_K K''$ with an integer $N \geq 2g + 1$. Let $\widehat{R}''/\widehat{R}'$ be the completion of a valuation ring R''_v of K''. The K''-valued points z_i extend to $\widehat{R}''$-valued points $\overline{z}_i$ of X' with reduction in the smooth part of $\widetilde{X}'$ meeting $\widetilde{X}'_i$. They give rise to a relative Cartier divisor

$$D := N \cdot \overline{z}_1 + \cdots + N \cdot \overline{z}_n$$

on $X'' := X' \otimes_{\widehat{R}'} \widehat{R}''$ which is relatively very ample. Now consider an $\widehat{R}''$-basis

$$\Gamma\big(X'', \mathcal{O}_{X''}(D)\big) = \widehat{R}'' f_0 \oplus \cdots \oplus \widehat{R}'' f_\ell .$$

The system $(f_0, \ldots, f_\ell)$ gives rise to a closed embedding

$$\underline{f} := (f_0, \ldots, f_\ell) : X''_\lambda \longrightarrow \mathbb{P}^\ell_{\widehat{R}''_\lambda}$$

of $X''_\lambda = X' \otimes_{\widehat{R}'} \widehat{R}''_\lambda$ over $\widehat{R}''_\lambda := \widehat{R}''/\widehat{R}''\pi^{\lambda+1}$ for all $\lambda \in \mathbb{N}$. The algebraization theorem of Grothendieck [39, III, 5.18] in the Noetherian case and [1, 2.13.9] in the general case, shows that X'' is associated to a flat projective $\widehat{R}''$-curve $\mathcal{X}''$ with special fiber $X'' \otimes_{\widehat{R}''} k$. Thus, $\mathcal{X}''$ is a semi-stable model of X_K over $\widehat{R}''$.

The sheaf $\mathcal{O}_{X_{K''}}(D_K)$ is defined over the finite Galois extension K'' of K. Therefore, one can choose a basis $(g_0, \ldots, g_\ell)$ of $\Gamma(X_{K''}, \mathcal{O}_{X_{K''}}(D_K))$. Thus, one can approximate the basis $(f_0, \ldots, f_\ell)$ by a basis $(h_0, \ldots, h_\ell)$ of $\Gamma(X_{K''}, \mathcal{O}_{X_{K''}}(D_K))$. Then, one can replace $\underline{f}$ by the morphism

$$\underline{h} := (h_0, \ldots, h_\ell) : X''_\lambda \longrightarrow \mathbb{P}^\ell_{R''_\lambda}.$$

Thus we obtain a semi-stable R''_v-model of $X_{K''}$ for the valuation v on K'' which is extended by $\widehat{R}''$. Since the Galois group of K''/K acts transitively on the valuations of K''/K, by gluing along the generic fiber we obtain a semi-stable R''-model of $X_{K''}$. Since the valuations of K''/K are inequivalent [10, 3.3.2] one can choose a basis $(h_0, \ldots, h_\ell)$ of $\Gamma(X_{K''}, \mathcal{O}_{X_{K''}}(D_K))$ such that $\underline{h}$ gives rise to an embedding of the model into $\mathbb{P}^\ell_{R''}$.

Uniqueness: Let X_1, X_2 be n-stable R-models and

$$\varphi_K : X_1 \otimes_R K \xrightarrow{\sim} X_2 \otimes_R K$$

the isomorphism of their generic fibers. Now consider the schematic closure $\Gamma \subset X_1 \times_R X_2$ of the graph of φ_K. Then it suffices to show that the projections $p_i : \Gamma \to X_i$ are isomorphisms for $i = 1, 2$. The latter can be checked after the π-adic completion. Thus, the assertion follows from the uniqueness in the formal case. □

4.6 Universal Covering of a Curve

By means of the formal semi-stable reduction theorem we can generalize the notion of a skeleton which we introduced in Definition 2.4.3.

In the following let K be a non-Archimedean field, not necessarily algebraically closed.

Definition 4.6.1. Let X be a smooth rigid-analytic curve over K.

A *semi-stable skeleton* of X is a surjective map $\rho : X \to S$ from X to a geometric graph S with the following properties:

(i) The inverse image $\rho^{-1}(v)$ of a vertex $v \in V(S)$ is a geometrically connected admissible subvariety of X which admits a smooth model over the valuation ring.
(ii) The inverse image $\rho^{-1}(e)$ of an edge $e \in E(S)$ is isomorphic to an open annulus $A(\varepsilon(e), 1)^-$ with $\varepsilon(e) \in |K^\times|$ and $\varepsilon(e) < 1$.
(iii) ρ is continuous; i.e., the inverse image $\rho^{-1}(\{v_1, e, v_2\})$ of an edge e with its two extremities v_1, v_2 is an admissible subvariety of X.

A *semi-stable skeleton* of X is said to *separate the points* $a_1, \ldots, a_n \in X$ if the points are mapped to vertices such that for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ either the points a_i, a_j are mapped to different vertices of S, or, if mapped to the same vertex $v \in V(S)$, the points a_i, a_j have different reduction under the canonical reduction map of $\rho^{-1}(v)$.

A semi-stable skeleton of X is called *stable* if, for every $v \in V(S)$ such that $\rho^{-1}(v)$ has rational reduction, one of the following properties holds

(i) v has 2 neighboring edges, where at least one is a loop of Definition A.1.4,
(ii) v has at least 3 neighbors; i.e., has index ≥ 3; cf. Definition A.1.7.

The existence of semi-stable skeletons follows from Theorem 4.4.3 immediately.

Proposition 4.6.2. *Let X be a geometrically connected smooth projective curve of genus g over a non-Archimedean field K. Then, after a suitable finite separable field extension of K, there exists a semi-stable skeleton of X. If the genus satisfies $g \geq 2$, there exists a unique stable one.*

Proposition 4.6.3. *Let X be a geometrically connected smooth projective curve over a non-Archimedean field K which admits a semi-stable reduction. Let $\rho : X \to S$ be a associated semi-stable skeleton. Then there exists a rigid analytic morphism $p_X : \widehat{X} \to X$ over K from a smooth rigid analytic curve $\widehat{X}$ and a semi-stable skeleton $\widehat{\rho} : \widehat{X} \to \widehat{S}$, where $p_X : \widehat{X} \to X$ is a covering in the topological sense and $p_S : \widehat{S} \to S$ is the universal covering in the sense of graphs, such that $p_S \circ \widehat{\rho} = \rho \circ p_X$.*

Proof. Let $p_S : \widehat{S} \to S$ be the universal covering of the graph S; cf. Proposition A.1.11. We remind the reader that $\widehat{S}$ is a tree. Let $e_1, \ldots, e_r$ be edges of S such that $S' := S - \{e_1, \ldots, e_r\}$ is a subtree of S, where r is the rank of $H_1(S, \mathbb{Z})$.

Choose an orientation on S. Let v_i^+ and v_i^- be the origin of e_i and the target of e_i, respectively, for $i = 1, \ldots, r$. Then let $X(0)$ be the result of the pasting of $X' := \rho^{-1}(S')$ with $\rho^{-1}(v_i^+ \cup e_i)$ along v_i^+ and with $\rho^{-1}(v_i^- \cup e_i)$ along v_i^-. Thus, we obtain a 1-dimensional rigid analytic curve with $2r$ open ends which are the open annuli associated to the e_i; note that we do not identify the e_i in the pasting process.

Now we use the universal covering $\widehat{S}$ as a recipe to glue copies of $X(0)$ along the open ends; again and again as in Proposition A.1.11. Thus, we obtain a 1-dimensional rigid analytic curve together with a skeleton $\widehat{\rho} : \widehat{X} \to \widehat{S}$ which is compatible with the projections $p_X : \widehat{X} \to X$ and $p_S : \widehat{S} \to S$. □

Definition 4.6.4. The rigid analytic curve $\widehat{X}$ is called the *universal covering* of X.

Proposition 4.6.5. *Let $\widehat{X}$ be the universal covering of a geometrically connected smooth projective curve X with a semi-stale reduction. Then every bounded holomorphic function on $\widehat{X}$ is constant.*

Proof. In order to keep notations simple, put $Z := \widehat{X}$ with reduction $\widetilde{Z}$. We may assume that K is algebraically closed. Let $z_0 \in Z$ be a closed point which reduces to a smooth point. Then we can replace f by $g := f - f(z_0)$. Thus, it remains to see that g vanishes identically. Let $(\widetilde{Z}_j ; j \in J)$ be the family of irreducible components of $\widetilde{Z}$ and let Z_j be the pre-image of $\widetilde{Z}_j - \mathrm{Sing}(\widetilde{Z})$. Let $j_0 \in J$ with $\tilde{z}_0 \in \widetilde{Z}_{j_0}$.

Assume now that g does not vanish identically. Then, there exist values $c_j \in |K^\times|$ such that $|g|_{Z_j}| = |c_j|$ for $j \in J$. Then the function $g_j := g/c_j$ has sup-norm 1 on Z_j and its reduction has a zero at $z_0 \in Z_{j_0}$, and hence a zero at $\tilde{z}_0 \in \widetilde{Z}_{j_0}$. Since the number of poles equals the number of zeros on a complete curve, there exists a point $\tilde{z}_1 \in \widetilde{Z}_{j_0}$, where $\tilde{g}_{j_0}$ has a pole. Since g is holomorphic, the pole must be a singular point $\tilde{z}_{j_1} \in \widetilde{Z}_{j_0} \cap \operatorname{Sing}(\widetilde{Z})$. By Proposition 4.3.1 now $\tilde{z}_{j_1}$ is a zero of $\tilde{g}_{j_1}$ and $|c_{j_0}| = \varepsilon^\nu \cdot |c_{j_1}|$. Here ε is the height of the annulus associated to the double point $\tilde{z}_{j_1}$, and ν is the sum of the pole order of $\tilde{g}_{j_0}$ at $\tilde{z}_{j_0}$ and of the number of zeros of g on the formal fiber of $\tilde{z}_{j_1}$. Thus we see

$$|c_{j_0}| \cdot \alpha^{-1} \leq |c_{j_1}|,$$

where $\alpha < 1$ is the maximum of all heights of annuli appearing in the skeleton of the curve X. Continuing this way, we obtain an infinite chain of indices $j_0, j_1, j_2, \ldots$ such that

$$|c_{j_0}| \cdot \alpha^{-n} \leq |c_{j_n}|$$

for all $n \in \mathbb{N}$. This contradicts the boundedness of the function g. Thus, we see that g is identically 0. □

Proposition 4.6.6. *Let X be a geometrically connected smooth projective curve X with a semi-stable reduction. Assume that X has at least three K-rational points $x_0, x_1, x_\infty \in X$ and that the reduction has only rational components. Then we have the following results*:

(i) *The universal covering $\widehat{X}$ can be embedded into $\mathbb{P}^1_K$.*
(ii) *The embedding $\widehat{X} \to \mathbb{P}^1_K$ that maps the points (x_0, x_1, x_∞) to $(0, 1, \infty)$ is uniquely determined.*
(iii) *Any automorphism of $\widehat{X}$ extends to a projective linear automorphism of $\mathbb{P}^1_K$. In particular, the deck transformation group Γ of $\widehat{X}/X$ can be regarded as a subgroup of* $\mathrm{PGL}(2, K)$.

Proof. (i) Let $\widehat{S} \to S$ be the universal covering of the skeleton S of X of Proposition 4.6.3 and assume that $\widehat{\rho}(x_i) \in \operatorname{vert}(\widehat{S})$ for $i = 0, 1, \infty$. For $n \in \mathbb{N}$ let $\widehat{S}(n)$ be the complete subtree which contains all vertices that have distance $\leq n$ from $\widehat{\rho}(x_0)$ and $\widehat{X}(n) := \widehat{\rho}^{-1}(\widehat{S}(n))$. There exists an index $N \in \mathbb{N}$ such that $x_0, x_1, x_\infty \in \widehat{X}(n)$ for all $n \geq N$. Then, as exercised in Proposition 3.7.1, we can paste all the holes of $\widehat{X}(n)$ by discs to obtain a smooth proper curve $X(n)$. By the genus formula in Proposition 4.2.6 we see that the genus of $X(n)$ is zero. Since $X(n)$ is projective algebraic due to Theorem 1.8.1, the curve $X(n)$ is isomorphic to the projective line. Thus, there exists an open immersion

$$\sigma_n : \widehat{X}(n) \xrightarrow{\sim} \Omega_n \subset \mathbb{P}^1_K$$

onto a subdomain of $\mathbb{P}^1_K$ sending the points (x_0, x_1, x_∞) to $(0, 1, \infty)$.

Next we assert that the sequence $(\sigma_m; n \in \mathbb{N})$ converges to an open immersion

$$\sigma := \lim_{n\to\infty} \sigma_n : \widehat{X} \longrightarrow \mathbb{P}^1_K.$$

It suffices to show that the sequence $(\sigma_n|_{\widehat{X}(N)}; n \in \mathbb{N})$ converges to an open immersion for every $N \in \mathbb{N}$. For showing this, consider $m, n \in \mathbb{N}$ with $n \geq m \geq N$. Since σ_n has no zeros or poles outside $\widehat{X}(N)$, the absolute value function associated to σ_n is constant due on $X(n) - \widehat{X}(N)$ due to Corollary 4.3.3. Thus, the meromorphic function

$$\frac{\sigma_n}{\sigma_m} : \widehat{X}(m) \longrightarrow \mathbb{G}_{m,K}$$

has neither zeros nor poles and hence it has a constant absolute value function by Corollary 4.3.4. Therefore, it can be written in the form $1 + h_{n,m}$, where $h_{n,m}$ is holomorphic on $\widehat{X}(m)$ with $|h_{n,m}| < 1$ and has a zero at the point x_1. Therefore, it holds

$$|h_{n,m}|_{\widehat{X}(N)}| < \varepsilon^{m-N}$$

for all $n \geq m \geq N$, where $\varepsilon < 1$ is the largest height of an annulus associated to an edge of the skeleton S, cf. Corollary 4.3.3. This shows that the sequence $(\sigma_n; n \in \mathbb{N})$ converges to an immersion of $\widehat{X}$ into $\mathbb{P}^1_K$.

(ii) If σ_1 and σ_2 are such immersions, the function σ_1/σ_2 is holomorphic and bounded on $\widehat{X}$. Then it follows from Proposition 4.6.5 that σ_1/σ_2 is constant. From the equality $\sigma_1(x_1) = 1 = \sigma_2(x_1)$ it follows $\sigma_1 = \sigma_2$.

(iii) Consider $\widehat{X}$ as an admissible open subvariety $\Omega \subset \mathbb{P}^1_K$ and assume that $0, 1, \infty \in \Omega$. Let $\sigma \in \Gamma$ be a deck transformation of $\widehat{X}/X$. We may view σ as a morphism $\sigma : \Omega \to \mathbb{P}^1_K$. Since a projective linear transformation τ in $\mathrm{PGL}(2, K)$ is equivalent to a mapping of the three points $0, 1, \infty$ to three distinct K-rational points in $\mathbb{P}^1_K$, there exists a transformation $\tau \in \mathrm{PGL}(2, K)$ such that the morphism $\tau^{-1} \circ \sigma : \Omega \to \mathbb{P}^1_K$ fixes the points $0, 1, \infty \in \Omega$. Then we have to show that $\varphi := \tau^{-1} \circ \sigma$ is the identity. Thus, consider the function

$$f := \frac{\varphi}{\zeta} : \Omega \longrightarrow \mathbb{P}^1_K,$$

where ζ is the coordinate on $\mathbb{P}^1_K$ with $\zeta(0) = 0$, $\zeta(1) = 1$ and $\zeta(\infty) = \infty$. Thus, f has no zeros and poles, and hence f is bounded, because it is bounded on an open neighborhood of $0, \infty$. Thus, we see by Proposition 4.6.5 that f is constant. As $f(1) = 1$, the function f is equal to 1. □

4.7 Characterization of Mumford Curves

Mumford curves were introduced in Theorem 2.3.1 via Schottky groups. Due to the Stable Reduction Theorem it is possible to characterize them by their reduction type. Moreover, due to the genus formula they can be determined by the first homology group.

Definition 4.7.1. A smooth projective curve X_K over K has a *split rational reduction* if there exists a projective semi-stable relative curve X over R such that its generic fiber $X_K := X \otimes_R K$ is isomorphic to X_K and its special fiber $X_s := X \otimes_R k$ is a configuration of rational curves with rational double points as singularities.

Theorem 4.7.2. *Let K be a non-Archimedean field and X_K a geometrically connected smooth projective smooth curve of genus $g \geq 2$. Assume that X_K admits a semi-stable reduction over K. Then the following conditions are equivalent*:

(a) *X_K is a Mumford curve.*
(b) *X_K has a split rational reduction.*
(c) $\mathrm{rk}(H^1(X_K, \mathbb{Z})) = g(X_K)$.

Proof. (a) → (b): Due to Proposition 2.4.11 there exists a semi-stable skeleton in the sense of Definition 2.4.2. The reduction of a rational disc minus finitely many maximal open discs is isomorphic to an affine line minus finitely many closed points. Thus, every semi-stable reduction of X_K has only rational components. The double points are rational, because the associated annuli are rational. Thus, we see that X_K has split rational reduction.

(b) → (c): This follows from the genus formula of Proposition 4.2.6.

(c) → (a): By the genus formula in Proposition 4.2.6 we see that the irreducible components of the semi-stable reduction of X_K are rational. Due to Proposition 4.6.6 the universal covering $\widehat{X}_K$ of X_K can be embedded into the projective line. Moreover, the deck transformation group $\Gamma := \mathrm{Deck}(\widehat{X}_K/X_K)$ can be viewed as a subgroup of $\mathrm{PGL}(2, K)$ in a canonical way with respect to an embedding of $\widehat{X}_K \hookrightarrow \mathbb{P}^1_K$. Thus, Γ acts on $\mathbb{P}^1_K$ discontinuously and the quotient $\widehat{X}_K/\Gamma$ is isomorphic to X_K. Since the action of Γ is related to the action of Γ on the universal covering of the skeleton, the action of Γ is free and Γ is finitely generated; cf. Proposition A.1.11. Thus, we see that Γ is a Schottky group; cf. Definition 2.2.2. Since X_K is proper, it is clear that $\widehat{X}_K \subset \mathbb{P}^1_K$ is the set of ordinary points of Γ. Therefore, X_K is a Mumford curve; cf. Theorem 2.3.1. □

Chapter 5
Jacobian Varieties

The main objective of this chapter is the uniformization of the Jacobian of a smooth projective curve X_K over a non-Archimedean field K and its relationship to a semi-stable reduction $\widetilde{X}$ of X_K.

We assume that the reader is familiar with the notion of the Jacobian variety of a smooth projective curve over a field; see for instance the article [68] or [15, Chap. 9]. For our purpose it is necessary to have analyzed the generalized Jacobian of a semi-stable curve $\widetilde{X}$, especially its representation as a torus extension of the Jacobian of its normalization $\widetilde{X}'$. In Sects. 5.1 and 5.2 we reassemble the main results we need in the sequel.

In Sect. 5.3 it is shown that the generalized Jacobian $\widetilde{J} := \operatorname{Jac}\widetilde{X}$ has a lifting $\overline{J}_K$ as an open rigid analytic subgroup of $J_K := \operatorname{Jac} X_K$ and that $\overline{J}_K$ has a smooth formal R-model $\overline{J}$ with semi-abelian reduction. $\overline{J}$ is a formal torus extension of a formal abelian R-scheme B with reduction $\widetilde{B} = \operatorname{Jac}\widetilde{X}'$.

The generic fiber $\overline{J}_K$ of $\overline{J}$ is the largest connected open subgroup of J_K which admits a smooth formal R-model; this is discussed in Sect. 5.4 in a more general context. The relationship between the maximal formal torus $\overline{T}$ of $\overline{J}$ and the group $H^1(X_K, \mathbb{Z})$ shows that the inclusion map $\overline{T}_K \hookrightarrow \overline{J}_K$ from the generic fiber $\overline{T}_K$ of the formal torus $\overline{T}$ to $\overline{J}_K$ extends to a rigid analytic group homomorphism $T_K \to J_K$, where T_K is the affine torus which contains $\overline{T}_K$ as the torus of units.

The push-out $\widehat{J}_K := T_K \amalg_{\overline{T}} \overline{J}_K$ is a rigid analytic group which contains $\overline{J}_K$ as an open rigid analytic subgroup and the inclusion $\overline{J}_K \hookrightarrow J_K$ extends to a surjective homomorphism $\widehat{J}_K \to J_K$ of rigid analytic groups. The kernel of the latter map is a lattice M in $\widehat{J}_K$ and makes $J_K = \widehat{J}_K/M$ into a quotient of the "universal covering" $\widehat{J}_K$. The representation $J_K = \widehat{J}_K/M$ is called the *Raynaud representation* of J_K.

Since every abelian variety is isogenous to a subvariety of a product of Jacobians, one can transfer the results to abelian varieties. This implies Grothendieck's semi-abelian reduction theorem for abelian varieties; cf. [42].

We want to mention that there are also contributions by Fresnel, Reversat and van der Put [30] and [84].

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5.1 Jacobian of a Smooth Projective Curve

In a first section we will give a short survey on results concerning the Jacobian variety of a smooth projective curve, which will be used in this chapter. Details can be found in [68] and [15, Chap. 9].

In the following let k be any field and let X/k be a connected smooth projective curve of genus g with a k-rational point x_0. If S is a k-scheme, then we denote by $x_S : S \to X_S := X \times_k S$ the induced S-valued point $x_S := (x_0, \mathrm{id}_S)$ of the S-scheme X_S. The Picard functor

$$\mathrm{Pic}^0_{X/k} : (k\text{-schemes}) \longrightarrow (\text{Sets}),$$

associates to a k-scheme S the set of isomorphism classes $(\mathcal{L}, r)$, where $\mathcal{L}$ is an invertible sheaf on X_S of degree 0 and where $r : \mathcal{O}_S \to x_S^*\mathcal{L}$ is an isomorphism. The latter is called a *rigidificator* of $\mathcal{L}$. Isomorphisms of such pairs are isomorphisms of invertible sheaves respecting the rigidificators.

Theorem 5.1.1. *The functor* $\mathrm{Pic}^0_{X/k}$ *is representable by a connected smooth projective k-group scheme* $\mathrm{Jac}\,X$. *More precisely, there exists a universal object* $(\mathcal{D}, \varrho)$ *on* $X \times \mathrm{Jac}\,X$ *with the following property*:

For every k-scheme S and every $(\mathcal{L}, r) \in \mathrm{Pic}^0_{X/k}(S)$ *there exists a unique morphism* $\varphi : S \to \mathrm{Jac}\,X$ *such that there is a unique isomorphism* $\lambda : \mathcal{L} \to (\mathrm{id}_X \times \varphi)^*\mathcal{D}$ *with* $x_S^*\lambda \circ r = \varphi^*\varrho$.

Definition 5.1.2. Let X be a connected smooth projective k-curve over k with a rational point. The projective k-variety $\mathrm{Jac}\,X$ is called the *Jacobian variety* of X.

If $k = K$ is a non-Archimedean field, then due to the results on GAGA in Theorem 1.6.11 the analytification of $\mathrm{Jac}\,X$ represents $\mathrm{Pic}^0_{X/K}$ on the category of rigid analytic spaces as well.

Depending on the chosen k-rational point $x_0 \in X$ we have the morphism

$$\iota : X \longrightarrow \mathrm{Jac}\,X,\ x \longmapsto [x - x_0],$$

where $[x - x_0]$ is the class of the invertible sheaf $\mathcal{O}_X(x - x_0)$ for k-rational points $x \in X$. The map ι is a closed immersion and gives rise to a morphism

$$\iota^{(n)} : X^{(n)} \longrightarrow \mathrm{Jac}\,X,\ x_1 + \cdots + x_n \longmapsto [x_1 - x_0] \otimes \cdots \otimes [x_n - x_0],$$

from the n-fold symmetric product $X^{(n)}$ to $\mathrm{Jac}\,X$ for each $n \in \mathbb{N}$.

If $n = g$ is equal to the genus of X, then the map $\iota^{(g)}$ is surjective and birational. The fiber of a point $\iota^{(g)}(D)$ is isomorphic to the projective linear system $|D| := \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$ associated to the divisor $D \in X^{(g)}$.

Definition 5.1.3. The image W_{g-1} of $\iota^{(g-1)}$ is a Weil divisor on $\mathrm{Jac}\,X$ and is called the *theta divisor* Θ of $\mathrm{Jac}\,X$.

A connected smooth proper k-group variety A is called an abelian variety. For example, $\operatorname{Jac} X$ is an abelian variety of dimension g, where g is the genus of X. A *translation by an S-valued point a* of A is the mapping

$$\tau_a : A_S \longrightarrow A_S,\ x \longmapsto \tau_a(x) := x + a,$$

where $A_S := A \times S$ for a k-scheme S. An invertible sheaf $\mathcal{L}$ on A is called *translation invariant* if for every $\overline{k}$-rational point a of A the translate $\tau_a^*\mathcal{L}$ is isomorphic to $\mathcal{L}$, where $\overline{k}$ is an algebraic closure of k. The latter is equivalent to require that

$$m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \cong \mathcal{O}_{A\times A}$$

is trivial, where $m : A \times A \to A$ is the group law and $p_i : A \times A \to A$ is the i-th projection. For an abelian variety A one has the functor

$$\operatorname{Pic}^{\tau}_{A/k} : (k\text{-schemes}) \longrightarrow (\text{Sets}),$$

which associates to a k-scheme S the set of isomorphism classes $(\mathcal{L}, r)$, where $\mathcal{L}$ is a translation invariant invertible sheaf and where r is a rigidificator along the unit section 0 of A. The functor $\operatorname{Pic}^{\tau}_{A/k}$ is representable; cf. [74, §13, p. 125].

Theorem 5.1.4. *Let A be an abelian variety over k. The functor $\operatorname{Pic}^{\tau}_{A/k}$ is representable by an abelian variety A'. More precisely, there exists a universal object $(\mathcal{P}_{A\times A'}, \varrho)$ on $A \times A'$ with the following property:*

For every k-scheme S and every couple $(\mathcal{L}, r) \in \operatorname{Pic}^{\tau}_{A/k}(S)$ there exists a unique morphism $\varphi : S \to A'$ of schemes such that there is a unique isomorphism $\lambda : \mathcal{L} \to (\mathrm{id}_A \times \varphi)^*\mathcal{P}_{A\times A'}$ which respects the rigidificators.

Definition 5.1.5. Let A be an abelian variety over k. The projective abelian variety A' is called the *dual abelian variety* of A.

If $k = K$ is a non-Archimedean field, then due to the GAGA Theorem 1.6.11 the analytification of A' represents $\operatorname{Pic}^{\tau}_{A/K}$ on the category of rigid analytic spaces as well. If $\mathcal{L}$ is an invertible sheaf on A, then by the universal property one obtains a morphism

$$\varphi_{\mathcal{L}} : A \longrightarrow A',\ a \longmapsto \tau_a^*\mathcal{L} \otimes \mathcal{L}^{-1},$$

which is in fact a group homomorphism. If $\mathcal{L}$ is ample, then $\varphi_{\mathcal{L}}$ is an isogeny; i.e., a finite surjective morphism. Furthermore, we want to mention the bi-duality of abelian varieties that A is the dual of A'; cf. [74, p. 132].

Now consider $J := \operatorname{Jac} X$ as an abelian variety and let J' be its dual. The theta divisor Θ of Definition 5.1.3 gives rise to a group homomorphism

$$\varphi_{\Theta} : J \longrightarrow J',\ x \longmapsto \tau_x^*\mathcal{O}_J(\Theta) \otimes \mathcal{O}_J(\Theta)^{-1} = \big[\mathcal{O}_J\big(\tau_{-x}(\Theta) - \Theta\big)\big].$$

This homomorphism is an isomorphism and does not depend on the base point x_0; it is called the *theta polarization* of J. There is an interesting relation between the universal sheaf $\mathcal{D}$ and the Poincaré sheaf $\mathcal{P}_{J\times J'}$ which is important for us in the following section. Details can be found in [68, §6].

Theorem 5.1.6. *Let X/k be a connected smooth projective curve of genus g with a k-rational point x_0. Let*

$$\iota : X \longrightarrow J := \mathrm{Jac}(X),\ \ x \longmapsto \iota(x) := [x - x_0],$$

be the canonical map which sends x to the class of the invertible sheaf $\mathcal{O}_X(x - x_0)$. Let $\Theta := \iota^{(g-1)}(X^{(g-1)}) \subset J$ be the theta divisor. Let $\mathcal{D}$ be the universal invertible sheaf on $X \times J$ of Theorem 5.1.1 and $\mathcal{P}_{J\times J'}$ the universal invertible sheaf on $J \times J'$ of Theorem 5.1.4. Then we have the following results:

(a) *If $D \in X^{(g)}$ with $a := \iota^{(g)}(D) \in J$ and $\Theta_a^- := \tau_a([-1]_J(\Theta))$, then*

$$\iota^*\mathcal{O}_J\left(\Theta_a^-\right) \cong \mathcal{O}_X(D)$$

are isomorphic, where $[-1]_J : J \to J, x \mapsto -x$, is the inverse map on J.

(b) *There exists a canonical isomorphism*

$$(\iota \times -\varphi_\Theta)^*\mathcal{P}_{J\times J'} \xrightarrow{\sim} \mathcal{D},$$

of rigidified invertible sheaves on $X \times J$.

(c) *There exists an isomorphism*

$$\mathcal{O}_{J\times J}\left(m^{-1}\Theta - \Theta \times J - J \times \Theta\right) \xrightarrow{\sim} (\mathrm{id}_J \times \varphi_\Theta)^*\mathcal{P}_{J\times J'},$$

of invertible sheaves on $J \times J$.

(d) *The invertible sheaf $(\iota \times \mathrm{id}_{J'})^*\mathcal{P}_{J\times J'}$ on $X \times J'$ gives rise to a canonical morphism $\varphi' : J' \to J, b' \mapsto [\iota^*\mathcal{P}_{J\times b'}]$, and a canonical isomorphism*

$$(\iota \times \mathrm{id}_{J'})^*\mathcal{P}_{J\times J'} \xrightarrow{\sim} \left(\mathrm{id}_X \times \varphi'\right)^*\mathcal{D}$$

of rigidified invertible sheaves on $X \times J'$.

(e) *The morphisms $\varphi' : J' \to J$ and $-\varphi_\Theta : J \to J'$ are inverse to each other.*
φ' is called the autoduality map.

Proof. (a) See [68, Lemma 6.8(a)].

(b) Due to [68, Lemma 6.8(b)] there exists an isomorphism

$$\left(\iota \times [-1]_J\right)^*\mathcal{O}_{J\times J}\left(m^{-1}\left(\Theta^-\right) - \Theta^- \times J - J \times \Theta^-\right) \xrightarrow{\sim} \mathcal{D}$$

of invertible sheaves on $X \times J$ with $\Theta^- = [-1]_J(\Theta)$. Then the assertion follows from (c), since we may assume that Θ is symmetric; cf. Lemma 2.9.14.

(c) See [68, Theorem 6.6].

(d) This follows from (e) and (b).

(e) See [68, Lemma 6.9]. □

For later application let us state some specializations.

Corollary 5.1.7. *In the situation of Theorem 5.1.6 we have the following results.*
If $\mathcal{L}$ is a rigidified invertible sheaf on X of degree 0, then

(a) $\varphi_\Theta^* \mathcal{P}_{[\mathcal{L}]\times J'} = \mathcal{P}_{J\times\varphi_\Theta([\mathcal{L}])}$.
(b) $\mathcal{L} = \iota^*\varphi_\Theta^*\mathcal{P}_{-[\mathcal{L}]\times J'}$.

If $\mathcal{N}$ is a rigidified translation invariant invertible sheaf on J, then

(c) $\iota^*\varphi_\Theta^*\mathcal{P}_{-[\iota^*\mathcal{N}]\times J'} = \iota^*\mathcal{N}$.
(d) $\mathcal{N} = \mathcal{P}_{J\times[\mathcal{N}]} = \varphi_\Theta^*\mathcal{P}_{-[\iota^*\mathcal{N}]\times J'}$.

Here "=" is a canonical isomorphism of rigidified invertible sheaves.

Proof. (a) This is true, because $(\mathrm{id}_J \times\varphi_\Theta)^*\mathcal{P}_{J\times J'}$ is symmetric by Theorem 5.1.6(c).

(b) Let $\ell \in J$ be the isomorphism class $[\mathcal{L}]$ of $\mathcal{L}$. Since $\mathcal{D}$ is the universal sheaf, we have that $\mathcal{L} \cong \mathcal{D}|_{X\times\ell}$. Thus, Theorem 5.1.6(b) and assertion (a) imply

$$\mathcal{L} = \mathcal{D}|_{X\times\ell} = (\iota \times -\varphi_\Theta)^*\mathcal{P}_{J\times J'}|_{X\times\ell} = \iota^*\mathcal{P}_{J\times-\varphi_\Theta(\ell)} = \iota^*\varphi_\Theta^*\mathcal{P}_{-\ell\times J'}.$$

(c) This follows from (b), because $\iota^*\mathcal{N}$ has degree 0.

(d) This follows from (c), because for every invertible sheaf $\mathcal{L}$ on X of degree 0 there is a unique class of an invertible sheaf $\mathcal{N}$ on J with $\mathcal{L} = \iota^*\mathcal{N}$ due to the autoduality in Theorem 5.1.6(e). □

5.2 Generalized Jacobian of a Semi-Stable Curve

In the following we will represent the generalized Jacobian of a semi-stable curve over a field k as a torus extension. This will be used in the sequel to understand $\mathrm{Pic}^0_{\widetilde{X}/k}$ of a semi-stable reduction $\widetilde{X}$ of a curve X_K over a non-Archimedean field K.

In the following let X be a projective semi-stable curve over a field k which is geometrically connected. We recall some facts about the functor

$$\mathrm{Pic}^0_{X/k} : (k\text{-schemes}) \longrightarrow (\mathrm{Sets}),$$

which associates to a k-scheme S the set of isomorphism classes of invertible sheaves $\mathcal{L}$ on $X \times S$ such that the degree of the pull back of $\mathcal{L}$ to every component X_ν of $X \times_S s$ is zero for all points $s \in S$. It is known that $\mathrm{Pic}^0_{X/k}$ is representable by a smooth k-variety J_K which is also called the *generalized Jacobian* of X/k; cf. [15, 9.2/8]. Actually, it is the 1-component of the representable functor $\mathrm{Pic}_{X/k}$ which is defined without the condition on the degrees. In particular, its representable space is smooth over k. In the sequel we will also write $\mathrm{Pic}^0_{X/k}$ for J if we have to indicate the curve.

Proposition 5.2.1. *Under the above conditions,* $\mathrm{Pic}^0_{X/k}$ *is an extension of an abelian variety by a torus.*

More precisely, let $X_1, \dots, X_n$ be the irreducible components of X and assume that all the singular points of X are k-rational. Let $p_i : X_i' \to X_i$ be the normalization of X_i for $i = 1, \dots, n$. Then the torus extension associated to $\mathrm{Pic}^0_{X/k}$ is given by the strict exact sequence

$$1 \longrightarrow T \longrightarrow J := \mathrm{Pic}^0_{X/k} \xrightarrow{\;q\;} \prod_{i=1}^{n} \mathrm{Pic}^0_{X_i'/k} \longrightarrow 1, \qquad (\dagger)$$

where q is induced via functoriality by the morphisms p_i for $i = 1, \dots, n$, and where T is an affine split torus. The rank of the torus is equal to the rank of the homology group $H_1(X \otimes_k \overline{k}, \mathbb{Z})$, where $\overline{k}$ is an algebraic closure of k; cf. Definition 5.2.2 below.

As explained in Sect. A.2 the torus extension ($\dagger$) is given by translation invariant line bundles on the abelian variety $\prod_{i=1}^{n} \mathrm{Pic}^0_{X_i'/k}$. More precisely, it is given by a group homomorphism from the character group of the torus T to the dual of $\prod_{i=1}^{n} \mathrm{Pic}^0_{X_i'/k}$ by Theorem A.2.8. The torus is related to the combinatorial configuration of the irreducible components of X. This is recorded by its graph $\Gamma(X \otimes_k \overline{k})$ of coincidence of its irreducible components.

Definition 5.2.2. If X is a semi-stable curve over a field k, then one associates a geometric graph $\Gamma = \Gamma(X)$ to X, which we call *graph of coincidence* of its irreducible components. The vertices of Γ are the irreducible components, say $X_1, \dots, X_n$, and the edges are given by the singular points of X. Each singular point lying on X_i and on X_j defines an edge joining the vertices X_i and X_j. Note that $X_i = X_j$ is allowed.

Note that this graph is equal to the graph of the skeleton associated to a semi-stable reduction which was introduced in Definition 4.6.1.

We denote by $H_1(X, \mathbb{Z})$ the first homology group $H_1(\mathrm{real}(\Gamma(X)), \mathbb{Z})$ of the realization of the graph $\Gamma(X)$, cf. Definition A.1.2.

The torus extension of J is made explicit by the following proposition; see also [97].

Proposition 5.2.3. *In the situation of Proposition 5.2.1 assume, in addition, that the irreducible components $X_1, \dots, X_n$ of X are smooth. Let x_i^0 be a k-rational point of X_i which does not belong to the singular locus of X for $i = 1, \dots, n$. Let $X' \to X$ be the normalization of X.*

A cycle $c \in H_1(X, \mathbb{Z})$ can be represented by a closed path as in Definition A.1.3, which we write as a formal sum of singular points,

$$c = z_1 + \cdots + z_N.$$

If $z \in X$ is a singular point, then there are two points $z', z'' \in X'$ over z. According to the orientation of c let z' be the point of the origin and z'' the one of the target *of the associated edge. As c is closed, there are on each vertex X_i as many origins as targets of the edges associated to $z_1, \dots, z_N$.*

Let $B_i := \mathrm{Pic}^0_{X_i/k}$ be the Jacobian of X_i and B_i' the dual of B_i. Put

$$B := B_1 \times \cdots \times B_n \quad \textit{and} \quad B' := B_1' \times \cdots \times B_n'.$$

Then B' is the dual of B and there is a canonical isomorphism

$$\varphi_\Theta := (\varphi_{\Theta_1}, \dots, \varphi_{\Theta_n}) : B_1 \times \cdots \times B_n \longrightarrow B_1' \times \cdots \times B_n',$$

where each φ_{Θ_i} is induced by the theta divisor Θ_i of B_i. Put

$$\phi' : H_1(X, \mathbb{Z}) \longrightarrow B \longrightarrow B',\ c \mapsto [c] \mapsto c' := -\varphi_\Theta\big([c]\big).$$

Here $[c] \in B$ is the class of the invertible sheaf

$$\left(\bigotimes_{j=1}^{N} [z_j'' - z_j']|_{X_1}, \dots, \bigotimes_{j=1}^{N} [z_j'' - z_j']|_{X_n}\right) \in \prod_{i=1}^{n} \mathrm{Jac}\, X_i'.$$

Then the torus extension $J := \mathrm{Pic}^0_{X/k}$ is given by ϕ' in the sense of Theorem A.2.8.

Thus, the group $H_1(X, \mathbb{Z})$ is canonically identified with the group of characters of the torus T of J by sending c to the tautological map $\tau_c : J \to P_{B \times c'}$.

Proof. The normalization X' decomposes into a disjoint union

$$X' = X_1 \dot\cup \cdots \dot\cup X_n$$

of the irreducible components of X, which are assumed to be smooth. Let

$$\iota_i : X_i \longrightarrow B_i := \mathrm{Jac}\, X_i,\ x \longmapsto \big[x - x_i^0\big],$$

be the canonical mapping of X_i to its Jacobian with respect to the chosen base point x_i^0. Then let

$$\iota := \iota_1 \amalg \cdots \amalg \iota_n : X' \longrightarrow B,\ X_i \ni x \longmapsto \big(0, \dots, \iota_i(x), \dots, 0\big),$$

be the induced map. The cycle $c = z_1 + \cdots + z_N$ induces the class of an invertible sheaf $[c] \in \mathrm{Pic}^0_{X'/k}(k)$ on X'. In terms of divisors the invertible sheaf on X' associated to c is defined by

$$[c] := \bigotimes_{j=1}^{N} \big[z_j'' - u_j^0\big] \otimes \bigotimes_{j=1}^{N} \big[-z_j' + v_j^0\big] \in B,$$

where u_j^0 is the unique point in $\{x_1^0, \dots, x_n^0\} \cap X_{\mu(j)}$ if $z_j'' \in X_{\mu(j)}$ and v_j^0 is the unique point in $\{x_1^0, \dots, x_n^0\} \cap X_{\nu(j)}$ if $z_j' \in X_{\nu(j)}$. Obviously, each backtracking on the path is canceled automatically.

In the following we prefer to work with line bundles instead of invertible sheaves, since the notations are easier to handle; cf. Definition 1.7.1. Since there are k-rational points $x_i^0 \in X_i - \operatorname{Sing} X$, we view $\operatorname{Pic}^0_{X'/k}$ as the functor which associates to a k-scheme S the set of isomorphism classes of line bundles on $X' \times S$ which are rigidified along the points x_i^0 and have degree 0 on the irreducible components X_i for $i = 1, \dots, n$. Let D_i' be the universal line bundle on $X_i \times B_i$ of degree zero, which is rigidified along x_i^0. Without changing the notations, we denote by D_i' the pull-back of D_i' to $X \times B$ under the projection $X_i \times B \to X_i \times B_i$ as well. Then

$$D_{X' \times B} := \left(D_1', \dots, D_n'\right)$$

is the universal line bundle on $X' \times B$ of degree zero on each X_i.

Next we define the pasting of the D_i' over the double points in a universal way. Let $P_{B_i \times B_i'}$ be the Poincaré bundle on $B_i \times B_i'$; cf. Theorem 5.1.4. We have a canonical isomorphism

$$D_i' \xrightarrow{\sim} (\iota_i \times -\varphi_{\Theta_i})^* P_{B_i \times B_i'} \xrightarrow{\sim} \left(\mathrm{id}_{B_i} \times \iota_i'\right)^* P^{-1}_{B_i \times B_i'}, \qquad (*)$$

where

$$\iota_i' := \varphi_{\Theta_i} \circ \iota_i : X_i \longrightarrow B_i'$$

for $i = 1, \dots, n$. Indeed, the first isomorphism is due to Theorem 5.1.6(b). Since the line bundle $(\mathrm{id}_{B_i} \times \varphi_{\Theta_i})^* P_{B_i \times B_i'}$ is symmetric due to Theorem 5.1.6(c), we obtain a canonical isomorphism

$$\xi : (\mathrm{id}_{B_i} \times \varphi_{\Theta_i})^* P_{B_i \times B_i'} \xrightarrow{\sim} \tau^* (\mathrm{id}_{B_i} \times \varphi_{\Theta_i})^* P_{B_i \times B_i'}$$

of symmetry, where $\tau : B_i \times B_i \to B_i \times B_i$ is the flipping of factors. This yields the second isomorphism.

The Poincaré bundle on $B \times B'$ is given by

$$P_{B \times B'} = p_1^* P_{B_1 \times B_1'} \otimes \cdots \otimes p_n^* P_{B_n \times B_n'},$$

where $p_i : B \times B' \to B_i \times B_i'$ are the projections for $i = 1, \dots, n$.

Now we define the torus extension

$$T \to E \xrightarrow{q} B \quad \widehat{=} \quad \phi' : H_1(X, \mathbb{Z}) \to B'$$

by the homomorphism ϕ' of the assertion. Thus it remains to define a universal line bundle on $X \times E$.

First assume that X is a tree-like configuration of smooth irreducible curves, then we have $B = \operatorname{Jac} X$ due to Proposition 5.2.1 and $H_1(X, \mathbb{Z}) = 0$. The universal line bundle is given in the following way:

Let $y_2, \dots, y_n$ be the singular points of X and let $y_i' \in X_i$ and $y_i'' \in X_j$ for some $j \neq i$ be the points in X' above y_i. Now we alter the universal line bundles $D_2', \dots, D_n'$ on $X \times B$ by line bundles induced from line bundles on the base B;

so they do not change the class of the line bundle on the fiber X. We proceed by induction:

For $n = 1$ set $D_1 := D_1'$.

For $n = 2$ assume that the orientation is directed from X_1 to X_2 and hence $y_2' \in X_1$ and $y_2'' \in X_2$; otherwise one interchanges the primes. Set

$$D_2 := D_2' \otimes p^* P^{+1}_{B\times\iota'(y_2'')} \otimes p^* P^{-1}_{B\times\iota'(y_2')},$$

where $p : X \times B \longrightarrow B$ is the projection. There is a canonical isomorphism

$$D_1|_{y_2'\times B} = P^{-1}_{B\times\iota'(y_2')} \xrightarrow{\sim} P^{-1}_{B\times\iota'(y_2'')} \otimes P^{+1}_{B\times\iota'(y_2'')} \otimes P^{-1}_{B\times\iota'(y_2')} = D_2|_{y_2''\times B},$$

by the canonical identification ($*$).

For $n \geq 3$ one considers the unique path from X_1 to X_n; say the path passes through $y_2, \ldots, y_m$ after a suitable renumbering. For simplicity assume that X_1 is a terminal component. Then put

$$D_n := D_n' \otimes \bigotimes_{\mu=2}^{m} p^* P^{+1}_{B\times\iota'(y_\mu'')} \otimes p^* P^{-1}_{B\times\iota'(y_\mu')} = D_n' \otimes \bigotimes_{\mu=2}^{m} p^* P_{B\times(\iota'(y_\mu'')-\iota'(y_\mu'))}.$$

Thus, for every double point y_μ which is the intersection point of X_i and X_j, there is a canonical isomorphism

$$\eta_\mu : D_i|_{y_\mu'\times B} \xrightarrow{\sim} D_j|_{y_\mu''\times B}.$$

We leave it to the reader to check that $(D_1, \ldots, D_n)$ equipped with the connecting isomorphisms $(\eta_2, \ldots, \eta_n)$ is a universal line bundle on $X \times B$, which is rigidified along the base point x_1^0.

Now consider the given X and let $r \geq 1$ be the rank of $H_1(X, \mathbb{Z})$. So there exist r double points $x_1, \ldots, x_r$ such that $X - \{x_1, \ldots, x_r\}$ is a tree-like configuration of smooth irreducible components. Furthermore, let $y_2, \ldots, y_n$ be the double points of $X - \{x_1, \ldots, x_r\}$. Then let $Y \to X$ be the normalization of X above the points $\{x_1, \ldots, x_r\}$; in other words, Y is defined as X but without the identification of the points x_ρ' and x_ρ'' for $\rho = 1, \ldots, r$. Thus, Y is a tree-like configuration of smooth irreducible curves. As constructed above there is a universal line bundle $D := D_{Y\times B}$ on $Y \times B$ which is given by line bundles $(D_1, \ldots, D_n)$ equipped with the connecting isomorphisms $(\eta_2, \ldots, \eta_n)$. To construct the universal line bundle on $X \times B$ we will introduce universal connecting isomorphisms

$$\tau_\rho : D|_{x_\rho'\times B} \xrightarrow{\sim} D|_{x_\rho''\times B}.$$

For this consider the cycle without backtracking

$$c_\rho = z_{\rho,1} + \cdots + z_{\rho,N_\rho},$$

which passes only once through x_ρ and does not meet any other point of $\{x_1, \dots, x_r\}$. Then we define a universal line bundle on $X \times E$ in the following way. We introduce coordinates on E by

$$q : E := P_{B\times c'_1} \times_B \dots \times_B P_{B\times c'_r} \longrightarrow B,$$

where $P_{B\times c'_\rho}$ is the $\mathbb{G}_m$-torsor associated to $c'_\rho := -\varphi_\Theta([c_\rho])$ and where c_ρ is a simple closed path, for $\rho = 1, \dots, r$.

Every line bundle $q^* P_{B\times c'_\rho}$ has a canonical tautological trivialization

$$\tau_\rho : E \longrightarrow P_{B\times c'_\rho},$$

which is defined by the projection onto the ρ-th component. We will define the connecting isomorphism of $D|_{x'_\rho \times B} \xrightarrow{\sim} D|_{x''_\rho \times B}$ via τ_ρ for each $\rho = 1, \dots, r$. Assume that x_ρ is the intersection point of X_i and X_j. Let $(u_1, \dots, u_\ell)$ be the double points which c_ρ passes through from X_1 to X_i and $(v_1, \dots, v_m)$ the ones from X_1 to X_j. Let $h \in \{1, \dots, \ell\}$ and $k \in \{1, \dots, m\}$ be indices such that

$$c_\rho = u_h + \dots + u_\ell + x_\rho - v_m - \dots - v_k.$$

The cycle has the same orientation as one of the paths starting at X_1 and leading to x_ρ. Thus, we may assume that this part is given by $(u_1, \dots, u_\ell)$. Then the orientation of the other part is opposite. The definition of the (vertical) isomorphism over E

$$\begin{array}{ccc}
D|_{x'_\rho \times B} & = & D_i|_{x'_\rho \times B} = P_{B\times \iota'(x'_\rho)} \otimes \bigotimes_{\lambda=1}^{\ell} P_{B\times(\iota'(u''_\lambda)-\iota'(u'_\lambda))} \\
\big\downarrow & & \\
D|_{x''_\rho \times B} & = & D_j|_{x''_\rho \times B} = P_{B\times \iota'(x''_\rho)} \otimes \bigotimes_{\mu=1}^{m} P_{B\times(\iota'(v''_\mu)-\iota'(v'_\mu))}
\end{array}$$

is equivalent to a section of the $\mathbb{G}_m$-torsor

$$E \longrightarrow P_{B\times(\iota'(x''_\rho)-\iota'(x'_\rho))} \otimes \bigotimes_{\mu=k}^{m} P_{B\times(\iota'(v''_\mu)-\iota'(v'_\mu))} \otimes \bigotimes_{\lambda=h}^{\ell} P_{B\times(\iota'(u'_\lambda)-\iota'(u''_\lambda))}.$$

This $\mathbb{G}_m$-torsor is associated to the line bundle $P_{B\times c'_\rho}$. Therefore, the canonical section $\tau_\rho : E \to P_{B\times c'_\rho}$ gives rise to a connecting isomorphism over the double point x_ρ for $\rho = 1, \dots, r$. Note that the change in the sign is forced by considering line bundles instead of invertible sheaves. Thus, we obtain a line bundle

$$L := D_{X\times E} \longrightarrow X \times E$$

via the pasting with $\tau_1, \ldots, \tau_r$, where we rigidified the pasting over the unit 1 section of E by requiring that $D_{X\times 1}$ is the trivial line bundle on X.

We obtain the line bundle L_ρ on $X \times E$ which is defined via the pasting with $\gamma_\rho := \tau_\rho : E \to P_{B\times c_\rho}$ and $\gamma_{\rho'} = \tau_{\rho'}(1)$ for $\rho' \neq \rho$. The pasting $\tau_{\rho'}(1) : E \to P_{B\times c'_{\rho'}}$ is the unique extension of the constant group homomorphism $T \to P_{0\times c'_\rho}$; cf. Sect. A.2. This is equivalent to the character on the torus $T = E_{1,0} \times \cdots \times E_{r,0}$, which is the fiber of $E \to B$ over the unit element 0 of B. This assignment is the identification of $H_1(X, \mathbb{Z})$ with the group of characters of T; cf. Theorem A.2.8.

Then $D_{X\times E}$ is the universal line bundle on $X \times E$.

Indeed, every line bundle L on X is equivalent to the isomorphism class of (L', τ), where L' is a line bundle on Y and $\tau = \{\tau_\rho; \rho = 1, \ldots, r\}$ is a system of isomorphisms of the stalks $\tau_\rho : L'(x'_\rho) \xrightarrow{\sim} L'(x''_\rho)$, where (x'_ρ, x''_ρ) are the points of Y lying above the singular point x_ρ. Since B represents the isomorphism classes of line bundles on Y which have degree zero on every component, the line bundle L' is of type $L' \cong D_{Y\times b}$ for some point b of B. Moreover, L' is equipped with gluing data $\tau_\rho : L(x'_\rho) \xrightarrow{\sim} L(x''_\rho)$ for $\rho = 1, \ldots, r$. Such gluing data are of the type as considered above. Thus, we see that the couple (L', τ) corresponds to a unique point of E.

We leave it to the reader to check that $(E, D_{X\times E})$ satisfies the universal property of the functor $\mathrm{Pic}^0_{X/k}$. Thus, we see that E represents $\mathrm{Pic}^0_{X/k}$. □

Corollary 5.2.4. *In the situation of Proposition 5.2.3 let $x_1, \ldots, x_r$ be double points of X for $r := \mathrm{rk}\, H_1(X, \mathbb{Z})$ such that $X - \{x_1, \ldots, x_r\}$ is a tree-like configuration of irreducible components. Then let $p : Y \to X$ be the normalization of X above the points $\{x_1, \ldots, x_r\}$. Then there is a canonical isomorphism*

$$\begin{array}{ccc} J := \mathrm{Pic}^0_{X/k} & \xrightarrow{\sim} & E := P_{B\times c'_1} \times_B \ldots \times_B P_{B\times c'_r} \\ \downarrow p^* & & \downarrow q \\ \mathrm{Pic}^0_{Y/k} & = & B := \mathrm{Pic}^0_{X'/k} \end{array}$$

which sends a point $\ell \in J$ to the point $\tau \in E$, where $q(\tau) := p^*\ell \in B$ is a line bundle on Y and $\tau := (\tau_1, \ldots, \tau_r)$ is the identification of the stalks

$$P_{q(\ell)\times\iota'(x'_\rho)} \longrightarrow P_{q(\ell)\times\iota'(x''_\rho)}, \quad \ell \longmapsto \tau_\rho(\ell).$$

Here $c'_\rho = -\varphi_\Theta([c_\rho])$ is the image of a simple path through x_ρ not meeting the other points x'_ρ for $\rho = 1, \ldots, r$ and the map τ_ρ is the multiplication with the tautological section $\tau_\rho : E \to P_{B\times c'_\rho}$.

Remark 5.2.5. In Theorem A.2.8 there is an interpretation of J without introducing a basis of $H_1(X,\mathbb{Z})$. Indeed, J can be represented as a B-scheme

$$q : J = \mathcal{S}pec\Big(\bigoplus_{m'\in H_1(X,\mathbb{Z})} \mathcal{P}_{B\times\phi'(m')}\Big) \longrightarrow B,$$

where $\mathcal{P}_{B\times\phi'(m')}$ is the invertible sheaf $\mathrm{Hom}_B(P_{B\times\phi'(m')}, \mathbb{A}^1_B)$ associated to the line bundle $P_{B\times\phi'(m')}$ and where the multiplication on the graded $\mathcal{O}_B$-algebra is given by the Theorem of the Square 7.1.6(b)

$$P_{B\times\phi'(m'_1)} \times P_{B\times\phi'(m'_2)} \longrightarrow P_{B\times\phi'(m'_1+m'_2)}.$$

Hereby the group $H_1(X,\mathbb{Z})$ is viewed as the group of characters of J

$$H_1(X,\mathbb{Z}) \longrightarrow \mathrm{Hom}_q(J, P_{B\times B'}), \quad m' \longmapsto \big[m' : t \longmapsto t^{m'}\big].$$

Note that an S-valued point t of J is a family consisting of sections $t^{m'}$ of $P_{q(t)\times\phi'(m')}(S)$ satisfying $t^{m'_1}\otimes t^{m'_2} = t^{m'_1+m'_2}$. After having introduced coordinates as in Corollary 5.2.4, the point t can be written as an r-tuple $t=(t_1,\ldots,t_r)$ and $m' = m'_1 c_1 + \cdots + m'_r c_r$. Thus we have that $t^{m'} = t_1^{\otimes m_1}\otimes\cdots\otimes t_r^{m_r}$ is a point of $P_{B\times\phi'(m')}$.

In the following we want to discuss the Weil construction of Jac X for later use.

Lemma 5.2.6. *In the situation of Proposition 5.2.3 let $\psi : X^* \to X$ be a proper morphism of connected semi-stable curves, which is an isomorphism above the complement $X - \{x_1,\ldots,x_r\}$ such that the inverse image of x_ρ under ψ is a projective line $P_\rho \cong \mathbb{P}^1_k$. Then set $A_\rho := P_\rho - \{x'_\rho, x''_\rho\}$, where x'_ρ, x''_ρ are the intersection points of P_ρ with the remaining part of X^*. Let $X_1,\ldots X_s$ be all the components of X with genus $g_j := g(X_j)\geq 1$. Then there exists an affine dense open part $W_j \subset X_j^{(g_j)}$ such that the morphism*

$$\iota : (A_1\times\cdots\times A_r)\times(W_1\times\cdots\times W_s) \longrightarrow \mathrm{Pic}^0_{X^*/k} = \mathrm{Pic}^0_{X/k}$$
$$(a_1,\ldots,a_r,w_1,\ldots,w_s) \longmapsto \bigotimes_{\rho=1}^{r}\big[a_\rho - a^0_\rho\big]\otimes\bigotimes_{j=1}^{s}\big[w_j - w^0_j\big]$$

is an open immersion, where $a^0_\rho \in A_\rho$ and $w^0_j \in X_j^{(g_j)}$ are rational points.

Proof. Let $X'\to X$ be the normalization. Then it is well-known that there exists a dense open affine subset $W_j \subset X_j^{(g_j)}$ such that

$$(W_1\times\cdots\times W_s) \longrightarrow \mathrm{Pic}^0_{X'/k}, \quad (w_1,\ldots,w_s) \longmapsto \bigotimes_{j=1}^{s}\big[w_j - w^0_j\big],$$

is an open immersion; cf. [15, 9.3.5]. The map

$$(A_1 \times \cdots \times A_r) \longrightarrow \mathrm{Pic}^0_{X/k}, \quad (a_1, \ldots, a_r) \longmapsto \bigotimes_{\rho=1}^{r} [a_\rho - a^0_\rho],$$

maps $A_1 \times \cdots \times A_r$ into the torus, since the composition of this map with the projection onto the abelian variety $\mathrm{Pic}^0_{X'/k}$ is constant. This map is a closed immersion onto the torus of J.

Indeed, a basis of $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^r$ is given by the circuits $c_1, \ldots, c_r$, where, for $\rho = 1, \ldots, r$, the circuit c_ρ passes x_ρ and meets no other $x_{\rho'}$ for $\rho' \neq \rho$. Every value $\alpha \in \mathbb{G}_{m,k}$ can be adjusted by a rational function f_ρ on P_ρ with value 1 in x'_ρ and α in x''_ρ which has a simple zero in a suitable $a_\rho \in P_\rho$ and a simple pole in a^0_ρ. Therefore, the divisor $a_\rho - a^0_\rho$ yields a unique line bundle which is obtained by twisting the trivial line bundle by α over the point x_ρ. Thus, the assertion follows. □

Using this lemma, one obtains a dense open part of Jac X. Moreover, there is a birational group law on this open part. Then, the associated algebraic group represents Jac X; cf. [15, §9.3].

Remark 5.2.7. The torus can be described in another way by topological cocycles with integer coefficients. Every cocycle $n = (n_{i,j}) \in H^1(X, \mathbb{Z})$ gives rise to a line bundle on $X \times \mathbb{G}_{m,k}$ via the cocycle

$$\zeta^n := \big(\zeta^{n_{i,j}}\big) \in H^1\big(X \times \mathbb{G}_{m,k}, \mathcal{O}^\times_{X \times \mathbb{G}_{m,k}}\big),$$

where ζ serves as a coordinate function on $\mathbb{G}_{m,k}$. Due to the universal property of the generalized Jacobian, this line bundle induces a group homomorphism

$$\mathbb{G}_{m,k} \longrightarrow \mathrm{Pic}^0_{X/k}, \quad t \longmapsto \zeta(t)^n.$$

If $n_1, \ldots, n_r$ is a basis of $H^1(X, \mathbb{Z})$, then the map

$$\mathbb{G}^r_{m,k} \longrightarrow \mathrm{Pic}^0_{X/k}, \quad t \longmapsto \zeta_1(t)^{n_1} \otimes \cdots \otimes \zeta_r(t)^{n_r},$$

gives rise to an isomorphism of the split torus $\mathbb{G}^r_{m,k}$ to the torus T, where $\zeta_1, \ldots, \zeta_r$ are the coordinate functions on $\mathbb{G}^r_{m,k}$.

Proposition 5.2.8. *In the situation of Proposition 5.2.3 there is a commutative diagram*

$$\begin{array}{ccc}
H^1(X, \mathbb{Z}) \times H_1(X, \mathbb{Z}) & \longrightarrow & \mathbb{Z} \\
\Big\downarrow{\scriptstyle h \times h'} & & \Big\downarrow{\scriptstyle \chi} \\
\mathrm{Hom}(\mathbb{G}_{m,k}, J) \times \mathrm{Hom}(T, \mathbb{G}_{m,k}) & \longrightarrow & \mathrm{Hom}(\mathbb{G}_{m,k}, \mathbb{G}_{m,k})
\end{array}$$

of isomorphisms of canonical pairings. Here h maps a cocycle m of $H^1(X,\mathbb{Z})$ *to the line bundle* ζ^m *and* h' *maps a cycle* $m' \in H_1(X,\mathbb{Z})$ *to the character in the sense of Remark 5.2.5. The pairing between* $H^1(X,\mathbb{Z})$ *and* $H_1(X,\mathbb{Z})$ *is the Kronecker product, when we view X as the realization* real $\Gamma(X)$ *of its graph of coincidence. The down-arrow on the right-hand side is the canonical one.*

Proof. To specify the pairing between $H^1(X,\mathbb{Z})$ and $H_1(X,\mathbb{Z})$ consider representations

$$H^1(X,\mathbb{Z}) = \mathbb{Z}z_1 \oplus \cdots \oplus \mathbb{Z}z_r,$$
$$H_1(X,\mathbb{Z}) = \mathbb{Z}z'_1 \oplus \cdots \oplus \mathbb{Z}z'_r.$$

In the first row each double point x_ρ gives rise to the cocycle $z_\rho \in Z^1(X,\mathbb{Z})$. Thus, $(\zeta(t)^{z_\rho})$ gives rise to a cocycle in $H^1(X \times \mathbb{G}_{m,k}, \mathcal{O}^\times_{X\times\mathbb{G}_{m,k}})$ for $t \in \mathbb{G}_{m,k}$ and hence to an invertible sheaf on $X \times \mathbb{G}_{m,k}$, which in turn is equivalent to a morphism $\mathbb{G}_{m,k} \to J$. In the second row z'_ρ represents the simple cycle passing through x_ρ and no other $x_{\rho'}$ for $\rho' \neq \rho$. Furthermore, in both cases we choose the same orientation on the double points; i.e., the orientation on the edge associated to the double point. Write $m = \sum_{\rho=1}^r m_\rho z_\rho$ and $m' = \sum_{\rho=1}^r m'_\rho z_\rho$. Then, the pairing in the upper row of the assertion yields $n := \sum_{\rho=1}^r m_\rho m'_\rho$. For the down maps one obtains

$$h(m)(t) = \left(t^{m_1}, \ldots, t^{m_r}\right) \quad \text{and} \quad h'\left(m'\right)(t_1, \ldots, t_r) = t_1^{m'_1} \cdot \ldots \cdot t_r^{m'_r}.$$

Indeed, by Remark 5.2.5 the elements of $\mathrm{Hom}(T, \mathbb{G}_{m,k})$ correspond one-to-one to the elements of $H_1(X,\mathbb{Z})$. Moreover, we know that $\mathrm{Hom}(\mathbb{G}_{m,k}, J)$ equals $\mathrm{Hom}(\mathbb{G}_{m,k}, T)$, since a morphism from $\mathbb{G}_{m,k}$ to an abelian variety is constant. By Lemma 5.2.9 the elements of $\mathrm{Hom}(\mathbb{G}_{m,k}, J)$ are of type $t \mapsto (\zeta(t)^m)$ for a uniquely determined $m \in H^1(X,\mathbb{Z})$ and $t \in \mathbb{G}_{m,k}$. Because of $\chi(n)(t) = t^n$ the assertion is clear. □

Lemma 5.2.9. *In the situation of Proposition 5.2.3 let* $T = \mathbb{G}^r_{m,k}$ *be a split torus with coordinate functions* $(\zeta_1, \ldots, \zeta_r)$*. Let* $\varepsilon_X : X \to X \times T$ *be the unit section of the torus and* $p_1 : X \times T \to X$ *the first projection. Then every invertible sheaf* $\mathcal{L}$ *on* $X \times T$ *has a unique representation*

$$\mathcal{L} \cong \left(t^m\right) \otimes p_1^* \varepsilon_X^* \mathcal{L},$$

where $m := m_1 z_1 + \cdots + m_r z_r$ *is a cocycle in* $H^1(X,\mathbb{Z}) = \mathbb{Z}z_1 \oplus \cdots \oplus \mathbb{Z}z_r$ *and* (t^m) *is the invertible sheaf given by the cocycle* $(\zeta_1(t)^{m_1} \otimes \cdots \otimes \zeta_r(t)^{m_r})$*.*

Proof. Let $p : Y \to X$ be the normalization of X over $\{x_1, \ldots, x_r\}$. Set $p_T := (p, \mathrm{id}_T)$. Then $p_T^*\mathcal{L}$ is isomorphic to $p_1^*\varepsilon_Y^* p_T^*\mathcal{L}$ due to the universal property of the Jacobian, since $T \to \mathrm{Pic}^0_{Y/k}$ is constant and $\mathcal{O}_T(T)$ is factorial. Then $\mathcal{L}$ is obtained from $\varepsilon_X^*\mathcal{L}$ by pasting the stalks over the couples $(T \times \{x'_\rho\}, T \times \{x''_\rho\})$ by $\zeta_1(t)^{m_1} \cdot \ldots \cdot \zeta_r(t)^{m_r}$, where $(m_1, \ldots, m_r) \in \mathbb{Z}^r$. □

The last statement follows from the universal property of $\mathrm{Pic}^0_{X/k}$ as well.

5.3 Lifting of the Jacobian of the Reduction

In the following let X_K be a connected smooth projective curve over a algebraically closed non-Archimedean field K. Then X_K has a formal R-model X, whose special fiber $\widetilde{X}$ is semi-stable; cf Theorem 4.4.3. Assume, in addition, that the irreducible components of $\widetilde{X}$ over the residue field k are smooth. Let $\widetilde{J}$ be the generalized Jacobian of $\widetilde{X}$; cf. Proposition 5.2.1.

Let g be the genus of X and r the rank of $H_1(\widetilde{X}, \mathbb{Z})$. Assume, in addition, that there exist rational components

$$\widetilde{A}_1, \ldots, \widetilde{A}_r$$

of $\widetilde{X} - \operatorname{Sing} \widetilde{X}$ which are isomorphic to $\mathbb{P}^1_k - \{0, \infty\}$ such that the complement $\widetilde{X} - (\widetilde{A}_1, \ldots, \widetilde{A}_r)$ is a tree-like configuration of irreducible components. Let

$$\tilde{a}^0_\rho \in \widetilde{A}_\rho$$

be a k-rational point for $\rho = 1, \ldots, r$. Furthermore, let $\widetilde{X}_1, \ldots, \widetilde{X}_s$ be all the irreducible components of $\widetilde{X}$ with genus $g_j := g(\widetilde{X}_j) \geq 1$. Then $\widetilde{X}_j^{(g_j)}$ is smooth over k as follows from the theorem on symmetric functions. Let

$$\tilde{w}^0_j \in \widetilde{X}_j^{(g_j)}$$

be a closed point, which is induced by g_j points of $\widetilde{X}_j$, which are contained in the smooth locus of $\widetilde{X}$ for $j = 1, \ldots, s$. Likewise let

$$\widetilde{W}_j \subset \widetilde{X}_j^{(g_j)}$$

be a dense open affine subscheme, which is disjoint from the singular locus of $\widetilde{X}^{(g_j)}$ such that

$$\widetilde{W}_j \longrightarrow \operatorname{Jac}(\widetilde{X}_j), \quad \tilde{w}_j \longmapsto \left[\tilde{w}_j - \tilde{w}^0_j\right],$$

is an open immersion; cf. [15, 9.3/5]. Moreover, we may assume $\tilde{w}^0_j \in \widetilde{W}_j$.

Let $\overline{X}$ be the generic fiber of the smooth part of the formal R-curve X. There are formal liftings of all these "$\sim$"-objects on X, which were introduced above. We denote them by the same symbol without "$\sim$" on top. Thus, we have open formal affine subdomains

$$A_\rho \subset \overline{X} \quad \text{and} \quad W_j \subset \overline{X}^{(g_j)},$$

which are smooth formal R-models [15, p. 255] with the given reductions $\widetilde{A}_\rho$ and $\widetilde{W}_j$. There exist K-rational points

$$a^0_\rho \in A_\rho \quad \text{and} \quad w^0_j \in \overline{X}_j^{(g_j)},$$

which are liftings of $\tilde{a}_\rho^0$ and of $\tilde{w}_j^0$, respectively. Put

$$A := A_1 \times \cdots \times A_r \quad \text{and} \quad W := W_1 \times \cdots \times W_s.$$

We view $A \times W \subset X_K^{(g)}$ as an open subset of the symmetric product $X_K^{(g)}$. Note that $g = r + g_1 + \cdots + g_s$ equals the genus of X_K; cf. Proposition 4.2.6.

Lemma 5.3.1. *With the above notations we have the following results*:

(a) *Let $\mathcal{L}_K = \mathcal{O}_{X_K}(D)$ be the invertible sheaf given by a divisor $D = \sum_{i=1}^n (x_i - y_i)$, where (x_i, y_i) is a pair of K-valued points of $\overline{X}$ which specialize into the same irreducible component of $\widetilde{X}$ for $i = 1, \ldots, n$. Then $\mathcal{L}_K$ extends to a formal invertible sheaf $\mathcal{L}$ on X with reduction $\widetilde{\mathcal{L}} = \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^n (\tilde{x}_i - \tilde{y}_i)) \in \mathrm{Pic}^0_{\widetilde{X}/k}(k)$.*

(b) *The map*

$$\iota : A \times W \longrightarrow \mathrm{Jac}(X_K),\ (a, w) \longmapsto \sum_{\rho=1}^{r} [a_i - a_i^0] \otimes \sum_{j=1}^{s} [w_j - w_j^0],$$

is an open immersion of rigid analytic varieties.

(c) *If $(a, w) \in A \times W$, then we have the following compatibility*

$$\widetilde{\iota(a, w)} = \tilde{\iota}(\tilde{a}, \tilde{w}),$$

where $\tilde{\iota}$ is the map defined in Lemma 5.2.6, which is associated to $(\tilde{a}^0, \tilde{w}^0)$.

(d) *The reduction map is compatible with the group law*

$$\widetilde{\iota(a, w) \otimes \iota(a', w')} = \widetilde{\iota(a, w)} \otimes \widetilde{\iota(a', w')}.$$

Proof. (a) The points x_i, y_i extend to R-valued points $\overline{x}_i, \overline{y}_i$ of X lying in the smooth locus of X/R. Then the Weil divisor $\sum_{i=1}^n (\overline{x}_i - \overline{y}_i)$ is a Cartier divisor, and hence it defines a formal line bundle $\mathcal{L}$ on X. Thus, the reduction of $\mathcal{L}$ is given by the Weil divisor $\sum_{i=1}^n (\tilde{x}_i - \tilde{y}_i)$, which is Cartier as well. Due to the very definition of $\mathrm{Pic}^0_{\widetilde{X}/k}$ the reduction $\widetilde{\mathcal{L}}$ belongs to $\mathrm{Pic}^0_{\widetilde{X}/k}(k)$.

(b) Since $\iota : X^{(g)} \to \mathrm{Jac}(X_K)$ is an open immersion on the open subset of $X^{(g)}$, where ι is injective, it suffices to show that $\iota|_{A \times W}$ is injective.

If $\iota(a, w) = \iota(a', w')$, then there exists a meromorphic function m on X_K with $(a - a' + w - w') = \operatorname{div} m$. By Corollary 4.3.4 the sup-norms of m on the irreducible components are equal. After adjusting the sup-norm to 1, the function m has a well-defined reduction as a rational function $\widetilde{m}$ on $\widetilde{X}$ and solves the divisor $\tilde{a} - \tilde{a}' + \tilde{w} - \tilde{w}'$. Thus, we see that the fiber of a point $\iota(a, w)$ under ι is contained in a formal fiber of $A \times W$, because $\tilde{\iota}$ is an open immersion by Lemma 5.2.6. Then the fiber of $\iota|_{A \times W}$ consists of a single point, because the fibers of $\iota : X^{(g)} \to \mathrm{Jac}(X_K)$ are projective spaces.

(c) and (d) These follow from (a). □

Lemma 5.3.2. *The group law of $J_K := \mathrm{Jac}(X_K)$ induces a group structure on the rigid analytic subvariety*

$$\overline{J}_+ := \iota\big((A \times W)_+\big)$$

of J_K, where $(A \times W)_+$ is the formal fiber of $A \times W$ at the point (a^0, w^0).

Moreover, $\overline{J}_+$ acts on $\iota(A \times W)$ and leaves the formal fibers of $\iota(A \times W)$ invariant. The action on the formal fibers is faithful and transitive.

In particular, $\overline{J}_+$ has the following property:

Let $\mathcal{L}_K = \mathcal{O}_{X_K}(D)$ be the class of an invertible sheaf given by a divisor $D = \sum_{i=1}^{n}(x_i - y_i)$, where (x_i, y_i) are pairs of K-valued points which specialize into the smooth part of $\widetilde{X}$ and into the same irreducible component. Then $\widetilde{\mathcal{L}}$ is trivial if and only if $[\mathcal{L}]$ belongs to $\overline{J}_+$.

Proof. Let $V := \iota(A \times W)$ and $\ell \in V$. Since V has a smooth formal R-model, $\overline{J}_+$ is isomorphic to an open ball by Corollary 4.1.10. Consider now a connected affinoid subdomain $U \subset \overline{J}_+$ with $1 \in U$ and the morphism

$$\varphi_\ell : U \longrightarrow J_K,\ a \longmapsto a \otimes \ell.$$

Then $U \cap \varphi_\ell^{-1}(V)$ is an affinoid subdomain of U. In particular, we have the restriction $\psi : U \cap \varphi_\ell^{-1}(V) \to V$. Due to Lemma 5.3.1(d) the canonical reduction of ψ is constant to $\tilde{\ell} \in \widetilde{J}$. Thus, ψ maps $U \cap \varphi_\ell^{-1}(V)$ to the formal fiber $V_+(\ell)$ of ℓ. Since U is connected, the whole U is mapped to $\overline{J}_+(\ell)$ by φ_ℓ, since $\{V, J_K - V_+(\ell)\}$ is an admissible covering of J_K by Remark 3.3.5.

Likewise one shows that $\overline{J}_+$ is invariant under the inverse map and that $\overline{J}_+$ acts transitively on each formal fiber of V. Thus we see that $\overline{J}_+$ is closed under the group law of J_K and the inverse map of J_K. Thus, we see that $\overline{J}_+$ is a subgroup of J_K.

If $[\widetilde{\mathcal{L}}] = 1$, then there exists a rational function $\widetilde{m}$ on $\widetilde{X}$ such that $\widetilde{D} = \mathrm{div}(\widetilde{m})$. By Corollary 4.3.6 there exists a meromorphic function m on X_K, which lifts $\widetilde{m}$. Then $E := D - \mathrm{div}(m)$ is of the same type as D with $\widetilde{E} = 0$. This means that on each formal fiber $X_+(\tilde{x})$ of $X \to \widetilde{X}$ the degree of $E|_{X_+(\tilde{x})}$ is zero. So it remains to see that, for points x, y of a smooth formal fiber, the class $[x - y]$ belongs to $\overline{J}_+$. Since such a formal fiber is isomorphic to an open ball by Proposition 4.1.12, there exists a closed disc B contained in $X_+(y)$ with $x, y \in B$. Now consider the mapping

$$\varphi : B \longrightarrow J_K,\ a \longmapsto [a - y].$$

As before consider the subvariety $B \cap \varphi^{-1}(V)$. The reduction of $[a - y]$ is zero as $\tilde{a} = \tilde{y}$. Since $[a - y]$ belongs to V and $\tilde{\iota}$ is an open immersion, we see $\varphi(a) \in \overline{J}_+$. Since B is connected, one concludes as above that φ maps the whole B to $\overline{J}_+$. □

Let us summarize the results of this section.

Proposition 5.3.3. *In the above situation the image $\iota(A \times W)$ generates a subgroup $\overline{J}_K \subset \mathrm{Jac}\, X_K$ which has a smooth formal R-model $\overline{J}$ such that $\iota|_{A \times W}$ is a formal* *analytic open immersion. The formal fiber at the unit section is* $\overline{J}_+$*. The reduction map induces the commutative diagram*

$$\begin{array}{ccccccc}
A \times W & \overset{\iota}{\hookrightarrow} & \overline{J}_K & \qquad & (a, w) & \longmapsto & [\sum_{\rho=1}^{r}(a_\rho - a_\rho^0) + \sum_{j=1}^{s}(w_j - w_j^0)] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{A} \times \widetilde{W} & \overset{\tilde{\iota}}{\hookrightarrow} & \widetilde{J} & & (\tilde{a}, \tilde{w}) & \longmapsto & [\sum_{\rho=1}^{r}(\tilde{a}_\rho - \tilde{a}_\rho^0) + \sum_{j=1}^{s}(\tilde{w}_j - \tilde{w}_j^0)]
\end{array}$$

where the vertical map on the right-hand side coincides with the reduction of formal invertible sheaves. $\overline{J}$ *represents all the divisor class of the form* $\bigotimes_{i=1}^{n}[x_i - y_i]$*, where* (x_i, y_i) *are pairs of* K*-valued points of* X *which specialize into the smooth part* $\overline{X}$ *and into same irreducible component.*

In particular, the reduction of $\overline{J}$ *is the generalized Jacobian* $\widetilde{J}$ *of* $\widetilde{X}$*, and hence semi-abelian.*

Proof. Put $V := \iota(A \times W)$. Its reduction $\widetilde{V}$ equals $\widetilde{A} \times \widetilde{W}$. Since $\widetilde{J}$ is quasi-compact, there exist finitely many K-valued points $\tau_1, \ldots, \tau_N$ in V such that

$$\widetilde{J} = \tilde{\tau}_1 \cdot \widetilde{V} \cup \cdots \cup \tilde{\tau}_N \cdot \widetilde{V}.$$

If we put

$$\overline{J}_K := \tau_1 \cdot V \cup \cdots \cup \tau_N \cdot V,$$

then $\{\tau_1 \cdot V, \ldots, \tau_N \cdot V\}$ is a formal analytic covering of $\overline{J}_K$. So this defines a formal R-model $\overline{J}$ of $\overline{J}_K$. The group law on $\mathrm{Jac}(X_K)$ restricts to a formal analytic group law on $\overline{J}$.

Indeed, let $\ell_1, \ell_2, \ell_3 \in \{\tau_1, \ldots, \tau_N\}$ and let $\widetilde{U} \subset \tilde{\ell}_1 \cdot \widetilde{V} \times \tilde{\ell}_2 \cdot \widetilde{V}$ be an open subset such that the group law on $\widetilde{J}$ maps $\widetilde{U}$ to $\tilde{\ell}_3 \cdot \widetilde{V}$. Let $U \subset \ell_1 \cdot V \times \ell_2 \cdot V$ be a lifting of $\widetilde{U}$. Then the group law in J_K maps U to $\ell_3 \cdot V$. In fact, consider (a_ν, w_ν) in $A \times W$ with $v_\nu := \iota(a_\nu, w_\nu) \in V$ for $\nu = 1, 2$ such that $(\ell_1 \cdot v_1, \ell_2 \cdot v_2) \in U$. Then there exists $v_3 \in V$ such that we have the reduction $\tilde{\ell}_1 \cdot \tilde{v}_1 \cdot \tilde{\ell}_2 \cdot \tilde{v}_2$ equals $\tilde{\ell}_3 \cdot \tilde{v}_3$. This implies by Lemma 5.3.1(d) that the invertible sheaf $\mathcal{L}$ associated to $\ell_1 \cdot v_1 \cdot \ell_2 \cdot v_2 \cdot \ell_3^{-1} \cdot v_3^{-1}$ has reduction $\widetilde{\mathcal{L}} = 1$. Then we see by Lemma 5.3.2 that there exists some $j \in \overline{J}_+$ such that the class of $\mathcal{L}$ is j, and hence we have $\ell_1 \cdot v_1 \cdot \ell_2 \cdot v_2 \cdot \ell_3^{-1} = j \cdot v_3$. Since $\overline{J}_+$ acts on the formal fibers of V, we obtain that $j \cdot v_3 = \iota(a_3, w_3)$ for some $(a_3, w_3) \in (A \times W)_+$. This shows that the group law on J_K restricts to a formal analytic law of composition on $\overline{J}_K$. Likewise one verifies that the covering is formal analytic and that the inverse map is also formal analytic. □

5.4 Morphisms to Rigid Analytic Groups with Semi-Abelian Reduction

In the following let G_K be a rigid analytic group and $\overline{G}_K \subset G_K$ an open subgroup which has a smooth formal R-model $\overline{G}$ with semi-abelian reduction $\widetilde{G}$; i.e., $\widetilde{G}$ is semi-abelian k-scheme.

Definition 5.4.1. A *semi-abelian group scheme* is a smooth separated commutative group scheme $G \to S$ with connected fibers such that each fiber G_s is an extension of an abelian variety by a torus T_s.

In this section we will consider rigid analytic morphisms $u_K : Z_K \to G_K$ from a connected rigid space Z_K with a K-rational point z_0 satisfying $u_K(z_0) = e$, where e is the unit element of G_K. We are especially interested in the case, where Z_K has a smooth formal model Z or where Z_K is an affine formal curve with semi-stable reduction.

Lemma 5.4.2. *Let $X := \operatorname{Spf} A$ and $Y := \operatorname{Spf} B$ be admissible affine formal R-scheme with geometrically reduced reductions. If $\varphi_K : X_K \to Y_K$ is a morphism of their generic fibers, then φ_K extends to a formal morphism $\varphi : X \to Y$.*

Proof. By Proposition 3.4.1 we have that $A = \mathring{A}_K$ and $B = \mathring{B}_K$. Obviously, the map φ_K^* maps B to A, and hence φ_K^* extends to a formal morphism. □

Lemma 5.4.3. *Let $\varphi_K : X_K \to Y_K$ be a morphism of separated rigid analytic spaces. Let U_K and V_K be an open affinoid subvarieties of X_K and of Y_K, respectively, which admit smooth formal models U and V, respectively. Let $x \in U_K$ be a point with reduction $\tilde{x} \in \widetilde{U}$ such that $\varphi_K(U_+(x)) \subset V_+(\varphi_K(x))$. Then there exists a formal open neighborhood U' of $\tilde{x}$ in U such that $\varphi_K|_{U'_K}$ extends to a formal morphism $\varphi : U' \to V$.*

Proof. Let $\Gamma_K \subset X_K \times_K Y_K$ be the graph of φ_K. Since φ_K is defined on X_K, the projection p_1 induces an isomorphism from the generic fiber Γ_K to X_K. In particular, we have that $\varphi_K = p_2 \circ (p_1|_{\Gamma_K})^{-1}$.

Consider now the schematic closure $\Gamma \subset U \times_R V$ of $\Gamma_K \cap (U_K \times V_K)$; cf. [15, §2.5]. Due to Theorem 3.2.1 the formal scheme Γ is locally of topologically finite presentation. Then look at the first projection $p_1|_\Gamma : \Gamma \to U$.

First we verify that $p_1|_\Gamma$ is flat at $\tilde{x}$. This can be checked after faithfully flat base change. The base change $\mathcal{O}_{U,\tilde{x}} \to \mathring{\mathcal{O}}_U(U_+(\tilde{x}))$ is faithfully flat due to Corollary 3.5.7, where the latter is the ring of holomorphic functions on $U_+(\tilde{x})$ which are bounded by 1. Since the schematic closure is compatible with flat base change [15, §2.5/2], the flatness of $p_1|_\Gamma$ at $\tilde{x}$ can be checked after that base change. Since φ_K maps on $U_+(\tilde{x})$ to $V_+(\varphi_K(x))$, the projection $p_1|_\Gamma : \Gamma \to U$ is an isomorphism over $U_+(\tilde{x})$, and hence $p_1|_\Gamma$ is flat at $\tilde{x}$.

Since the flat locus is open and a flat map is open, there exists a formal open affine $U' \subset U$, which is contained in the image of $p_1|_\Gamma$. Thus, we obtain the morphism $\varphi|_K|_{U'} := p_2 \circ (p_1|_{\Gamma_K})^{-1} : U'_K \to V_K$ of the affinoid subdomains. Then $\varphi|_K|_{U'}$ extends to a morphism $\varphi : U' \to V$ by Lemma 5.4.2. □

Corollary 5.4.4. *Let $\varphi_K : X_K \to Y_K$ be a morphism of rigid analytic spaces. Assume that X_K and Y_K admit smooth formal models X and Y, respectively. If φ_K respects formal fibers, then φ_K extends to a formal morphism $\varphi : X \to Y$.*

In the following we will make use of an extension theorem for holomorphic functions, which we will be presented only in a very simple setting. Actually the statement can be generalized to a more general situation.

Proposition 5.4.5. *Let $\mathbb{B}^n$, $n \geq 2$, be an n-dimensional ball with coordinate functions $\zeta_1, \ldots, \zeta_n$. Let $g := (\zeta_1 - c_1) \cdot \ldots (\zeta_1 - c_m) \in R[\zeta_1]$ be a polynomial, where the reductions $\tilde{c}_1, \ldots, \tilde{c}_m \in k$ are pairwise distinct. For some $\varepsilon \in |K^\times|$ put*

$$U := \{z \in \mathbb{B}^n;\ |\zeta_\nu(z)| \leq \varepsilon \text{ for } \nu = 2, \ldots, n\},$$
$$L := \{z \in \mathbb{B}^n;\ |g(z)| = 1\}.$$

Then every holomorphic function on $U \cup L$ extends to a holomorphic function on $\mathbb{B}^n$ in a unique way.

Proof. Because of the identity principle it suffices to show that every holomorphic function $f \in \mathcal{O}_{\mathbb{B}^n}(U \cup L)$ is induced by an element of $K\langle\zeta_1, \ldots, \zeta_n\rangle$. First, we remark that $f|_L$ admits a unique partial fraction expansion

$$f = \sum_{i=0}^{\infty} a_i \zeta_1^i + \sum_{\mu=1}^{m} \sum_{j_\mu=1}^{\infty} a_{j_\mu,\mu} (\zeta_1 - c_\mu)^{-j_\mu}$$

with coefficients in $K\langle\zeta_2, \ldots, \zeta_n\rangle$ and the usual conditions on convergence. Restricting the expansion to U shows that the fractional part vanishes. This implies that the extension of f is a holomorphic function on $\mathbb{B}^n$. □

Now we turn to the extension theorem for morphisms to group varieties.

Proposition 5.4.6. *Let G_K be a rigid analytic group and $\overline{G}_K \subset G_K$ an open subgroup, which has a smooth formal model $\overline{G}$ with semi-abelian reduction $\widetilde{G}$. Let $u_K : Z_K \to G_K$ be a rigid analytic morphism, where Z_K is a connected rigid space with a K-rational point z_0.*

If Z_K admits a smooth formal model Z, then u_K maps Z_K to the translate $u_K(z_0) \cdot \overline{G}_K$. If $u_K(z_0) \in \overline{G}_K$, then $u_K : Z_K \to \overline{G}_K$ extends to a formal morphism $u : Z \to \overline{G}$.

This, especially, applies to morphisms $\varphi_K : \overline{\mathbb{G}}_{m,K} \to G_K$, where

$$\overline{\mathbb{G}}_{m,K} := \{t \in \mathbb{G}_{m,K};\ |t| = 1\}$$

is the torus of units. *Its R-model is the* formal torus $\overline{\mathbb{G}}_{m,R} := \operatorname{Spf} R\langle\zeta, \zeta^{-1}\rangle$.

Obviously, we may assume that K is algebraically closed. We start with the proof in a special case.

Lemma 5.4.7. *In the situation of Proposition 5.4.6 assume, in addition, that Z_K is the 1-dimensional unit disc $\mathbb{D}_K$, $z_0 = 0$ and $u_K(z_0) = e$. Then u_K maps Z_K into the formal fiber $\overline{G}_+(e)$ of $\overline{G}_K$ of the unit element e of G_K.*

Proof. Since the reduction $\widetilde{G}$ of $\overline{G}$ is semi-abelian, $\widetilde{G}$ is an extension of an abelian variety $\widetilde{B}$ by a torus $\widetilde{T}$. In particular, $\widetilde{G}$ is a $\widetilde{T}$-torsor over $\widetilde{B}$; cf. Theorem A.2.8. Thus, there exists an affine open neighborhood $\widetilde{V}$ of $\widetilde{T}$ in $\widetilde{G}$. Let $V \subset \overline{G}$ be the lifting of $\widetilde{V}$. Then the generic fiber V_K is affinoid.

Let us first consider the special case, where u_K maps Z_K into V_K. Then u_K extends to a formal morphism $u : Z \to V$ by Lemma 5.4.2, and hence u induces a morphism $\tilde{u} : \widetilde{Z} \to \widetilde{V}$. Since the reduction $\widetilde{Z}$ is the affine line and a map from a rational variety to an abelian variety is constant, $\tilde{u}$ maps $\widetilde{Z}$ into the torus $\widetilde{T}$. Since a map from the affine line to a torus is constant, the map $\tilde{u}$ is constant. Thus, we see that u_K maps Z_K into $\overline{G}_+(e)$.

Now consider the general case. Let $U_K \subset u_K^{-1}(V_K)$ be the connected component of $u_K^{-1}(V_K)$ which contains z_0. Due to Corollary 2.4.7 the subdomain U_K is a closed disc $\mathbb{D}_K(r)$ minus finitely many open discs, where $\mathbb{D}_K(r)$ is a subdisc of $Z_K = \mathbb{D}_K$ of radius $r \in |K^\times|$. Let us first assume that all the open discs are contained in the boundary $\{z \in \mathbb{D}_K ; |z| = r\}$. Once we have verified the assertion in this case, we are done in general. In fact, that case implies that $\mathbb{D}_K(r)$ is not punctured by open discs. Thus, we have $U_K = \mathbb{D}_K(r)$ is a disc, and $u_K(U_K) \subset \overline{G}_+(e)$, as was seen above. Then the maximum principle implies $U_K = \mathbb{D}_K$.

Therefore, we can consider the case, where $U_K \subset \mathbb{D}_K(r)$ and that U_K contains a non-empty formal open subdomain W_K of $\mathbb{D}_K(r)$. Thus, W_K is the generic fiber of an open formal subscheme W of $\mathbb{D}_R(r)$. Let ζ be a coordinate on $\mathbb{D}_K(r)$, which is adjusted to sup-norm 1. Then there exists a polynomial

$$g := (\zeta - c_1) \cdot \ldots \cdot (\zeta - c_m) \in R[\zeta]$$

as in Proposition 5.4.5 such that every $z \in \mathbb{D}_K(r)$ with $|g(z)| = 1$ belongs to W_K.

For $c \in |K^\times|$ with $c < 1$ consider the subdomain

$$\Delta(c) := \left\{ z \in \mathbb{D}_K^2(r); \left|\zeta_1(z) - \zeta_2(z)\right| \leq c \right\},$$

where ζ_1, ζ_2 are the coordinate functions of $\mathbb{D}_K^2(r)$, and the morphism

$$v_K : \mathbb{D}_K(r) \times \mathbb{D}_K(r) \longrightarrow G_K,\ (z_1, z_2) \longmapsto u_K(z_1) \cdot u_K(z_2)^{-1}.$$

By arguments on continuity we see that there exists an $\varepsilon \in |K^\times|$ with $v_K(\Delta(\varepsilon)) \subset \overline{G}_+(e)$. Since the reduction of W is rational, one concludes as above that u_K maps W_K into a subset which reduces to the torus $\widetilde{T}$. Therefore, v_K maps $W_K \times W_K$ to a subset of $\overline{G}_K$, which reduces to the torus as well and hence v_K maps $W_K \times W_K$ to V_K. So it follows from Proposition 5.4.5 that $v_K(\Delta(c)) \subset V_K$ for all

$c < 1$. Indeed, introduce the coordinate functions $\xi_1 := \zeta_1, \xi_2 := \zeta_1 - \zeta_2$ on $\mathbb{D}^2_K(r)$. Note that, for $c < 1$, it holds

$$\begin{aligned}\Delta(c) \cap (W_K \times W_K) &= \big\{(z_1, z_2) \in \Delta(c); \big|g(z_1)\big| = 1, \big|g(z_2)\big| = 1\big\} \\ &= \big\{(z_1, z_2) \in \Delta(c); \big|g(z_1)\big| = 1\big\},\end{aligned}$$

since $|g(z_1)| = 1$ implies $|g(z_2)| = 1$ if $|z_1 - z_2| < 1$. Therefore, the morphism $v_K : \Delta(c) \dashrightarrow V_K$ is defined everywhere for all $c < 1$.

Then it follows from Lemma 5.4.3 that v_K maps a formal open neighborhood D_K of the diagonal of $\mathbb{D}^2_K(r)$ to V_K. Let D_K be the generic fiber of an open subdomain $D \subset \mathbb{D}^2_R(r)$. By Lemma 5.4.2 we conclude that the morphism v_K extends to a formal morphism $v : D \to V$. Then it follows by the usual techniques that u_K extends to formal morphism $u : \mathbb{D}_R(r) \to \overline{G}$. Indeed, the domain of definition of a rational map can be checked after faithfully flat base change [15, 2.5/5]. Then consider the diagram

$$\begin{array}{ccc} D' & & \\ \downarrow & \searrow & \\ \mathbb{D}_R(r) & \longrightarrow & \overline{G}; \end{array} \qquad \begin{array}{ccc} (x_1, x_2) & & \\ \downarrow & \searrow & \\ x_1 & \vdash - - \to & u(x_1) = v(x_1, x_2) \cdot u(x_2), \end{array}$$

where $D' = D \cap (\mathbb{D}_R(r) \times W)$; cf. [15, 4.4/2]. The vertical map is faithfully flat, the down right map is defined everywhere and the horizontal map coincides with u. Restricting the diagram to infinitesimal levels mod π^{n+1}, the induced horizontal map is defined everywhere due to [15, 2.5/5]. Thus, we see that $u_K : W_K \to V_K$ extends to a morphism $u : \mathbb{D}_R(r) \to V$. Now as above we see that $\tilde{u}$ is constant and hence u_K maps $\mathbb{D}^1_R(r)$ into the formal fiber $\overline{G}_+(e)$. □

Now we turn to the proof of Proposition 5.4.6.

Proof of Proposition 5.4.6. Since Z_K has smooth reduction, every formal fiber is an open n-dimensional ball due to Corollary 4.1.10. We may assume that $u_K(z_0) = e$. Then it follows from Lemma 5.4.7 that the formal fiber $Z_+(\tilde{z}_0)$ is mapped into $\overline{G}_+(e)$. By Lemma 5.4.3 we have that Z_K contains a formal open neighborhood U of z_0 such that $u_K|_{U_K}$ extends to a formal map $U \to \overline{G}$. Likewise we have for every other point $z \in Z_K$

$$u_k\big(Z_+(\tilde{z})\big) \subset u_K(z) \cdot \overline{G}_+(e).$$

Since we may assume that Z_K is quasi-compact, there exist finitely many open subsets $U_1, \ldots, U_\ell$ of Z_K which cover Z_K such that

$$u_K(Z_K) \subset u_K(U_1) \cup \cdots \cup u_K(U_\ell) \subset u_K(z_1) \cdot \overline{G}_K \cup \cdots \cup u_K(z_\ell) \cdot \overline{G}_K.$$

Two residue classes $u_K(z_i) \cdot \overline{G}_K$ and $u_K(z_j) \cdot \overline{G}_K$ either coincide or are disjoint. Since Z_K is connected, we see that $u_K(Z_K) \subset \overline{G}_K$. Due to Corollary 5.4.4 the map u_K extends to a formal morphism $u : Z \to \overline{G}$. □

Next we want to discuss the case, where Z_K is an affinoid curve with semi-stable reduction.

Proposition 5.4.8. *Let G_K be a rigid analytic group and $\overline{G}_K \subset G_K$ an open subgroup which has a smooth formal model $\overline{G}$ with semi-abelian reduction $\widetilde{G}$. Assume that the embedding of the maximal formal torus $\overline{T} \hookrightarrow \overline{G}$ extends to a rigid analytic homomorphism of the associated affine torus $T_K \to G_K$.*

Let Z_K be a connected affine rig-smooth formal curve, which has a semi-stable model Z with precisely one singular point $\tilde{z}_0$. Let $\zeta : Z_K \to \mathbb{G}_{m,K}$ be a holomorphic function which restricts to a coordinate function on the formal fiber of $Z_+(\tilde{z}_0)$ of the double point $\tilde{z}_0$. Let $z_0 \in Z_+(\tilde{z}_0)$ be a K-rational point with $\zeta(z_0) = 1$.

If $u_K : Z_K \to G_K$ is a rigid analytic morphism, then there exists a unique group homomorphism $\varphi : \mathbb{G}_{m,K} \to G_K$ such that u_K factorizes into

$$u_K = (\varphi \circ \zeta) \cdot \overline{u} \cdot u_K(z_0),$$

where $\overline{u} : Z_K \to \overline{G}_K$ is a rigid analytic morphism.

Proof. Let $\varrho \in \sqrt{|K^\times|}$ and let $\varrho^2 < 1$ be the height of the annulus $Z_+(\tilde{z}_0)$; cf. Proposition 4.1.12. After removing finitely many smooth formal fibers of Z_K we may assume that ζ has no zeros. Thus we have

$$Z_+(\tilde{z}_0) = \left\{z \in Z_K; \varrho < \left|\zeta(z)\right| < \varrho^{-1}\right\},$$

where ζ takes absolute values ϱ or ϱ^{-1} on the smooth parts of Z. For $\varrho < \rho \leq 1$ we put

$$A(\rho) := \left\{z \in Z_K; \rho \leq \left|\zeta(z)\right| \leq \rho^{-1}\right\},$$
$$\overline{A}(\rho) := \left\{z \in Z_K; \left|\zeta(z)\right| = \rho\right\}.$$

We may assume that $z_0 \in \overline{A}(1)$ and $u_K(z_0) = e$. Then ζ gives rise to an isomorphism $\zeta : \overline{A}(1) \xrightarrow{\sim} \overline{\mathbb{G}}_{m,K}$ with $\zeta(z_0) = 1 \in \overline{\mathbb{G}}_{m,K}$. Then u_K maps $\overline{A}(1)$ into $\overline{G}_K$ and extends to a morphism $u : A \to \overline{G}$ by Proposition 5.4.6, where A is a smooth R-model of $\overline{A}(1)$.

Since morphisms from rational variety to an abelian varieties are constant, the reduction map $\tilde{u} : \widetilde{A} \to \widetilde{G}$ maps $\widetilde{A}$ into the torus $\widetilde{T}$ of $\widetilde{G}$. Then there exists a group homomorphism $\tilde{\chi} : \widetilde{\mathbb{G}}_{m,k} \to \widetilde{T}$ such that $\tilde{u} = \tilde{\chi} \circ \tilde{\zeta}$. Now $\tilde{\chi} : \widetilde{\mathbb{G}}_{m,k} \to \widetilde{T}$ lifts to a homomorphism $\overline{\chi} : \overline{\mathbb{G}}_{m,K} \to \overline{T}_K$ by Proposition 5.6.7 or by Lemma 5.5.1, and hence it extends to a group homomorphism $\chi : \mathbb{G}_{m,K} \to T_K$.

By our assumption we obtain a rigid analytic morphism $\varphi \circ \zeta : Z_K \to G_K$, which restricts to $\overline{\chi} \circ \zeta|_{\overline{A}(1)}$. By multiplying u_K by $(\varphi \circ \zeta)^{-1}$, we may assume that u_K maps $\overline{A}(1)$ to the formal fiber $\overline{G}_+(e)$.

Then we want to show that $u_K(Z_+(z_0)) \subset \overline{G}_+(e)$. Let U_K be a connected component of $u_K^{-1}(\overline{G}_K)$ which contains z_0. If $U_K \neq Z_K$, then there are only three possibilities for U_K with suitable ρ, ρ_1, ρ_2 in $|K^\times|$:

$$U_K := \left\{z \in Z_K;\, \rho_1 \leq \left|\zeta(z)\right| \leq \rho_2\right\} \quad \text{with } \varrho < \rho_1 < 1 < \rho_2 < \varrho^{-1},$$
$$U_K := \left\{z \in Z_K;\, \rho \leq \left|\zeta(z)\right|\right\} \quad \text{with } \varrho < \rho < \varrho^{-1},$$
$$U_K := \left\{z \in Z_K;\, \left|\zeta(z)\right| \leq \rho\right\} \quad \text{with } \varrho < \rho < \varrho^{-1}.$$

We have to show that all three cases cannot occur. Let us treat the second case. Concentrate on the map $u_K|_{\overline{A}(\rho)}$. By what we saw for $u_K|_{\overline{A}(1)}$ we know that there is a factorization $u_K|_{\overline{A}(\rho)} = (\chi \circ \zeta) \cdot \overline{u} \cdot g$, where $\chi : \mathbb{G}_{m,K} \to T_K$ is a group homomorphism, $\overline{u} : \overline{A}(\rho) \to \overline{G}_+(e)$ and $g \in G$ is a point. By continuity for every ρ' close to ρ we would have a similar representation of $u_K|_{\overline{A}(\rho')}$. But this can happen only if $\chi \circ \zeta$ is constant, because the torus $\overline{T}_K$ is a closed subset of $\overline{G}_K$ and hence the induced map $\chi \circ \zeta|_{A(\rho')}$ maps to a torus of units, where $A(\rho')$ is an annulus of height less than one. Thus, χ is trivial and hence u_K maps $Z_+(\tilde{z}_0)$ into $\overline{G}_+(e)$. Then it is obvious that the second case cannot occur.

By similar arguments one excludes the other cases. □

5.5 Uniformization of Jacobians

In the following let X_K be geometrically connected smooth projective curve of genus $g \geq 1$. Due to Theorem 4.5.3 there exists a finite separable field extension K'/K such that, after replacing K by K', there exists a semi-stable reduction $X_K \to \widetilde{X}$ such that $\widetilde{X}$ satisfies all the conditions mentioned in the beginning of Sect. 5.3. In this section we assume that such a semi-stable reduction of X_K exists already over the given field; i.e., all statements are true only after a suitable finite separable field extension. To avoid frequently extending the base field, we assume that K is algebraically closed. In the following let

$$J_K := \mathrm{Jac}(X_K)$$

be the Jacobian of X_K. We have shown in Proposition 5.3.3 that there exists an open analytic subgroup $\overline{J}_K$ of J_K which has a smooth formal structure with semi-abelian reduction $\widetilde{J}$.

Lemma 5.5.1. *If $n_1, \ldots, n_r$ be a basis of $H^1(\widetilde{X}, \mathbb{Z})$, then we have the map*

$$\Phi : \mathbb{G}^r_{m,K} \longrightarrow J_K,\ t \longmapsto \zeta_1(t)^{n_1} \otimes \cdots \otimes \zeta_r(t)^{n_r},$$

where $(\zeta_\rho(t)^{n_\rho})$ is the invertible sheaf given by the cocycle $n_\rho \in H^1(\widetilde{X}, \mathbb{Z})$ and the coordinate functions $\zeta_1, \ldots, \zeta_r$ on $\mathbb{G}^r_{m,K}$. The morphism Φ restricts to a morphism

$\overline{\Phi} := \Phi|_{\overline{\mathbb{G}}^r_{m,K}}$, *where*

$$\overline{\mathbb{G}}_{m,K} := \{t \in \mathbb{G}_{m,K};\, |t| = 1\}$$

is the torus of units of $\mathbb{G}_{m,K}$. *The map* $\overline{\Phi}$ *induces an isomorphism to the maximal torus of* $\overline{J}_K$. *The reduction of* $\overline{\Phi}$ *is a closed immersion onto the maximal subtorus* $\widetilde{T}$ *of* $\widetilde{J}$.

Proof. If $n = (n_{\mu,\nu}) \in H^1(\widetilde{X}, \mathbb{Z})$ is a cocycle with respect to an open covering $\widetilde{\mathfrak{U}} = \{\widetilde{U}_1, \ldots, \widetilde{U}_N\}$, then it gives rise to a cocycle in $H^1(X_K, \mathbb{Z})$ with respect to a lifting $\{U_1, \ldots, U_N\}$ of the covering $\widetilde{\mathfrak{U}}$. Thus, for every $t \in \mathbb{G}_{m,K}$ and a coordinate function ζ of $\mathbb{G}_{m,K}$ one obtains a cocycle $(\zeta(t)^{n_{\mu,\nu}})$ in $Z^1(X_K, \mathcal{O}^\times_{X_K})$, and hence, due to the universal property of $\mathrm{Jac}(X_K)$, a morphism

$$\varphi : \mathbb{G}_{m,K} \longrightarrow \mathrm{Jac}(X_K),\ t \longmapsto \zeta(t)^n.$$

Therefore, a basis $n_1, \ldots, n_r$ of $H^1(\widetilde{X}, \mathbb{Z})$ gives rise to a morphism

$$\Phi : \mathbb{G}^r_{m,K} \longrightarrow J_K,\ t \longmapsto \zeta_1(t)^{n_1} \otimes \cdots \otimes \zeta_r(t)^{n_r},$$

which restricts to a morphism

$$\overline{\Phi} : \overline{\mathbb{G}}^r_{m,K} \longrightarrow \overline{J}_K,\ t \longmapsto \zeta_1(t)^{n_1} \otimes \cdots \otimes \zeta_r(t)^{n_r},$$

by Proposition 5.4.6. The latter is a lifting of the torus of $\widetilde{J}$; cf. Remark 5.2.7. □

The next objective is the quotient $\overline{J}/\overline{T}$. There is a general result.

Lemma 5.5.2. *For* $n \in \mathbb{N}$ *put* $R_n := R/R\pi^{n+1}$ *for some* $\pi \in \mathfrak{m}_R$ *with* $\pi \neq 0$. *Let* $\overline{G}$ *be a smooth formal* R*-group scheme with semi-abelian reduction* $\widetilde{G}$ *and let* $\overline{T} \subset \overline{G}$ *be a formal torus which reduces to the maximal torus* $\widetilde{T}$ *of* $\widetilde{G}$. *Put* $\overline{G}_n := \overline{G} \otimes_R R_n$ *and* $\overline{T}_n = \overline{T} \otimes_R R_n$. *Then the quotient* $\overline{G}_n/\overline{T}_n$ *is representable by an abelian* R_n*-scheme for all* $n \in \mathbb{N}$ *and the inductive limit* $B := \varinjlim B_n$ *is a smooth formal abelian* R*-scheme.*

In particular, every B_n *has a dual* B'_n *and* $B' := \varinjlim B'_n$ *is the dual of* B; *i.e.,* B' *represents the functor of translation invariant formal line bundles on* B.

Proof. The existence of the quotient is shown in [41, Exp. $\mathrm{VI_A}$, 3.2]; see also [42, Exp. IX, 7.1]. But it is buried in a very general context. In our case it is easier to show. Indeed, we know that the reduction $\widetilde{G}$ is an extension of an abelian variety $\widetilde{B}$ by the torus $\widetilde{T}$; cf. Sect. A.2. Moreover, there exists a quasi-section $\sigma : \widetilde{B} \dashrightarrow \widetilde{J}$, which is defined on a dense open affine subscheme U_k of $\widetilde{B}$; cf. Proposition A.2.5. Since the schemes are of finite presentation there exists a element $\pi \in \mathfrak{m}_R$ with $\pi \neq 0$ such that all the objects are defined over $R/R\pi$. Now denote the new objects by a subindex "0".

Let $V_0 := \sigma_0(U_0)$ be the image of σ_0; this is a closed smooth subscheme of $q^{-1}(U_0)$, where $q_0 : \overline{G}_0 \to B_0$ is the projection. Now V_0 lifts to a smooth closed formal subscheme $V_n \subset q^{-1}(U_n)$ for every $n \in \mathbb{N}$ in a coherent way. Moreover, the group law of $\overline{G}$ gives rise to an isomorphism $\overline{T}_n \times V_n \tilde{\longrightarrow} q^{-1}(U_n)$. Thus, V_n is a chart for the quotient and can be equipped with a birational group law. Such a birational group law can be realized as smooth group scheme B_n over R_n by gluing translates, because R_n is strictly Henselian; cf. [41, Exp. XVIII] and [15, Chap. 5]. The associated group B_n is the quotient $\overline{G}_n/\overline{T}_n$ for all $n \in \mathbb{N}$. Since $\widetilde{B}$ is proper, so B_n is a proper R_n-scheme.

For the existence of the dual see Theorem 6.1.1. □

Proposition 5.5.3. *In the situation of Proposition 5.3.3 the group $\overline{J}$ is an extension of a formal abelian R-scheme B by a formal torus $\overline{T}$*

$$1 \to \overline{T} \to \overline{J} \to B \to 1 \quad \widehat{=} \quad \overline{\phi} : H_1(\widetilde{X}, \mathbb{Z}) \to B', \tag{$\overline{\dagger}$}$$

where B' is the dual of B. The extension ($\overline{\dagger}$) is a lifting of the torus extension

$$1 \to \widetilde{T} \to \widetilde{J} \to \widetilde{B} := \prod_{j=1}^{s} \widetilde{B}_i \to 1 \quad \widehat{=} \quad \widetilde{\phi} : H_1(\widetilde{X}, \mathbb{Z}) \to \widetilde{B}', \tag{$\widetilde{\dagger}$}$$

where $\widetilde{B}_j = \mathrm{Jac}(\widetilde{X}'_j)$ is the Jacobian of the normalization $\widetilde{X}'_j$ of the irreducible component $\widetilde{X}_j$ for $i = 1, \ldots, s$ and $\widetilde{B}'$ is dual of $\widetilde{B}$.

Proof. The assertion follows from Proposition 5.3.3, Lemmas 5.5.1 and 5.5.2. The characterization via $\overline{\phi}$ and $\widetilde{\phi}$ follow from Theorem A.2.8. The explicit definition of the homomorphism $\widetilde{\phi}$ is given in Proposition 5.2.3. □

Furthermore, $\overline{J}$ can be represented as the total space

$$\overline{J} = \overline{E}_1 \times_B \ldots \times_B \overline{E}_r$$

of translation invariant formal $\overline{\mathbb{G}}_{m,K}$-torsors $\overline{E}_1, \ldots, \overline{E}_r$ over B; cf. Theorem A.2.8. We write here $\overline{E}_i$ in order to indicate that these are formal $\overline{\mathbb{G}}_{m,R}$-torsors on B. They are given by cocycles $(\varepsilon_{\mu,\nu}) \in Z^1(B, \mathcal{O}_B^\times)$ with respect to the formal topology on B. Their absolute value functions satisfy $|\varepsilon_{\mu,\nu}(b)| = 1$ for every $b \in B_K$. We can also restrict them to the generic fiber B_K. Thus, we obtain a rigid analytic line bundle E_i and hence a $\mathbb{G}_{m,K}$-torsor over B_K. Thus we have an affine torus extension

$$\widehat{J}_K := E_1 \times_B \ldots \times_B E_r,$$

which contains $\overline{J}_K$ as an open rigid subgroup. On $\widehat{J}_K$ there is a value map

$$v : \widehat{J}_K \longrightarrow \mathbb{R}^r, \ (z_1, \ldots, z_r) \longmapsto \big(|z_1|, \ldots, |z_r|\big), \tag{$*$}$$

since the transition functions have constant absolute value functions equal to 1. One can also interpret $\widehat{J}_K$ as the push-forward of $\overline{J}_K$ with respect to the inclusion map

$\overline{T}_K \hookrightarrow T_K$ of the torus of units into the affine torus like $\overline{\mathbb{G}}_{m,K} \hookrightarrow \mathbb{G}_{m,K}$. Thus we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{T}_K & \longrightarrow & \overline{J}_K & \xrightarrow{\overline{q}} & B_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T_K & \longrightarrow & \widehat{J}_K & \xrightarrow{q} & B_K & \longrightarrow & 1, \end{array}$$

where the down arrow in the middle is induced by the down arrow on the left; cf. Sect. A.2. Thus we obtain the following result.

Corollary 5.5.4. *In the above situation we have the commutative diagram*

$$\begin{array}{ccccc} \operatorname{Hom}(T_K, \mathbb{G}_{m,K}) & \xrightarrow{\sim} & \operatorname{Hom}(\overline{T}, \overline{\mathbb{G}}_{m,R}) & \xrightarrow{\sim} & \operatorname{Hom}(\widetilde{T}, \mathbb{G}_{m,k}) \\ \| & & \| & & \| \\ \operatorname{Hom}_{B_K}(\widehat{J}_K, P_{B_K \times B'_K}) & \xrightarrow{\sim} & \operatorname{Hom}_B(\overline{J}, P_{B \times B'}) & \xrightarrow{\sim} & \operatorname{Hom}_{\widetilde{B}}(\widetilde{J}, P_{\widetilde{B} \times \widetilde{B}'}) \end{array}$$

of canonical isomorphisms, where $P_{B \times B'}$ denotes the Poincaré $\mathbb{G}_m$-torsor over $B \times B'$, etc. In each row, the first horizontal map is the restriction and the second is the reduction. The down arrows are defined via the push-outs; cf. Notation 6.1.7 and Sect. A.2.

Proposition 5.4.8 implies the following result.

Proposition 5.5.5. *Let $u_K : Z_K \to J_K$ be a rigid analytic morphism from a connected rigid space Z_K with a K-rational point z_0 satisfying $u_K(z_0) = 1$, where 1 is the unit element of J_K.*

If Z_K admits a smooth formal R-model, then u_K maps Z_K to $\overline{J}_K$ and $u_K : Z_K \to \overline{J}_K$ extends to a formal morphism $u : Z \to \overline{J}$.

As an application the proposition has the following corollary.

Corollary 5.5.6. *The canonical map $H^1(\widetilde{X}, \mathbb{Z}) \longrightarrow H^1(X_K, \mathbb{Z})$ is bijective. In particular, all morphisms in the canonical commutative diagram*

$$\begin{array}{ccccc} H^1(\widetilde{X}, \mathbb{Z}) & \longrightarrow & & \longrightarrow & \operatorname{Hom}(\mathbb{G}_{m,k}, \widetilde{J}) \\ \downarrow & & & & \uparrow \\ H^1(X_K, \mathbb{Z}) & \longrightarrow & \operatorname{Hom}(\mathbb{G}_{m,K}, J_K) & \longrightarrow & \operatorname{Hom}(\overline{\mathbb{G}}_{m,K}, \overline{J}_K) \end{array}$$

are bijective.

Proof. Due to Proposition 5.2.8 the upper horizontal map is bijective. The second map in the lower row is the restriction to the formal torus $\overline{\mathbb{G}}_{m,K}$, which is well defined by Proposition 5.5.5. This map is obviously injective. The vertical map on the right-hand side is injective. Indeed, a group homomorphism $\overline{\mathbb{G}}_{m,K} \to \overline{J}_K$ with constant reduction is constant, because $J_+(\tilde{1})$ is of unipotent type; for example $J_+(\tilde{1})$ does not contain any non-trivial ℓ-torsion point for ℓ prime to $\operatorname{char} k$. Thus, it remains to show that

$$H^1(X_K, \mathbb{Z}) \longrightarrow \operatorname{Hom}(\overline{\mathbb{G}}_{m,K}, J_K)$$

is injective. So consider a cocycle of $n = (n_{\mu,\nu}) \in H^1(X_K, \mathbb{Z})$ with respect to an admissible covering $\{U_\mu\}$ which is mapped to the constant homomorphism $1 \in \operatorname{Hom}(\overline{\mathbb{G}}_{m,K}, J_K)$. Thus, by the universal property of J_K we see that the invertible sheaf associated to the cocycle

$$\left(\zeta^{n_{\mu,\nu}}\right) \cong p_2^*\mathcal{L} \in H^1\left(X_K \times_K \overline{\mathbb{G}}_{m,K}, \mathcal{O}^\times_{X_K \times_K \overline{\mathbb{G}}_{m,K}}\right)$$

is a pull-back of an invertible sheaf $\mathcal{L}$ on $\overline{\mathbb{G}}_{m,K}$, where p_2 is the second projection $X_K \times_K \overline{\mathbb{G}}_{m,K} \to \overline{\mathbb{G}}_{m,K}$, and where ζ be a coordinate function on $\overline{\mathbb{G}}_{m,K}$. Since the coordinate ring of $\overline{\mathbb{G}}_{m,K}$ is factorial, $\mathcal{L}$ is trivial, and hence the cocycle $(\zeta^{n_{\mu,\nu}})$ is solvable. Thus, there exist units ε_μ in $\mathcal{O}^\times_{X_K \times_K \overline{\mathbb{G}}_{m,K}}(U_\mu \times_K \overline{\mathbb{G}}_{m,K})$ such that

$$\zeta^{n_{\mu,\nu}} = \varepsilon_\mu \cdot \varepsilon_\nu^{-1}$$

for all μ, ν. Due to Proposition 1.3.4 the units are of type

$$\varepsilon_\mu = c_\mu \cdot \zeta^{n_\mu} \cdot (1 + h_\mu),$$

where c_μ is a constant and h_μ is a holomorphic function on $U_\mu \times_K \overline{\mathbb{G}}_{m,K}$ with sup-norm $|h_\mu| < 1$. Thus, we have that $n_{\mu,\nu} = n_\mu - n_\nu$, and hence that the cocycle $n = (n_{\mu,\nu})$ is trivial. □

Corollary 5.5.6 implies that the morphism $\bar{\iota} : \overline{T}_K \to \overline{J}_K \to J_K$ extends to a morphism $T_K \to J_K$. Thus, we obtain the important result:

Corollary 5.5.7. *In the above situation the inclusion map $\overline{J}_K \hookrightarrow J_K$ extends to a homomorphism $\widehat{J}_K \to J_K$ of rigid analytic groups.*

Moreover, Proposition 5.4.8 yields the mapping property:

Proposition 5.5.8. *Let Z_K be a connected affine rig-smooth formal curve which has a semi-stable model Z with precisely one singular point $\tilde{z}_0$. Let $\zeta : Z_K \to J_K$ be a holomorphic function which restricts to a coordinate function on the formal fiber of $Z_+(\tilde{z}_0)$ of the double point $\tilde{z}_0$. Let $z_0 \in Z_+(\tilde{z}_0)$ be a K-rational point with $\zeta(z_0) = 1$.*

If $u_K : Z_K \to J_K$ is a rigid analytic morphism, then there exists a unique group homomorphism $\varphi : \mathbb{G}_{m,K} \to J_K$ such that u_K factorizes into

$$u_K = (\varphi \circ \zeta) \cdot \overline{u} \cdot u_K(z_0),$$

where $\overline{u} : Z_K \to \overline{J}_K$ is a rigid analytic morphism.

As was explained above there is a canonical value map on $\widehat{J}_K$. Thus, for every $c \in |K^\times|$ with $c \geq 1$, we can define the open subvariety

$$\widehat{J}_K(c) := \big\{(z_1, \ldots, z_r) \in E_1 \times_B \ldots \times_B E_r; c^{-1} \leq |z_i| \leq c \text{ for } i = 1, \ldots, r\big\}.$$

Corollary 5.5.9. *In the above situation we have the following*:

(a) *There exists an element $c \in |K^\times|$, $c > 1$, such that the inclusion $\overline{J}_K \hookrightarrow J_K$ extends to a surjective rigid analytic group homomorphism*

$$p : \widehat{J}_K(c) \longrightarrow J_K.$$

(b) *There exists an element $\varepsilon \in \sqrt{|K^\times|}$ with $c > \varepsilon > 1$ such that*

$$\widehat{J}_K(\varepsilon) \cap p^{-1}(\overline{J}_K) = \overline{J}_K.$$

Proof. (a) Let $p_0 \in X_K$ be a K-rational point. Then consider the mapping

$$j : X_K^{(g)} \longrightarrow J_K, \ (q_1 + \cdots + q_g) \longmapsto [q_1 - p_0] \cdot \ldots \cdot [q_g - p_0].$$

Every point $\ell \in J_K$ is of type

$$\ell = j(p_1 + \cdots + p_g) = [p_1 + \cdots + p_g - g \cdot p_0],$$

where $p_1, \ldots, p_g \in X_K$ are closed points and g is the genus of X_K. Now we move the points $q_1, \ldots, q_g$ one by one from p_0 to p_i. So we will study the morphism

$$u : X_K \longrightarrow J_K, \ z \longmapsto [z - p_0].$$

It suffices to see that there exists a constant c such that $[q - p_0] \in \widehat{J}(c)$ for all $q \in X_K$. For this take one of the shortest paths in the graph $\Gamma(\widetilde{X})$ which leads from the reduction $\tilde{p}_0$ to $\tilde{q}$. This path passes through finitely many double points of $\widetilde{X}$, where each double point is involved only one time. It follows from Proposition 5.5.8 that passing with q through the annulus associated to a double point requires a certain amount of the torus T_K to keep $[q - p_0]$ inside $\widehat{J}(c)$. This amount of T_K depends only on the height of the annulus. Furthermore, it follows from Proposition 5.5.5 that moving with q inside a connected smooth formal part of X_K one stays inside a translate of $\overline{J}$. Since there are only finitely many double points in $\widetilde{X}$, the assertion follows.

(b) Let c be the element from (a) and consider the morphism

$$p_c := p|_{\dots} : p^{-1}(\overline{J}_K) \cap \widehat{J}_K(c) \longrightarrow \overline{J}_K.$$

Then $\overline{J}_K$ is an open subset of $p^{-1}(\overline{J}_K) \cap \widehat{J}_K(c)$ and the restriction of p_c to $\overline{J}_K \to \overline{J}_K$ is an isomorphism and, in particular, finite. Therefore, the inclusion $\overline{J}_K \to p^{-1}(\overline{J}_K) \cap \widehat{J}_K(c)$ is finite. Thus, $\overline{J}_K$ is a closed analytic subset of $p^{-1}(\overline{J}_K) \cap \widehat{J}_K(c)$. Thus, it is a connected component of $p^{-1}(\overline{J}_K) \cap \widehat{J}_K(c)$. Due to the maximum principle there exists an $\varepsilon > 1$ such that $\widehat{J}_K(\varepsilon) \cap p^{-1}(\overline{J}_K) = \overline{J}_K$. □

Definition 5.5.10. The $\mathbb{G}^r_{m,K}$-extension

$$\widehat{J}_K := E_i \times_B \times \dots \times_B E_r$$

is called the *universal covering* of J_K. It is the push-forward $T_K \times_{\overline{T}} \overline{J}_K$ of $\overline{J}_K$ by the inclusion of the formal torus $\overline{T}_K \hookrightarrow T_K$.

The notion "*universal covering*" will become clear in Corollary 6.3.4, since $H^1(\widehat{J}_K, \mathbb{Z})$ vanishes. Let us summarize the results of this section.

Theorem 5.5.11. *In the situation of above we have the following results*:

(a) *The group homomorphism $p : \widehat{J}_K \to J_K$ is surjective.*
(b) *The kernel $M :=$ ker p is discrete in $\widehat{J}_K$ with $M \cap \widehat{J}_K(\varepsilon) = \{1\}$ for $\varepsilon > 1$ and close to one. M is free of rank $r =$ rk $H_1(\widetilde{X}, \mathbb{Z}) =$ rk $H^1(X_K, \mathbb{Z})$.*
(c) *The rigid analytic quotient $\widehat{J}_K/M$ is isomorphic to the Jacobian of X_K. The morphism $\widehat{J}_K \to J_K$ is a covering map in the topological sense.*

Proof. (a) The map is surjective by Corollary 5.5.9(a).

(b) The kernel $M :=$ ker p satisfies $M \cap \widehat{J}_K(\varepsilon) = \{1\}$ for $\varepsilon > 1$ close to 1 by Corollary 5.5.9(b). Therefore, $\widehat{J}_K(\varepsilon) \to J_K$ is an open immersion.

The absolute value on $\widehat{J}_K$ gives rise to a group homomorphism

$$\sigma : \widehat{J}_K \xrightarrow{\quad v \quad} \widehat{J}_K/\overline{J}_K = |K^\times|^r \xrightarrow{\quad -\log \quad} \mathbb{R}^r,$$

which maps M bijectively to a lattice of $\mathbb{R}^r$. Thus, we see that M is free of rank $\leq r$. In order to show that $\sigma(M)$ has rank r, consider the induced homomorphism

$$\widehat{J}_K/\overline{J}_K \xrightarrow{\qquad} |K^\times|^r/v(M) \xrightarrow{\quad -\log \quad} \mathbb{R}^r/\sigma(M).$$

Then we see by Corollary 5.5.9(b) that a bounded part of $|K^\times|^r$ has dense image in $\mathbb{R}^r/\sigma(M)$. Hence, a bounded part of $\mathbb{R}^r$ has dense image in $\mathbb{R}^r/\sigma(M)$, and it follows that $M \cong \sigma(M)$ has rank r.

(c) The rigid analytic quotient $\widehat{J}_K/M$ exists. Indeed, we have the charts of type $x \cdot \widehat{J}_K(\varepsilon)$, where $x \in \widehat{J}_K$ is a K-rational point, which itself can be covered by affinoid spaces. Obviously, there are finitely many points $x_1, \dots, x_N$ such that the charts $x_i \cdot \widehat{J}_K(\varepsilon)$ for $i = 1, \dots, N$ cover the set $\widehat{J}_K(c)$, and hence their images cover the quotient set $\widehat{J}_K/M$. Then the inverse image $p^{-1}(x_i \cdot \widehat{J}_K(\varepsilon))$ is a disjoint union $\bigcup_{m \in M} m \cdot x_i \cdot \widehat{J}_K(\varepsilon)$. Thus $p : \widehat{J}_K \to J_K$ is a covering in the topological sense. □

Corollary 5.5.12. *Let X_K be a smooth projective curve assumed to be geometrically connected. Then the following conditions are equivalent*:

(a) *X_K is a Mumford curve.*
(b) Jac(X_K) *is a rigid analytic torus.*

Proof. (a) → (b) This was shown in Theorem 2.8.7.

(b) → (a) Due to Theorem 5.5.11 we know the structure of J_K. There is an open analytic subgroup $\overline{J}_K$ of J_K which is an extension of a formal abelian variety B by a formal torus and B is a lifting of the Jacobian of the normalization of the stable reduction $\widetilde{X}$ of X_K. In order to show that X_K is a Mumford curve it suffices to show that B is trivial; cf. Theorem 4.7.2. Since J_K is a rigid analytic torus, the formal abelian part is trivial. □

5.6 Applications to Abelian Varieties

The uniformization of Jacobians can be generalized to abelian varieties, since every abelian variety is a quotient of a product of Jacobians by a connected subgroup; cf. [88, VII, §13]. We want to mention that one can achieve the results of this section by the methods of Chap. 7 without using Jacobians; however one needs the stable reduction theorem for smooth projective curves.

Proposition 5.6.1. *If A is an abelian variety over a perfect field K, then there exists a product J of Jacobians and an abelian subvariety N of J such that J is isogenous to $A \times N$.*

Proof. As explained in [88, VII, §13], there exists a product J of Jacobians and a quotient morphism $\beta : J \to A$ with connected kernel N'. Since K is perfect, the reduced subvariety $N := N'_{\text{red}} \subset J$ is an abelian variety. Due to Poincaré's complete irreducibility theorem [74, §19] there exists an abelian subvariety $Z \subset J$ such that J is isogenous to the product $Z \times N$. Then $\beta|_Z : Z \to A$ is an isogeny, and hence there exists an isogeny $\alpha : A \to Z$ satisfying $\beta \circ \alpha = n \cdot \text{id}_A$ for some integer $n \geq 1$; cf. [74, §18]. □

Theorem 5.6.2 (Formal semi-abelian reduction theorem). *Let A_K be an abelian variety over an algebraically closed non-Archimedean field. Then there exists a connected open rigid analytic subgroup $\overline{A}_K \subset A_K$ with the following properties*:

(a) *$\overline{A}_K$ extends to a formal smooth R-group scheme $\overline{A}$ with semi-abelian reduction.*
(b) *$\overline{A}$ is an extension of a formal abelian R-group scheme B by a split formal torus $\overline{T}$.*
(c) *The inclusion map $\overline{T}_K \hookrightarrow A_K$ extends to a homomorphism $T_K \to A_K$ from the affine torus which contains the formal torus $\overline{T}_K$ as torus of units.*

The group $\overline{A}_K$ is uniquely determined by condition (a)*. Also the torus $\overline{T}_K$ is unique.*

Proof. (a) We know the assertion for Jacobians; cf. Proposition 5.5.3, and hence for a product J_K of Jacobians as well. Let Z_K be an abelian subvariety of a product J_K of Jacobians and let $\alpha : A_K \to Z_K$ be the isogeny as in Proposition 5.6.1. Then $\alpha^{-1}(\overline{J}_K)$ is an open analytic subgroup which inherits the formal structure from $\overline{J}$, because α is finite. Now let $\overline{A}_K = \alpha^{-1}(\overline{J}_K)^0$ be its 1-component. Then $\overline{A}_K$ is a connected formal analytic smooth group variety and satisfies the assertion (a). Indeed, the reduction $\widetilde{A}$ is reduced, and hence smooth as a group variety. The reduction map $\widetilde{\alpha} : \widetilde{A} \to \widetilde{J}$ is finite, and so $\widetilde{A}$ is semi-abelian as well.

(b) Let $\widetilde{T} \subset \widetilde{A}$ be the maximal torus which is a smooth subgroup of $\widetilde{A}$. Let $\widetilde{B}$ be the quotient $\widetilde{A}/\widetilde{T}$. Due to the lifting of tori [41, II, Theorem 3.6] or Proposition 5.6.7 below, the group $\widetilde{T}$ lifts to a formal torus $\overline{T}$ of $\overline{A}$ and its quotient $B = \overline{A}/\overline{T}$ is a formal abelian R-scheme due to Lemma 5.5.2. Therefore, the sequence

$$0 \to \overline{T} \to \overline{A} \to B \to 0$$

is strict exact, and hence $\overline{A}$ is a formal torus extension of B cf. Sect. A.2.

The uniqueness follows from Proposition 5.4.6.

(c) follows from Lemma 5.6.4 below.

The assertion on the uniqueness follows from Proposition 5.4.6. □

Theorem 5.6.2 implies the Semi-Abelian Reduction Theorem of Grothendieck; see [42].

Theorem 5.6.3 (Algebraic semi-abelian reduction theorem). *Let R be a discrete valuation ring of height* 1 *and let A_K be an abelian variety over its field of fractions $K = Q(R)$.*

Then there exists a finite Galois extension K'/K such that the Néron model $A_{R'}$ of $A_{K'} := A_K \otimes_K K'$ over the ring of integers R' of K' has semi-abelian reduction.

Proof. Let $\widehat{R}$ be the completion of R and $\widehat{K}$ its field of fractions. One shows as in the proof of Lemma 3.1.13 that there exists a finite separable extension $\widehat{K}'/\widehat{K}$ such that $A_{\widehat{K}'} := A_K \otimes_K \widehat{K}'$ has an open subgroup $\overline{A}_{\widehat{K}'}$ as asserted in Theorem 5.6.2. Due to the lemma of Krasner [10, 3.2.4/5] there exists a finite separable field extension K'/K such that the completion of K' is isomorphic to $\widehat{K}'$. Now consider the Néron model $A_{R'}$ of $A \otimes_K K'$ over R'. Since the extension $R' \to \widehat{R}'$ is of ramification index 1, the canonical morphism

$$A_{R'} \otimes_{R'} \widehat{R}' \longrightarrow A_{\widehat{R}'}$$

to the Néron model $A_{\widehat{R}'}$ of $A_{\widehat{K}'}$ is an isomorphism due to [15, 7.2/1].

The completion of $A_{\widehat{R}'}$ with respect to the 1-component of the special fiber is canonically isomorphic to $\overline{A}_{\widehat{K}'}$, since its étale $\widehat{R}'$-points are dense in $\overline{A}_{\widehat{K}'}$. Thus we see that $A_{R'}$ has semi-abelian reduction. □

We continue towards the uniformization of abelian varieties.

Lemma 5.6.4. *In the situation of Theorem 5.6.2 let A_K be an abelian variety and let A'_K be its dual. Then there are canonical morphisms*

$$H^1\big(A'_K, \mathbb{Z}\big) \to \mathrm{Hom}(\mathbb{G}_{m,K}, A_K) \to \mathrm{Hom}(\overline{\mathbb{G}}_{m,K}, A_K) = \mathrm{Hom}(\overline{\mathbb{G}}_{m,K}, \overline{A}_K)$$

and these morphisms are bijective.

Proof. The first morphism maps a cocycle $n = (n_{i,j}) \in H^1(\mathfrak{U}, \mathcal{O}^{\times}_{A'_K})$ to the cocycle $\zeta^n \in H^1(\mathfrak{U}, \mathcal{O}^{\times}_{A'_K \times \mathbb{G}_{m,K}})$, where $\mathfrak{U}$ is an admissible covering of A'_K and ζ is a coordinate function on $\mathbb{G}_{m,K}$. Thus, we can regard ζ^n as an invertible sheaf on $A'_K \times \mathbb{G}_{m,K}$. Since A_K is the dual of A'_K, the cocycle gives rise to a morphism $\varphi : \mathbb{G}_{m,K} \to A_K$, which is obviously a group homomorphism. The second map is the restriction of morphisms to the open subgroup $\overline{\mathbb{G}}_{m,K}$, so the second map mapping is injective.

So it remains to see that every homomorphisms $\varphi : \overline{\mathbb{G}}_{m,K} \to A_K$ is induced by a unique cocycle $n \in H^1(A'_K, \mathbb{Z})$. As argued before, φ is equivalent to an invertible sheaf $\mathcal{L}$ on $A'_K \times \overline{\mathbb{G}}_{m,K}$ which is rigidified along the unit section of A'_K. Now we need an argument that is shown in Corollary 6.2.6 later. In fact, $\mathcal{L}$ trivializes locally on A'_K; i.e., there exists an admissible covering $\mathfrak{U} = (U_1, \ldots, U_r)$ of A'_K such that $\mathcal{L}|_{U_i \times \overline{\mathbb{G}}_{m,K}}$ is trivial. So we see that $\mathcal{L}$ can be represented by a cocycle $(\lambda_{i,j})$ in $H^1(\mathfrak{U} \times \overline{\mathbb{G}}_{m,K}, \mathcal{O}^{\times}_{A'_K \times \overline{\mathbb{G}}_{m,K}})$. Thus, $\lambda_{i,j}$ is of the form

$$\lambda_{i,j} = \zeta^{n_{i,j}} \cdot (1 + h_{i,j}),$$

where $n := (n_{i,j}) \in H^1(A'_K, \mathbb{Z})$ and $h_{i,j}$ are functions on $(U_i \cap U_j) \times \overline{\mathbb{G}}_{m,K}$ with sup-norm $|h_{i,j}| < 1$; cf. Proposition 1.3.4. One easily shows that the cocycle $(1 + h_{i,j})$ is a group homomorphism if and only if $h_{i,j} = 0$ for all i, j. Thus, the morphism $H^1(A'_K, \mathbb{Z}) \to \mathrm{Hom}(\overline{\mathbb{G}}_{m,K}, A_K)$ is surjective. The injectivity follows by similar reasoning on such units.

The last identity follows from Proposition 5.4.6. □

Theorem 5.6.5 (Uniformization theorem). *Let A_K be an abelian variety over an algebraically closed non-Archimedean field K. Let $\overline{A}$ and $\overline{T}$ be as in Theorem 5.6.2* *and T_K the affine torus containing $\overline{T}_K$ as a torus of units. Then we have the following results*:

(a) *The push-out $\widehat{A}_K$ of $\overline{A}_K$ by $\overline{T}_K \hookrightarrow T_K$ is an extension of the generic fiber B_K of a formal abelian R-scheme B by the torus T_K.*
(b) *The open immersion $\overline{A}_K \hookrightarrow A_K$ extends to a map $p : \widehat{A}_K \to A_K$, which is a covering map in the topological sense.*
(c) *The kernel $M := \ker p$ is a discrete subgroup on $\widehat{A}_K$, which intersects $\overline{A}_K$ in the unit element. The abelian group M is free of rank equal to the dimension of the torus part $\overline{T}$ of $\overline{A}$. In particular, there is a value map v, which on the level of K-valued points*

$$\widehat{A}_K/\overline{A}_K \xrightarrow{\quad v \quad} |K^{\times}|^r/v(M) \xrightarrow{\quad -\log \quad} \mathbb{R}^r/\log(v(M))$$

induces a bijection, and $-\log$ is a dense inclusion into a compact real torus.
(d) *As a rigid analytic variety A_K is isomorphic to the quotient $\widehat{A}_K/M$.*

Proof. (a) This follows from Theorem 5.6.2(b) due to its construction.

(b) It follows from Theorem 5.6.2(c) that the inclusion $\overline{A}_K \hookrightarrow A_K$ extends to a homomorphism $p : \widehat{A}_K \to A_K$. To show the surjectivity of p, we go back to the construction of $\overline{A}_K$ in Theorem 5.6.2(a). We considered a product J_K of Jacobians and a surjective morphism $\beta : J_K \to A_K$; cf. Proposition 5.6.1. Then we applied the uniformization of J_K. Due to Proposition 5.4.6 the morphism β maps $\overline{J}_K$ to $\overline{A}_K$. Obviously the torus part of $\widehat{J}_K$ is mapped to the torus T_K of $\widehat{A}_K$. Since β and $\widehat{J}_K \to J_K$ are surjective, the map $p : \widehat{A}_K \to A_K$ is surjective. As in the case of Jacobians there is a value map $v : \widehat{A}_K \to \mathbb{R}^r$ as $\widehat{A}_K$ is a product of $\mathbb{G}_{m,K}$-torsors given by formal line bundles. Thus, one can conclude as in Corollary 5.5.9 and Theorem 5.5.11 that p is a covering map in the topological sense.

(c) and (d) Follow from (b) as in Theorem 5.5.11. □

Definition 5.6.6. The representation $A_K = \widehat{A}_K/M$ is called the *Raynaud representation* of the abelian variety A_K.

We will show in Theorem 6.4.8 that the B_K of Theorem 5.6.2(a) is in fact an abelian variety with good reduction and $\widehat{A}_K$ is an algebraic torus extension of B_K. For the convenience of the reader we add the proof of the lifting of tori as we used it here. A main point in the proof is the vanishing of a certain Hochschild homology group [23, II, Sect. 3, Prop. 4.2], but in our case it can easily be verified by checking coefficients. The lifting of tori is also treated in [95].

Proposition 5.6.7. *Assume that K is algebraically closed. Let G be a smooth formal R-group scheme and assume that its reduction $\widetilde{G}$ is a torus extension of a commutative smooth group variety $\widetilde{B}$ by a split torus $\widetilde{T}$. Then there is a unique lifting $\varphi : T \to G$ of the closed immersion $\widetilde{\varphi} : \widetilde{T} \to \widetilde{G}$, where T is a split formal torus with reduction $\widetilde{T}$.*

In particular, φ is a closed immersion, and G is a formal torus extension of a commutative smooth formal group scheme B which is a lifting of $\widetilde{B}$.

Proof. We start with the torus extension

$$0 \to \widetilde{T} \xrightarrow{\widetilde{\varphi}} \widetilde{G} \xrightarrow{\widetilde{q}} \widetilde{B} \to 0.$$

Denote by

$$m : G \times G \longrightarrow G$$

the group law of G and by $\widetilde{m}$ the induced group law on the reduction. Since $\widetilde{G}$ is a torus extension of $\widetilde{B}$, there exists an open affine neighborhood $\widetilde{U}$ of the unit section of $\widetilde{B}$ and a section $\sigma : \widetilde{U} \to \widetilde{G}$ of $\tilde{q}$ over $\widetilde{U}$; cf. Proposition A.2.5. Thus, the map

$$\widetilde{W} := \widetilde{T} \times \widetilde{U} \xrightarrow{\sim} q^{-1}(\widetilde{U}), \quad (t, u) \longmapsto \widetilde{m}\big(t, \sigma(u)\big),$$

is an isomorphism. In particular, $q^{-1}(\widetilde{U})$ is affine.

Let $W \subset G$ be the formal open subscheme of G lifting $\widetilde{W}$. The unit section of $\widetilde{B}$ is a smooth subvariety, so there exists a system of parameters $\widetilde{g}_1, \ldots, \widetilde{g}_s$ on $\widetilde{U}$ whose locus is the unit section; eventually after a shrinking of $\widetilde{U}$. Let $g_1, \ldots, g_s$ be functions on W whose reductions coincide with the pull-backs of $\widetilde{g}_1, \ldots, \widetilde{g}_s$. Then the locus $V(g_1, \ldots, g_s) \subset W$ is smooth with reduction $\widetilde{T}$. Let T be a formal torus with reduction $\widetilde{T}$. Since W is smooth, there exists a lifting

$$\varphi : T \longrightarrow W$$

of $\widetilde{\varphi}$ with $\varphi(1)$ equal to the unit section of A. Note that φ is in general not a group homomorphism from T to G. We will stepwise transform φ into a homomorphism.

The image $\sigma(\widetilde{U}) \subset \widetilde{W}$ is a smooth subscheme of $\widetilde{W}$. It also lifts to a smooth closed subscheme $U \subset W$ of W. We may assume that the unit section 0 of G is a point of U. Then the formal fiber $U_+(0)$ is isomorphic to an r-dimensional open ball due to Corollary 4.1.10. The morphism

$$T \times U \longrightarrow W, \quad (t, u) \longmapsto m\big(\varphi(t), u\big),$$

is étale and finite. Since it is an isomorphism in the reduction, it is an isomorphism. It is clear that the group law m restricts to a group law of

$$W_+ := \big\{a \in A_K ; \big|g_1(a)\big| < 1, \ldots, \big|g_s(a)\big| < 1\big\}.$$

In particular, we have an isomorphism

$$T_K \times U_+(0) \xrightarrow{\sim} W_+.$$

Let $\xi_1, \ldots, \xi_r$ be coordinates on T which induce coordinates on the torus $\widetilde{T}$. Since $U_+(0)$ is isomorphic to an open s-dimensional unit ball, the parameters

$g_1, \ldots, g_s$ give rise to coordinates $(\eta_1, \ldots, \eta_s)$ on $U_+(0)$. Moreover, we may assume that the unit element of G coincides with the origin of W_+. The restriction of the formal group law induces a group law on W_+. Therefore $m|_{W_+ \times W_+}$ can be written as an $(r+s)$-tuple of Laurent series in the coordinates of $T_K \times T_K$ with coefficients in $\mathcal{O}_{U_K}(U_+(0))$:

$$m|_{W_+ \times W_+}\left(\begin{pmatrix} \zeta_1 \\ \eta_1 \end{pmatrix}, \begin{pmatrix} \zeta_2 \\ \eta_2 \end{pmatrix}\right) = \begin{pmatrix} \zeta_1 \cdot \zeta_2 \\ \eta_1 + \eta_2 \end{pmatrix} + \sum_{\mu,\nu \in \mathbb{Z}^r} \begin{pmatrix} u_{\mu,\nu}^{(1)} \zeta_1^\mu \zeta_2^\nu \\ u_{\mu,\nu}^{(2)} \zeta_1^\mu \zeta_2^\nu \end{pmatrix}$$

with coefficients $u_{\mu,\nu}^{(i)} \in \mathring{\mathcal{O}}_{U_K}(U_+(0))$; these depend on η_1, η_2. The product and the sum are defined on the components. Since the reduction of m is the group law on $\widetilde{T}$, the absolute value of coefficients of $u_{\mu,\nu}^{(1)}$ are less than 1 and of $u_{\mu,\nu}^{(2)}$ are less or equal to 1. Thus, we see that there exists an element c of $K^\times$ with $|c| < 1$ such that $m|_{W_+ \times W_+}$ restricts to a group law on

$$W_c := \left\{x \in W_K; \left|g_1(x)\right| \leq |c|, \ldots, \left|g_s(x)\right| \leq |c|\right\}.$$

This is a smooth formal group with reduction $\mathbb{G}_{m,k}^r \times \mathbb{G}_{a,k}^s$. For proving our theorem, we can replace G by this group. Now we start some calculations where we normalize the coordinates η by η/c.

We have to study the pull-back of m by $\varphi \times \varphi$ as a function of the coordinates $\zeta_i := (\zeta_{i,1}, \ldots, \zeta_{i,r})$ of the two factors of $T_K \times T_K$

$$m\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) = \begin{pmatrix} \zeta_1 \cdot \zeta_2 \\ 0 \end{pmatrix} + \sum_{\mu,\nu \in \mathbb{Z}^r} \begin{pmatrix} r_{\mu,\nu}^{(1)} \\ r_{\mu,\nu}^{(2)} \end{pmatrix} \zeta_1^\mu \zeta_2^\nu,$$

where $r_{\mu,\nu}^{(1)} \in R^r$ and $r_{\mu,\nu}^{(2)} \in R^s$, whose components have absolute value less than 1, because the reduction of φ is an isomorphism to $\widetilde{T}$. By our choice of the coordinate functions we have $r_{0,0}^1 = r_{0,0}^2 = 0$. Since the Laurent series converges, there exists an element $\pi \in R$ with $|\pi| < 1$ such that, with respect to the maximum norm on components, it holds

$$r_{\mu,\nu} = 0 \bmod \pi^n$$

for all $\mu, \nu \in \mathbb{Z}^r$ and $n = 1$. We write here n, because we will iterate a process, which will be explained below.

An important point of the proof is the associativity of the group law. We obtain modulo π^{n+1}:

$$m\big(\zeta_1, m(\zeta_2, \zeta_3)\big) = \begin{pmatrix} \zeta_1(\zeta_2\zeta_3 + u^{(1)}(\zeta_2, \zeta_3)) \\ u^{(2)}(\eta_2, \eta_3)(\zeta_2, \zeta_3) \end{pmatrix} + \begin{pmatrix} u^{(1)}(\zeta_1, \zeta_2\zeta_3) \\ u^{(2)}(\eta_1, \eta_2 + \eta_3)(\zeta_1, \zeta_2\zeta_3) \end{pmatrix}$$

$$m\big(m(\zeta_1, \zeta_2), \zeta_3\big) = \begin{pmatrix} (\zeta_1\zeta_2 + u^{(1)}(\zeta_1, \zeta_2))\zeta_3 \\ u^{(2)}(\eta_1, \eta_2)(\zeta_1, \zeta_2) \end{pmatrix} + \begin{pmatrix} u^{(1)}(\zeta_1\zeta_2, \zeta_3) \\ u^{(2)}(\eta_1 + \eta_2, \eta_3)(\zeta_1\zeta_2, \zeta_3) \end{pmatrix}$$

where we omitted the pull-back morphism given by φ and where $u^{(1)}$ are the first r and $u^{(2)}$ are the last s components of u.

Thus, for the first r components we obtain the relation

$$\zeta_1 u^{(1)}(\zeta_2,\zeta_3) + u^{(1)}(\zeta_1,\zeta_2\zeta_3) = u^{(1)}(\zeta_1,\zeta_2)\zeta_3 + u^{(1)}(\zeta_1\zeta_2,\zeta_3) \bmod \pi^{n+1}.$$

Comparing the coefficients of ζ_1 yields

$$u^{(1)}_{\mu,\nu}\zeta_2^{\mu}\zeta_3^{\nu} + u^{(1)}_{1,\nu}\zeta_2^{\nu}\zeta_3^{\nu} = u^{(1)}_{1,\nu}\zeta_2^{\nu}\zeta_3 + u^{(1)}_{1,\nu}\zeta_2\zeta_3^{\nu} \bmod \pi^{n+1}.$$

We see that $u^{(1)}_{\mu,\nu} \in \pi^{n+1}$ if $(\mu,\nu) \neq (1,1)$, and hence

$$u^{(1)}(\zeta_1,\zeta_2) = u^{(1)}_{1,1}\zeta_1\zeta_2 = r^{(1)}_{1,1}\zeta_1\zeta_2 \bmod \pi^{n+1},$$

where $r^{(1)}_{1,1} = u^{(1)}_{1,1}(1) \in R^r$ is the evaluation at the unit element of T.

For the last s components we have to specify the dependency on the coordinates η_1,η_2,η_3. Therefore we obtain modulo π^{n+1}:

$$\begin{aligned}&\sum_{\mu,\nu} u^{(2)}_{\mu,\nu}(\eta_2,\eta_3)\zeta_2^{\mu}\zeta_3^{\nu} + u^{(2)}_{\mu,\nu}(\eta_1,\eta_2+\eta_3)\zeta_1^{\mu}\zeta_2^{\nu}\zeta_3^{\nu}\\ &\quad = \sum_{\mu,\nu} u^{(2)}_{\mu,\nu}(\eta_1,\eta_2)\zeta_1^{\mu}\zeta_2^{\nu} + u^{(2)}_{\mu,\nu}(\eta_1+\eta_2,\eta_3)\zeta_1^{\mu}\zeta_2^{\mu}\zeta_3^{\nu}.\end{aligned}$$

This yields the following relations modulo π^{n+1}:

$$u^{(2)}_{\mu,\nu} = 0 \quad \text{if } \mu \neq 0,\ \nu \neq 0 \text{ and } \mu \neq \nu.$$

Then, by looking at the coefficients of ζ_3^0 one obtains for $\mu \neq 0$

$$\begin{aligned} u^{(2)}_{\mu,0}(\eta_2,\eta_3) &= u^{(2)}_{0,\mu}(\eta_1,\eta_2)\\ u^{(2)}_{\mu,0}(\eta_1,\eta_2+\eta_3) &= u^{(2)}_{\mu,0}(\eta_1,\eta_2)\\ u^{(2)}_{\mu,0}(\eta_1+\eta_2,\eta_3) &= -u^{(2)}_{\mu,\mu}(\eta_1,\eta_2)\end{aligned}$$

and for $\mu = \nu = 0$

$$u^{(2)}_{0,0}(\eta_2,\eta_3) + u^{(2)}_{0,0}(\eta_1,\eta_2+\eta_3) = u^{(2)}_{0,0}(\eta_1,\eta_2) + u^{(2)}_{0,0}(\eta_1+\eta_2,\eta_3)) \bmod \pi^{n+1}.$$

For our purpose we are interested in the value of $u^{(2)}$ modulo π^{n+1}. Since the coefficients of $u^{(2)}$ vanish modulo π and our values η fulfill $\eta \equiv 0 \bmod \pi^n$, we get from the above relations modulo π^{n+1}:

$$u^{(2)}(\zeta_1,\zeta_2) = \sum_{\mu\in\mathbb{Z}^r} r^{(2)}_{\mu,0}\zeta_1^{\mu} + \sum_{\mu\in\mathbb{Z}^r} r^{(2)}_{\mu,\mu}\zeta_1^{\mu}\zeta_2^{\mu} + \sum_{\mu\in\mathbb{Z}^r} r^{(2)}_{0,\mu}\zeta_2^{\mu},$$

where $r^{(2)}_{\mu,\nu} \in R$ with $r^{(2)}_{0,0} = 0$, $r^{(2)}_{\mu,0} = -r^{(2)}_{\mu,\mu} = r^{(2)}_{0,\mu}$. Putting

$$v(\zeta) = \sum_{\mu \in \mathbb{Z}^r} r^{(2)}_{\mu,0} \zeta^{\mu},$$

implies

$$v(\zeta_1) + v(\zeta_2) - v(\zeta_1\zeta_2) = u^{(2)}(\zeta_1, \zeta_2) \bmod \pi^{n+1}.$$

Then we replace $\varphi : T \to W$ by

$$\psi : T \longrightarrow W_+, \ \zeta \longmapsto \begin{pmatrix} \zeta - r^{(1)}_{1,1}\zeta \\ -v(\zeta) \end{pmatrix}.$$

Thus, computing modulo π^{n+1}, we arrive at

$$\begin{aligned} m\big(\psi(\zeta_1), \psi(\zeta_2)\big) &= \begin{pmatrix} (\zeta_1 - u^{(1)}_{1,1}\zeta_1)(\zeta_2 - u^{(1)}_{1,1}\zeta_2) \\ -v(\zeta_1) - v(\zeta_2) \end{pmatrix} + \begin{pmatrix} u^{(1)}_{1,1}\zeta_1\zeta_2 \\ u^{(2)}(\zeta_1, \zeta_2) \end{pmatrix} \\ &= \begin{pmatrix} \zeta_1\zeta_2 - u^{(1)}_{1,1}\zeta_1\zeta_2 \\ -v(\zeta_1\zeta_2) \end{pmatrix} \\ &= \psi(\zeta_1\zeta_2) \bmod \pi^{n+1}. \end{aligned}$$

Starting the process at $n = 1$, iteration yields the lifting of the torus.

The uniqueness follows easily. Indeed, let $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_r$ be coordinates of the torus $\widetilde{T}$ and let $\xi_{1,1}, \ldots, \xi_{1,r}$ resp. $\xi_{2,1}, \ldots, \xi_{2,r}$ be liftings of the coordinates. Then the correction term $u^{(1)}_{1,1}$ is uniquely determined modulo π^{n+1}. Therefore, $\xi_{1,j} - \xi_{2,j}$ vanishes modulo π^{n+1}. Then the uniqueness follows by induction. □

Chapter 6
Raynaud Extensions

In the last chapter we presented the uniformization $J_K = \widehat{J}_K/M$ of the Jacobian variety J_K of a connected smooth projective curve. The universal covering $\widehat{J}_K$ is a *Raynaud extension*; i.e. an affine torus extension of the generic fiber of a formal abelian R-scheme. The new topic in this chapter is the algebraization result for $\widehat{J}_K$; i.e., that $\widehat{J}_K$ is an algebraic torus extension of an abelian variety with good reduction.

We study this in the more general setting of uniformized abeloid varieties; i.e., of rigid analytic groups in Raynaud representation E_K/M, where E_K is a Raynaud extension and where $M \subset E_K$ is a lattice of rank equal to the torus part of E_K. This requires a systematic study of Raynaud extensions and their line bundles with M-action. Thus, one is led to the construction of the dual of a uniformized abeloid variety. The algebraization of a uniformized abeloid variety is related to the existence of a polarization.

Of special interest are the polarizations of Jacobians $\mathrm{Jac}(X)$. There are two, the usual theta polarization and the canonical polarization which is related to a pairing on the homology group $H_1(X, \mathbb{Z})$ of the curve X. In Sect. 6.5 we discuss these polarizations. This is related to Riemann's vanishing theorem Corollary 2.9.16 for Mumford curves.

In Sect. 6.6, following the article [13] we discuss the results of this chapter on the degeneration data of abelian varieties and compare them with the ones established in [27]. Prerequisites on torus extensions and cubical structures are surveyed in the Appendix.

6.1 Basic Facts

Let us fix the notation for the following sections. For simplicity we assume in this chapter that K is an algebraically closed non-Archimedean field. Let

$$B := \varinjlim B_n$$

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be a formal abelian scheme over Spf R; i.e., a direct limit of abelian schemes B_n over $R_n := R/R\pi^{n+1}$, where $0 \neq \pi \in \mathfrak{m}_R$. The functor $\mathrm{Pic}^{\tau}_{B_n/R_n}$ of translation invariant line bundles is representable by an abelian R_n-scheme B'_n due to [2, 7.3]. Recall the statement for abelian varieties of Theorem 5.1.4.

Theorem 6.1.1. *In the above situation the formal abelian group scheme* $B' := \varinjlim B'_n$, *where* B'_n *represents the functor* $\mathrm{Pic}^{\tau}_{B_n/R_n}$, *represents the functor of translation invariant formal line bundles on* B.

The generic fiber $B'_K := B' \otimes_R K$ *of* B' *represents the functor* $\mathrm{Pic}^{\tau}_{B_K/K}$ *of translation invariant line bundles on the generic fiber* B_K *of* B.

Proof. The first statement follows from [2, 7.3]. The second assertion follows from Proposition 6.2.5; see below. Indeed, if S_K is an affinoid space, then every rigid analytic line bundle on $B_K \times_R S_K$ extends to a formal line bundle on $B \times_R S$, where S is a suitable formal R-model of S_K. □

The universal line bundle $P_{B\times B'}$ on $B \times B$ is called the *Poincaré bundle*. It is rigidified both-sided along $0 \times B'$ and $B \times 0'$, where 0 and $0'$ are the unit section of B and B', respectively. In the sequel we will denote the associated $\mathbb{G}_m$-torsor by $P_{B\times B'}$ as well; see [42, Exp. VII] for details. If we consider the invertible sheaf associated to $P_{B\times B'}$, we write $\mathcal{P}_{B\times B'}$. The formal scheme B' is called the *dual of* B.

The biduality theorem states that the canonical map

$$B \longrightarrow B'', \quad b \longmapsto [P_{b\times B'}];$$

is an isomorphism; this follows from [74, p. 133], since B and B' are smooth formal R-schemes. Thus, $P_{B\times B'}$ is also the Poincaré bundle of B'.

On the $\mathbb{G}_m$-torsor $P_{B\times B'}$ there are two compatible partial group laws which are induced by the tensor product of line bundles on $B \times B'$ depending on which one of the two factors B and B' is viewed as the base scheme.

A *formal torus extension* of B by a split formal torus $\overline{T} = \overline{\mathbb{G}}^r_{m,R}$ is a strict exact sequence

$$1 \to \overline{T} \to \overline{E} \to B \to 1. \qquad (\overline{\dagger})$$

$\overline{\mathbb{G}}_{m,R} = \mathrm{Spf}\, R\langle \zeta, \zeta^{-1}\rangle$ is the formal 1-dimensional torus. Its generic fiber

$$\overline{\mathbb{G}}_{m,K} := \left\{ z \in \mathbb{G}_{m,K};\, |z| = 1 \right\}$$

is called *torus of units*. Associated to the extension ($\overline{\dagger}$) we denote by

$$M' := \mathrm{Hom}(\overline{T}, \overline{\mathbb{G}}_{m,R})$$

the character group of $\overline{T} := \overline{\mathbb{G}}^r_{m,R}$. Due to Theorem A.2.8 the extension ($\overline{\dagger}$) is equivalent to a group homomorphism $\phi' : M' \to B'$. This is indicated by

$$1 \to \overline{T} \to \overline{E} \to B \to 1 \quad \widehat{=} \quad \phi' : M' \to B'.$$

Via the push-forward $\overline{T}_K := \overline{\mathbb{G}}^r_{m,K} \hookrightarrow T_K := \mathbb{G}^r_{m,K}$ one obtains the affine torus extension of B_K. Since $\operatorname{Hom}(\overline{T}, \overline{\mathbb{G}}_{m,R}) = \operatorname{Hom}(T_K, \mathbb{G}_{m,K})$, we have

$$1 \to T_K \to E_K \to B_K \to 1 \quad \widehat{=} \quad \phi' : M' \to B'_K. \tag{†}$$

Definition 6.1.2. In the above situation, the rigid analytic group variety E_K is called *Raynaud extension*. Note that M' is a free abelian group.

Every character $m' \in M'$ gives rise to a commutative diagram

$$\begin{array}{ccccc} T_K & \longrightarrow & E_K & \longrightarrow & B_K \\ \downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle \langle \_, m' \rangle} & & \| \\ \mathbb{G}_{m,K} & \longrightarrow & P_{B_K \times \phi'(m')} & \longrightarrow & B_K, \end{array}$$

where the down arrow in the middle is push out by m'; cf. Sect. A.2. In other terms, if one has introduced coordinates, say

$$E_K = P_{B_K \times \phi'(e'_1)} \times_B \cdots \times_B P_{B_K \times \phi'(e'_r)}$$

and $m' = m'_1 e'_1 + \cdots + m'_r e'_r$, the map $\langle \_, m' \rangle$ is given by the tensor product

$$(t_1, \ldots, t_r) \longmapsto t_1^{\otimes m'_1} \otimes \cdots \otimes t_r^{m'_r}.$$

Remark 6.1.3. Let S_K be a rigid analytic space. Then the set of S_K-valued points $\sigma : S_K \to E_K$ of E_K bijectively corresponds to the set of families of S_K-valued points $(\sigma_{m'} : S_K \to P_{B_K \times \phi'(m')}; m' \in M')$ satisfying the relations $\sigma_{m'_1 + m'_2} = \sigma_{m'_1} \otimes \sigma_{m'_2}$ for all $m'_1, m'_2 \in M'$.

If L is a rigidified formal line bundle on B, then we have a well-defined absolute value

$$|\_| : L_K \longrightarrow \mathbb{R}_{\geq 0}$$

on the generic fiber L_K of L, because the cocycle of its transition functions $\lambda_{i,j}$ in $Z^1(\mathfrak{U}, \mathcal{O}_B^\times)$ satisfy $|\lambda_{i,j}(x)| = 1$ for all rigid points $x \in U_i \cap U_j$ and all i, j, where $\mathfrak{U} = (U_i; i \in I)$ is an open covering of B.

The absolute value on formal line bundles extends uniquely to a value map

$$|\_| : E_K \longrightarrow \mathbb{R}^r, \ (z_1, \ldots, z_r) \longmapsto \big(|z_1|, \ldots, |z_r|\big).$$

A subgroup $M \subset E_K$ is called a *lattice* if the map

$$-\log|\_| : E_K \longrightarrow \mathbb{R}^r, \ (z_1, \ldots, z_r) \longmapsto \big(-\log|z_1|, \ldots, -\log|z_r|\big),$$

sends M to a lattice of $\mathbb{R}^r$ bijectively.

Proposition 6.1.4. *In the above situation we have the following results*:

(a) *The rigid analytic quotient* $p : E_K \to A_K := E_K/M$ *exists and is a smooth rigid analytic group variety. The quotient map is a covering in the topological sense.*

(b) M *has full rank if and only if* E_K/M *is a proper rigid analytic space.*

Proof. It follows as in the case of tori in Proposition 2.7.3. Note that a proper formal scheme has a proper rigid analytic generic fiber; cf. Theorem 3.3.12. □

We associate to a *quotient of a Raynaud extension* E_K by a lattice M the following diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow h & \searrow \phi & \\ T_K & \longrightarrow & E_K & \xrightarrow{q} & B_K \\ & & \downarrow p & & \\ & & E_K/M & & \end{array} \qquad \widehat{=} \qquad \phi' : M' \longrightarrow B'_K,$$

where $h : M \to E_K$ is the inclusion of a lattice, $\phi := q \circ h : M \to B_K$ the induced map and $p : E_K \to E_K/M$ the rigid analytic quotient map.

Definition 6.1.5. A connected proper smooth rigid analytic group variety A_K is called an *abeloid variety*; cf. Sect. 7.1.

If A_K admits a representation $A_K = E_K/M$ as above, where M is a lattice in E_K of full rank, then the quotient E_K/M is called *Raynaud representation* of A_K.

It will be shown in Theorem 7.6.4 that every abeloid variety over a non-Archimedean field admits a Raynaud representation after a suitable finite field extension.

The invertible sheaf associated to a line bundle L on a space Z is denoted by the associated calligraphic letter

$$\mathcal{L} := \mathcal{H}om_Z\big(L, \mathbb{A}^1_Z\big).$$

Note that this association is contravariant; cf. Remark 1.7.2. Using this notation, one can write

$$E_K = \mathcal{S}pec\bigg(\bigoplus_{m' \in M'} \mathcal{P}_{B_K \times \phi'(m')}\bigg),$$

where the multiplication of homogeneous elements in the graded $\mathcal{O}_{B_K}$-module is induced by the relations explained above.

Convention 6.1.6. In order to keep the symbols simple, we omit the subindex "K" at the symbols T and E if no confusion is possible. Later in Sect. 6.2 when dealing with the relation between formal and rigid geometry we will return to the precise notation again. Furthermore, the groups M and M' are always considered as discrete rigid analytic spaces; no subindex "K" will be introduced.

Notation 6.1.7. If $\sigma : S \to E$ is an S-valued point of E and $m' \in M'$ is a character, then we define

$$\langle \sigma, m' \rangle := \sigma_{m'} : S \longrightarrow P_{B \times \phi'(m')}$$

as an S-valued point of the $\mathbb{G}_{m,K}$-torsor $P_{B\times\phi'(m')}$. The mapping

$$\langle \sigma, \_ \rangle : M' \longrightarrow P_{B\times B'},\ m' \longmapsto \langle \sigma, m' \rangle,$$

is a group homomorphism lifting the map $\phi' : M' \to B'$.

In particular, the inclusion $\iota : T \hookrightarrow E$ gives rise to the character $\iota_{m'} = m'$ and the identity map $e := \mathrm{id}_E$ induces the canonical morphism

$$e_{m'} : E \longrightarrow P_{B\times\phi'(m')},\ z \longmapsto \langle z, m' \rangle.$$

Proposition 6.1.8. *Let B be a formal abelian scheme and B' its dual. Regard B as the dual of B'. Let M and M' be free abelian groups of rank r. Consider two Raynaud extensions*

$$1 \to T \to E \xrightarrow{q} B \to 1 \quad \widehat{=} \quad \phi' : M' \to B'$$

$$1 \to T' \to E' \xrightarrow{q'} B' \to 1 \quad \widehat{=} \quad \phi : M \to B.$$

Then the following data correspond bijectively to each other:

(a) *group homomorphisms $h : M \to E$ satisfying $\phi = q \circ h$,*
(b) *bilinear forms from $M \times M'$ to the biextension $P_{B\times B'}$ over $\phi \times \phi'$*

$$\begin{array}{ccc} & & P_{B\times B'} \\ & \overset{\langle \_,\_ \rangle}{\nearrow} & \downarrow \\ M \times M' & \xrightarrow{\ \phi\times\phi'\ } & B \times B'. \end{array}$$

(c) *group homomorphisms $h' : M' \to E'$ satisfying $\phi' = h' \circ q'$.*

Under the above correspondence the following properties are equivalent:

(i) *$h : M \to E$ maps into a lattice.*
(ii) *The absolute value $|\langle \_, \_ \rangle|$ of the bilinear form is non-degenerate*; *i.e.*, $-\log|\langle \_, \_ \rangle| : M \times M' \longrightarrow \mathbb{R}$ *is non-degenerate.*
(iii) *$h' : M' \to E'$ maps into a lattice.*

Proof. The map h associates to an $m \in M$ a point $h(m) \in E$ which can be considered as a group homomorphism

$$h(m) : M' \longrightarrow P_{B\times B'},\ m' \longmapsto h(m)_{m'} \in P_{\phi(m)\times\phi'(m')},$$

lifting $\phi' : M' \to B'$. Then

$$\langle\_,\_\rangle : M \times M' \longrightarrow P_{M\times B'},\ \big(m, m'\big) \longmapsto \big\langle m, m'\big\rangle := h(m)_{m'},$$

is bilinear, because h and each $m' \in M'$ are group homomorphisms. Moreover, h gives rise to a group homomorphism

$$h' : M' \longrightarrow P_{B\times B'},\ m' \longmapsto \begin{bmatrix} h'(m') : M \longrightarrow P_{B\times B'} \\ \qquad m \longmapsto \langle m, m'\rangle \end{bmatrix},$$

and h' fulfills the rule $h'(m_1'+m_2') = h'(m_1') \otimes h'(m_2')$. This, in turn, defines a group homomorphism $h' : M' \to E'$ by Remark 6.1.3.

Again by Remark 6.1.3 a bilinear form $\langle\_,\_\rangle$ gives rise to a group homomorphism

$$h : M \longrightarrow E,\ m \longmapsto \begin{bmatrix} M \longrightarrow E \\ m \longmapsto (\langle m, m'\rangle; m' \in M') \end{bmatrix}.$$

Similarly, h' induces the bilinear form and conversely, a bilinear form induces a group homomorphism $h' : M' \to E'$.

The additional assertions (i)–(iii) follow by the definition of a lattice. □

Proposition 6.1.9. *Consider two Raynaud extensions*

$$1 \to T_1 \to E_1 \to B_1 \to 1 \quad \widehat{=} \quad \phi_1' : M_1' \to B_1'$$
$$1 \to T_2 \to E_2 \to B_2 \to 1 \quad \widehat{=} \quad \phi_2' : M_2' \to B_2'.$$

(a) *Then there is a canonical bijection between the following sets*:

(i) *The set of homomorphisms $\Lambda : E_1 \to E_2$ of rigid analytic groups.*

(ii) *The set of pairs of morphisms (λ', φ), where $\lambda' : M_2' \to M_1'$ is a homomorphism of the character groups and where $\varphi : B_1 \to B_2$ is a morphism of formal abelian schemes such that the diagram*

$$\begin{array}{ccc} M_1' & \xrightarrow{\ \phi_1'\ } & B_1' \\ \uparrow{\scriptstyle \lambda'} & & \uparrow{\scriptstyle \varphi'} \\ M_2' & \xrightarrow{\ \phi_2'\ } & B_2' \end{array}$$

is commutative, where $\varphi' : B_2' \to B_1'$ is the dual map associated to φ.

In particular, the image of a point x_1 *of* E_1 *is given by the family*

$$\big\langle \Lambda(x_1), m_2' \big\rangle = pr\big(\big\langle x_1, \lambda'\big(m_2'\big)\big\rangle\big) \quad \text{for } m_2' \in M_2',$$

where $pr : P_{B_1 \times \phi_1'(\lambda'(m_2'))} = \varphi^* P_{B_2 \times \phi_2'(m_2')} \to P_{B_2 \times \phi_2'(m_2')}$ *is the projection.*

(b) *Let* $\Lambda : E_1 \to E_2$ *be a morphism and* (λ', φ) *the corresponding couple in the sense of* (a). *Let* $h_i : M_i \to E_i$ *be a group homomorphism and set* $\phi_i := q_i \circ h_i$ *for* $i = 1, 2$ *and let* $\lambda : M_1 \to M_2$ *be a group homomorphism. Then the condition* $\Lambda \circ h_1 = h_2 \circ \lambda$ *is equivalent to the compatibility* $\langle h_2 \circ \lambda(m_1), m_2' \rangle = pr(\langle h_1(m_1), \lambda'(m_2') \rangle)$ *for all* $m_1 \in M_1$ *and* $m_2' \in M_2'$.

Proof. (i) → (ii): Consider a morphism $\Lambda : E_1 \to E_2$. By Proposition 5.4.6 the map Λ restricts to a morphism $\overline{\Lambda} : \overline{E}_1 \to \overline{E}_2$ of the formal extensions. Thus, it restricts to a morphism $\overline{\Lambda}_T : \overline{T}_1 \to \overline{T}_2$ of the tori, and hence it gives rise to a group homomorphism $\lambda' : M_2' \to M_1'$ of their character groups; cf. Proposition 2.7.1(b). In particular, we have $\lambda'(m_2') = m_2' \circ \Lambda|_T$.

Taking quotients by $\overline{T}_1$ and $\overline{T}_2$, we see that Λ induces a morphism $\varphi : B_1 \to B_2$. By taking push-outs with respect to $\lambda'(m_2') : T_1 \to \mathbb{G}_{m,K}$ and $m_2' : T_2 \to \mathbb{G}_{m,K}$, the morphism Λ produces the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_{m,K} & \longrightarrow & P_{B_1 \times \phi_1'(\lambda'(m_2'))} & \longrightarrow & B_1 & \longrightarrow & 1 \\ & & \Vert & & \downarrow{\scriptstyle \Lambda_{\lambda'(m_2')}} & & \downarrow{\scriptstyle \varphi} & & \\ 1 & \longrightarrow & \mathbb{G}_{m,K} & \longrightarrow & P_{B_2 \times \phi_2'(m_2')} & \longrightarrow & B_2 & \longrightarrow & 1. \end{array}$$

The latter is only possible if the upper extension is the pull-back of the lower extension. Due to the universal property of the Poincaré bundle there is a canonical identification $\varphi^* P_{B_2 \times b_2'} = P_{B_1 \times \varphi'(b_2')}$ for points $b_2' \in B_2'$ and hence

$$P_{B_1 \times \phi_1'(\lambda'(m_2'))} = \varphi^* P_{B_2 \times \phi_2'(m_2')} = P_{B_1 \times \varphi'(\phi_2'(m_2'))}.$$

Thus, we see $\phi_1' \circ \lambda' = \varphi' \circ \phi_2'$ for $m_2' \in M_2'$.

(ii) → (i): An S-valued point $\sigma : S \to E_1$ of E_1 is a family

$$\big(\sigma_{m_1'} : S \to P_{B_1 \times \phi_1'(m_1')}; m_1' \in M_1'\big)$$

of morphisms satisfying the relation $\sigma_{m_1' + n_1'} = \sigma_{m_1'} \otimes \sigma_{n_1'}$ for $m_1', n_1' \in M_1'$. Then, one can define $\Lambda(\sigma)$ as the family $(\Lambda(\sigma)_{m_2'}; m_2' \in M_2')$, where

$$\Lambda(\sigma)_{m_2'} : S \xrightarrow{\sigma_{\lambda'(m_2')}} P_{B_1 \times \phi_1'(\lambda'(m_2'))} = \varphi^* P_{B_2 \times \phi_2'(m_2')} \xrightarrow{pr} P_{B_2 \times \phi_2'(m_2')}$$

and pr is the projection. Indeed, due to (ii) there are canonical identifications $P_{B_1 \times \phi_1'(\lambda'(m_2'))} = P_{B_1 \times \varphi'(\phi_2'(m_2'))} = \varphi^* P_{B_2 \times \phi_2'(m_2')}$. Applying this reasoning to the universal point $\sigma := \mathrm{id}_{E_1}$ yields the desired morphism $\Lambda : E_1 \to E_2$. It is clear that this correspondence is bijective.

In particular, if $x_1 \in E_1$ and $m_2' \in M_2'$, then we have seen that

$$\langle \Lambda(x_1), m_2' \rangle = pr\big(\langle x_1, \lambda'(m_2') \rangle\big),$$

where $pr : P_{B_1 \times \phi_1'(\lambda'(m_2'))} = \varphi^* P_{B_2 \times \phi_2'(m_2')} \to P_{B_2 \times \phi_2'(m_2')}$ is the projection.

(b) If $\Lambda \circ h_1 = h_2 \circ \lambda$, then it follows from (a)

$$\langle h_2 \circ \lambda(m_1), m_2' \rangle = \langle \Lambda \circ h_1(m_1), m_2' \rangle = pr\big(\langle h_1(m_1), \lambda'(m_2') \rangle\big).$$

Conversely, if the formula holds, then again by (a) we have that

$$\langle h_2 \circ \lambda(m_1), m_2' \rangle = pr\big(\langle h_1(m_1), \lambda'(m_2') \rangle\big) = \langle \Lambda \circ h_1(m_1), m_2' \rangle$$

for all $m_2' \in M_2$, and hence $\Lambda \circ h_1 = h_2 \circ \lambda$. □

Remark 6.1.10. The assertion of Proposition 6.1.9 is of particular importance in the case, when $B_1 = B$ and $B_2 = B'$, and hence $B_2' = B''$ and $B_1' = B'$. There is a canonical isomorphism

$$\iota : B \xrightarrow{\sim} B'', \ b \longmapsto [P_{b \times B'}],$$

where the brackets indicate the class of the line bundle; cf. [74, p. 132]. Moreover, a morphism $\varphi : B \longrightarrow B'$ induces the morphism

$$\varphi' : B'' \longrightarrow B', \ b'' = [P_{b \times B'}] \longmapsto \big[(\mathrm{id}_b \times \varphi)^* P_{b \times B'}\big] = \big[\varphi^* P_{b \times B'}\big]. \qquad (\dagger)$$

Due to the universal property of B' there is a canonical identification

$$(\varphi \times \mathrm{id}_B)^* P_{B' \times B} = \big(\mathrm{id}_B \times \varphi' \circ \iota\big)^* P_{B \times B'}.$$

Indeed, for a point $b \in B$ we have that

$$\varphi^* P_{B' \times b} = P_{B \times [\varphi^* P_{B' \times b}]} = \big(\mathrm{id}_B \times \varphi' \circ \iota\big)^* P_{B \times B'}|_{B \times b}.$$

In the following we identify $\iota : B \xrightarrow{\sim} B''$ and write φ' instead of $\varphi' \circ \iota$.

Thus, if $\varphi = \varphi'$, then we have the identification

$$(\mathrm{id}_B \times \varphi)^* P_{B \times B'} = (\varphi \times \mathrm{id}_B)^* P_{B' \times B}.$$

Then we obtain an isomorphism of symmetry ξ on $(\mathrm{id}_B \times \varphi)^* P_{B \times B'}$

$$\begin{array}{ccc} (\mathrm{id}_B \times \varphi)^* P_{B \times B'} & \xrightarrow{\ \xi\ } & (\mathrm{id}_B \times \varphi)^* P_{B \times B'} \\ \downarrow & & \downarrow \\ B \times B & \xrightarrow{\ \tau\ } & B \times B, \end{array} \qquad \begin{array}{ccc} P_{b_1 \times \varphi(b_2)} & \xrightarrow{\ \xi\ } & P_{b_2 \times \varphi(b_1)} \\ \downarrow & & \downarrow \\ (b_1, b_2) & \longmapsto & (b_2, b_1), \end{array}$$

where τ flips the factors of $B \times B$.

In the special case, where $\varphi = \varphi_N : B \to B'$, $b \mapsto \tau_b^* N \otimes N^{-1}$ for a line bundle N on B, one has the symmetry $\varphi = \varphi'$. Indeed, consider in (†) the universal point $b = \mathrm{id}_B$. In this case

$$(\mathrm{id}_B \times \varphi)^* P_{B\times B'} = m^* N \otimes p_1^* N^{-1} \otimes p_2^* N^{-1} \otimes 0^* N =: \mathcal{D}_2(N),$$

where $m, p_1, p_2, 0 : B \times B \to B$ are the group law, the projections and the zero map. The symmetry is given by the canonical symmetry on $\mathcal{D}_2(N)$. □

Corollary 6.1.11. *Consider two Raynaud extensions*

$$1 \to T \to E \xrightarrow{q} B \to 1 \quad \widehat{=} \quad \phi' : M' \to B'$$

$$1 \to T' \to E' \xrightarrow{q'} B' \to 1 \quad \widehat{=} \quad \phi : M \to B,$$

where B' is the dual of B. Let $\Lambda : E \to E'$ be a homomorphism, which corresponds to a couple of morphisms $(\lambda : M \to M', \varphi : B \to B')$ in the sense of Proposition 6.1.9; i.e., $\phi' \circ \lambda = \varphi' \circ \phi$. Assume that $\varphi = \varphi'$ is symmetric.

Then, for every S-valued point $\sigma : S \to E$, the image $\Lambda(\sigma) := \Lambda \circ \sigma$ in E' is characterized by the formula

$$\big\langle m, \Lambda(\sigma)\big\rangle = \xi\big(\langle \sigma, \lambda(m)\rangle\big)$$

for all $m \in M$. Here the map $\xi : (\mathrm{id}_B \times \varphi)^* P_{B\times B'} \longrightarrow (\mathrm{id}_B \times \varphi)^* P_{B\times B'}$ is the symmetry associated to the condition $\varphi = \varphi'$; cf. Remark 6.1.10.

Proof. Since the situation may look confusing, adopt the notation from Proposition 6.1.9; i.e., $B_1 = B$, $B_2 = B'$, $M_1' = M'$, $M_2' = M$, $\phi_1' = \phi'$ and $\phi_2' = \phi$. By Proposition 6.1.9 we have the following commutative diagram

$$\begin{array}{ccc} M' & \xrightarrow{\phi'} & B' \\ \uparrow{\scriptstyle \lambda} & & \uparrow{\scriptstyle \varphi'=\varphi} \\ M & \xrightarrow{\phi} & B. \end{array}$$

An S-valued point $\sigma : S \to E$ consists of a family of compatible morphisms $(\sigma_{m'} : S \to P_{B\times\phi'(m')})_{m'\in M'}$. Due to Proposition 6.1.9 the morphism $\Lambda : E \to E'$ sends σ to the point $\Lambda(\sigma) = (\Lambda(\sigma)_m)_{m\in M}$, where

$$\begin{array}{llll} \Lambda(\sigma)_m : S \xrightarrow{\sigma_{\lambda(m)}} & P_{B\times\phi'(\lambda(m))} = (\mathrm{id}_B \times \varphi)^* P_{B\times B'}|_{B\times\phi(m)} & & \\ & \downarrow{\scriptstyle \xi} & & \\ & (\mathrm{id}_B \times \varphi)^* P_{B\times B'}|_{\phi(m)\times B} & \xrightarrow{pr} & P_{\phi(m)\times B'}. \end{array}$$

Note that ξ is an isomorphism of the fibers

$$\xi : P_{q(\sigma)\times\varphi(\phi(m))} \xrightarrow{\sim} P_{\phi(m)\times\varphi(q(\sigma))}.$$

Indeed, put $m_2' = m$ in the proof of Proposition 6.1.9. Thus, we obtain

$$\big\langle m, \Lambda(\sigma)\big\rangle = \Lambda(\sigma)_m = \xi(\sigma_{\lambda(m)}) = \xi\big(\big\langle \sigma, \lambda(m)\big\rangle\big)$$

for all $m \in M$. □

Proposition 6.1.12. *In the situation of Corollary 6.1.11 assume that there are morphisms $h : M \to E$ and $h' : M' \to E'$ such that $\phi = q \circ h$ and $\phi' = q' \circ h'$. Then a morphism $\Lambda = (\lambda, \varphi) : E \to E'$ sends M to M' in the sense that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{h} & E \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\Lambda} \\ M' & \xrightarrow{h'} & E' \end{array}$$

is commutative if and only if we have for all $m_1, m_2 \in M$ that

$$\big\langle m_1, h' \circ \lambda(m_2)\big\rangle = \xi\big(\big\langle h(m_2), \lambda(m_1)\big\rangle\big). \qquad (*)$$

Proof. If we have the formula (∗), then we obtain for $m_2 \in M$ that

$$\big\langle m_1, \Lambda\big(h(m_2)\big)\big\rangle = \xi\big(\big\langle h(m_2), \lambda(m_1)\big\rangle\big) = \big\langle m_1, h' \circ \lambda(m_2)\big\rangle$$

for all $m_1 \in M_1$. Indeed, the first equation follows from Corollary 6.1.11 and the second is the formula (∗). Since M is the character group of E', we obtain $\Lambda(h(m_2)) = h'(\lambda(m_2))$ for all $m_2 \in M$. Thus, we see $\Lambda \circ h = h' \circ \lambda$.

If the diagram is commutative, then we have that $\Lambda(h(m_2)) = h'(\lambda(m_2))$ for all $m_2 \in M$. Then Corollary 6.1.11 yields

$$\big\langle m_1, h'\big(\lambda(m_2)\big)\big\rangle = \big\langle m_1, \Lambda\big(h(m_2)\big)\big\rangle = \xi\big(\big\langle h(m_2), \lambda(m_1)\big\rangle\big)$$

for all $m_1, m_2 \in M$. □

Remark 6.1.13. The symmetry condition (∗) of Proposition 6.1.12 means that $\langle \_, \_ \rangle$

$$\begin{array}{ccccc} & \overset{\langle \_,\_ \rangle}{\nearrow} & (\phi \times (\phi' \circ \lambda))^* P_{B\times B'} & \longrightarrow & P_{B\times B'} \\ & & \downarrow & & \downarrow \\ M \times M & \longrightarrow & M \times M & \xrightarrow{\phi\times(\phi'\circ\lambda)} & B \times B'. \end{array}$$

is a symmetric trivialization of the biextension $(\phi \times (\phi' \circ \lambda))^* P_{B\times B'}$.

6.2 Line Bundles

In this section we will study line bundles on Raynaud extensions. We will consider a situation as introduced in Corollary 6.1.11. Note Convention 6.1.6. Let

$$\begin{aligned} 1 \to T \to E \xrightarrow{q} B \to 1 \quad &\widehat{=} \quad \phi' : M' \to B' \\ 1 \to T' \to E' \xrightarrow{q'} B' \to 1 \quad &\widehat{=} \quad \phi : M \to B, \end{aligned} \tag{$\dagger$}$$

be two Raynaud extensions, where B' is the dual of B. In addition, let

$$h : M \longrightarrow E$$

be a homomorphism of groups such that $\phi = q \circ h$. Equivalent to these data is a trivialization

$$t : M \times M' \longrightarrow (\phi \times \phi')^* P_{B \times B'},\ (m, m') \longmapsto \langle m, m' \rangle := h(m)_{m'},$$

of the biextension. Then we will consider line bundles L on E equipped with an M-linearization; cf. Definition 1.7.11. In the following we write $\tau_m : E \to E$ for the translation $\tau_{h(m)} : E \to E$ by $h(m)$ and $\tau_{q(m)} : B \to B$ for the translation $\tau_{q(h(m))} : B \to B$ by $q \circ h(m)$. There is a fundamental example.

Example 6.2.1. In the above situation let $(N, r(0))$ be a rigidified line bundle on B, where $r(0) \in N_0$ is the rigidificator at the unit element 0 of B. Due to the Theorem of the Square 7.1.6 we have the canonical homomorphism

$$\varphi_N : B \longrightarrow B',\ b \longmapsto \varphi_N(b) := \tau_b^* N \otimes N^{-1},$$

which associates to a point b of B the isomorphism class of the translation invariant line bundle $\tau_b^* N \otimes N^{-1}$, where $\tau_b : B \to B$ is the translation by the point b. Let $L := q^* N$ be the pull-back of N to E which is rigidified by $r(0)$ along the unit section of E. We start with the following data:

$$\begin{aligned} r : M \longrightarrow \phi^* N \quad & \text{trivialization of the torsor } \phi^* N, \\ \lambda : M \longrightarrow M' \quad & \text{map of sets.} \end{aligned}$$

We will show how and under which conditions the data (r, λ) give rise to an M-linearization of $q^* N$. For each $m \in M$ we have fixed the element $r(m) \in N_{\phi(m)}$. Moreover, $\tau_{\phi(m)}^* N \otimes N^{-1}$ is translation invariant. Due to the definition of φ_N there is a canonical isomorphism

$$\tau_{\phi(m)}^* N \otimes N^{-1} \xrightarrow{\sim} N_{\phi(m)} \otimes N_0^{-1} \otimes P_{B \times \varphi_N(\phi(m))},$$

where N_b is the pull-back of N under the constant map $B \to b \in B$ and the identification is determined by the sending the rigidificator $r(m) \otimes r(0)^{-1}$ to

$r(m) \otimes r(0)^{-1} \otimes 1$. Thus, one can identify

$$\mathrm{Hom}\big(N, \tau^*_{\phi(m)} N\big) = \tau^*_{\phi(m)} N \otimes N^{-1} = N_{\phi(m)} \otimes N_0^{-1} \otimes P_{B \times \varphi_N(\phi(m))}. \qquad (*)$$

As a first condition we assume $\varphi_N \circ \phi = \phi' \circ \lambda$.

An M-linearization on q^*N is a family $(c_m; m \in M)$ of isomorphisms

$$c_m : q^*N \longrightarrow \tau^*_m q^* N$$

satisfying compatibility relations, which are discussed below. Such an isomorphism c_m corresponds to a trivialization of $q^* P_{B \times \phi'(\lambda(m))}$. There is a trivialization given by the canonical section of Notation 6.1.7

$$e_{\lambda(m)} : E \longrightarrow P_{B \times \phi'(\lambda(m))} = P_{B \times \varphi_N(\phi(m))}, \ z \longmapsto \big\langle z, \lambda(m)\big\rangle.$$

Thus, by using the identification $(*)$ an M-linearization c can be given by the data (r, λ) in the following way:

$$c_m : q^*N \longrightarrow \tau^*_m q^* N, \ f \longmapsto r(m) \otimes r(0)^{-1} \otimes e_{\lambda(m)} \otimes f.$$

In particular, $r(0)$ is mapped to $r(m)$.

Next we will discuss the compatibility condition. The family $(c_m; m \in M)$ is an M-linearization of q^*N if and only if we have

$$\tau^*_{m_1}(c_{m_2}) \circ c_{m_1} = c_{m_1 + m_2}$$

for all $m_1, m_2 \in M$. In terms of the data (r, λ) the compatibility condition can be rephrased by the following conditions:

1. $\varphi_N \circ \phi = \phi' \circ \lambda$,
2. $\lambda : M \to M'$ is a group homomorphism,
3. $r(m_1 + m_2) \otimes r(m_1)^{-1} \otimes r(m_2)^{-1} \otimes r(0) = \langle m_1, \lambda(m_2)\rangle$

for all $m_1, m_2 \in M$. The factor $\langle m_1, \lambda(m_2)\rangle$ is taken into account, since the canonical section $e_{\lambda(m)}$ is not translation invariant if $\lambda(m)$ is not trivial. Indeed, for $m' \in M'$ and $z \in E$ there is the following relation

$$\tau^*_m e_{m'}(z) = e_{m'}(m \cdot z) = \big\langle m \cdot z, m'\big\rangle = \big\langle m, m'\big\rangle \otimes e_{m'}(z);$$

here "$\cdot$" is the group law on E. Then it follows

$$\begin{aligned} \tau^*_{m_1}(c_{m_2}) \circ c_{m_1} &= r(m_2) \otimes \big\langle m_1, \lambda(m_2)\big\rangle \otimes r(m_1) \otimes r(0)^{\otimes -2} \otimes e_{\lambda(m_2)} \otimes e_{\lambda(m_1)} \\ c_{m_1+m_2} &= r(m_1 + m_2) \otimes r(0)^{-1} \otimes e_{\lambda(m_1+m_2)}. \end{aligned}$$

So $\lambda : M \to M'$ is group homomorphism and r satisfies the condition 3.

Conversely, if all the conditions are fulfilled, then the couple (r, λ) gives rise to an M-linearization on q^*N. Especially, when N is translation invariant and $\lambda = 0$, then $r : M \to N$ is a group homomorphism from M to the torus extension of B given by N. In this case we have that $N = P_{B \times b'}$ for some $b' \in B'$ and r is a point of E'_K; cf. Remark 6.1.3.

The main task of this section is to show that all important M-linearization on line bundles on E are of the type discussed in Example 6.2.1. For this we need to know that the line bundles under consideration can be written as pull-backs of formal line bundles on B; i.e., we need informations on the trivialization of rigid analytic line bundles on smooth formal R-schemes and on the descent of line bundles on E with respect to $q : E \to B$.

As in the case of schemes, it is convenient to work with cubical structures; this notion was invented by Breen [19]. The basic facts we need are reassembled from [70] in Sect. A.3 and will be used here freely.

If L is a line bundle on a rigid analytic S-group space X over a rigid analytic space S, we write as in Notation A.3.1

$$\mathcal{D}_n(L) := \bigotimes_{\emptyset \neq I \subset \{1,\dots,n\}} \mu_I^* L^{\otimes(-1)^{n+\mathrm{card}(I)}},$$

where the tensor product runs through all non-empty subsets I of $\{1, \dots, n\}$ and where $\mu_I : X^n \to X$ is the morphism

$$\mu_I : X^n \longrightarrow X, \ (x_1, \dots, x_n) \longmapsto \sum_{i \in I} x_i .$$

A cubical structure on a $\mathbb{G}_m$-torsor L consists of a section τ of the associated $\mathbb{G}_m$-torsor of $\mathcal{D}_3(L)$, which satisfies a symmetry and a cocycle condition of Definition A.3.3. This can be interpreted as the structure of a symmetric biextension on $\mathcal{D}_2(L)$; cf. Proposition A.3.5.

A cubical line bundle is automatically rigidified and, conversely, every rigidified line bundle on a connected proper group space like B has a canonical cubical structure due to the Theorem of the Cube 7.1.6. Over non-proper group spaces cubical structures are useful to make rigidified line bundles even more rigid in the sense that there are restrictions on automorphisms.

Viewing formal analytic spaces X over R as direct limits of schemes X_n over $R_n = R/R\pi^{n+1}$, we can apply scheme-theoretic results about cubical line bundles on each level X_n. Hereby we obtain a result on the descent of cubical line bundles on formal group schemes. In particular, we can apply it to the formal situation

$$1 \to \overline{T} \to \overline{E} \xrightarrow{\overline{q}} B \to 1 \qquad (\overline{\dagger})$$

which is associated to ($\dagger$) of the beginning of this section.

In the following the subindex "K" indicates the rigid analytic space which is associated to a formal R-scheme.

Lemma 6.2.2. *Let L be a cubical rigid analytic line bundle on $\overline{E}_K$. Then L extends to a formal cubical line bundle $\overline{L}$ on $\overline{E}$.*

For showing the representability of the Picard-functor and especially the assertion of Theorem 6.3.2, we need a more general result than Lemma 6.2.2, which will be dealt with in Proposition 6.2.5. We start the proof with preliminary statements.

Lemma 6.2.3. *Let $Y = \operatorname{Spf} B \to X = \operatorname{Spf} A$ be a smooth morphism of affine formal R-schemes with connected fibers of relative dimension d. Let $\mathfrak{b}_K \subset B_K$ be an ideal associated to a relative Cartier divisor D_K. Let $\mathfrak{b} \subset B$ be a coherent ideal with $\mathfrak{b} \otimes_R K = \mathfrak{b}_K$. Assume that $B/\mathfrak{b}$ is flat over A. Then $\mathfrak{b}$ induces a formal relative Cartier divisor D which extends D_K.*

Proof. Since $B/\mathfrak{b}$ is flat over A, the sequence

$$0 \to \mathfrak{b} \otimes_A k(s) \to B \otimes_A k(s) \to (B/\mathfrak{b}) \otimes_A k(s) \to 0$$

is exact for the residue field $k(s)$ of every closed point s of $\operatorname{Spec} A$. The flatness of $B/\mathfrak{b}$ over A implies that the relative dimension of $(B/\mathfrak{b}) \otimes_A k(s)$ is $d-1$. If $\mathfrak{b}(s) := \mathfrak{b} \otimes_A k(s)$ has no embedded primes, then $\mathfrak{b}(s)$ gives rise to a relative Cartier divisor on the fiber $Y(s) := Y \times_X \operatorname{Spec} k(s)$, because $Y \to X$ is smooth. Since a local generator of $\mathfrak{b}(s)$ lifts to a local generator of $\mathfrak{b}$ as $\mathfrak{b}$ is a coherent ideal of B, we see that $\mathfrak{b}$ induces a relative Cartier divisor on Y/X; i.e., on every level Y_n/X_n for all $n \in \mathbb{N}$.

Thus, it remains to show that $\mathfrak{b}(s)$ has no embedded components. Therefore, consider the canonical inclusion

$$j : \mathfrak{b} \longrightarrow \mathfrak{b}^{**} := \operatorname{Hom}_B\big(\operatorname{Hom}_B(\mathfrak{b}, B), B\big) \subset B$$

from $\mathfrak{b}$ to its reflexive closure $\mathfrak{b}^{**} \subset B$. The B-module $\mathfrak{b}^{**}$ is regarded as a subset of B via $\beta \longmapsto \beta(\iota)$, where $\iota : \mathfrak{b} \hookrightarrow B$ is the inclusion. Since $\mathfrak{b}_K$ is locally principal, $j \otimes_R K$ is an isomorphism. So $\mathfrak{b}^{**}$ can be multiplied by a power π^n into $\mathfrak{b}$ and hence $\mathfrak{b}^{**}/\mathfrak{b}$ is $\pi^{\mathbb{N}}$-torsion. Since $B/\mathfrak{b}$ is flat over A, and hence flat over R, the multiplication by π on $B/\mathfrak{b}$ is injective. Then $\mathfrak{b} \to \mathfrak{b}^{**}$ is bijective, because $\mathfrak{b}^{**}/\mathfrak{b}$ is a submodule of $B/\mathfrak{b}$. Thus, we see that $\mathfrak{b}$ is reflexive, and hence that $\mathfrak{b}(s)$ is reflexive as well. Therefore, $\mathfrak{b}(s)$ has no embedded components. □

Lemma 6.2.4. *Let $q : Y \to X$ be a smooth morphism of admissible formal R-schemes with connected fibers of relative dimension d. Assume that Y is quasi-compact. If $\mathcal{L}_K$ is an invertible sheaf on Y_K, then there exists an admissible formal blowing-up $X' \to X$ such that $\mathcal{L}_K$ extends to an invertible sheaf $\mathcal{L}'$ on Y' where $Y' = Y \times_X X'$.*

Proof. There exists an admissible formal blowing-up $X' \to X$ and a finite open covering $\{Y^1, \ldots, Y^n\}$ of $Y' := Y \times_X X'$ such that $\mathcal{L}_K|_{Y_K^\nu}$ extends to an invertible sheaf $\mathcal{L}^\nu$ on Y^ν, which is isomorphic to relative Cartier divisor, for $\nu = 1, \ldots, n$.

Indeed, first consider an affine situation $q : V = \operatorname{Spf} B \to U = \operatorname{Spf} A$, where $V \subset Y$ and $U \subset X$ are open and affine. Then we have a non-trivial global section f of $\mathcal{L}_K|_{V_K}$, and hence $\mathcal{L}_K|_{V_K}$ is isomorphic to a coherent sheaf $\mathcal{J}_K$ of ideals on Y_K. We see by Proposition 3.4.21 that there is a Zariski closed subset Z_K^1 of U_K such that $\mathcal{J}_K$ gives rise to a relative Cartier divisor outside Z_K^1. By altering the global section f, one can find an open covering $\{X_K^1, \ldots, X_K^n\}$ of U_K such that $\mathcal{L}_K|_{q^{-1}(X_K^i)}$

is equivalent to a relative Cartier divisor for $i = 1, \ldots, n$. By Theorem 3.3.4 there exists an admissible formal blowing-up $X' \to X$ such that $\{X_K^1, \ldots, X_K^n\}$ is induced by a formal open covering of $U \times_X X'$. So we may assume that $\mathcal{L}_K|_{V_K}$ is already given by an ideal $\mathfrak{b}_K$ of B_K, which induces a relative Cartier divisor.

Then we choose a finitely generated ideal $\mathfrak{b} \subset B$ with $\mathfrak{b} \otimes_R K = \mathfrak{b}_K$. Due to the flattening technique of Theorem 3.3.7 there exists an admissible formal blowing-up $X' \to X$ such that the proper transform of $\mathcal{O}_V/\mathfrak{b}\mathcal{O}_V$ becomes flat after the base change by $X' \to X$. Then, it follows from to Lemma 6.2.3 that $\mathfrak{b}_K$ induces an invertible sheaf $\mathcal{B}'$ of ideals on $V' := V \times_X X'$, which gives rise to a relative Cartier divisor on V' over X'.

Since Y is quasi-compact, by Proposition 3.2.7(g) we can arrange this for an open covering of Y. Summarizing, we see that $\mathcal{L}_K|_{Y_K^\nu}$ extends to an invertible sheaf $\mathcal{L}_\nu \cong \mathcal{O}_Y \cdot f_\nu$ on Y_ν for $\nu = 1, \ldots, n$, where f_ν induces a relative Cartier divisor on Y^ν/X, as asserted.

Thus, the invertible sheaf is associated to the cocycle $\lambda_{\mu,\nu} := f_\mu/f_\nu$. These functions are invertible on the generic fiber and on an X-dense open subscheme of Y. Since the fibers of $Y \to X$ are smooth and connected, we obtain by Lemma 3.4.17(a) that the transition functions $\lambda_{\mu,\nu}$ are invertible over $Y^\mu \cap Y^\nu$. This shows that $\mathcal{L}_K$ extends to a formal invertible sheaf $\mathcal{L}$ on $Y \times_X X'$. □

For showing the representability of $\mathrm{Pic}^\tau_{A_K/K}$ for a uniformized abeloid variety A_K as in Definition 6.1.5, one has to deal with line bundles not only on A_K or E_K but also on products $A_K \times S_K$, where S_K is an affinoid space. Therefore, we will consider a quasi-compact, separated rigid space S_K in the following. Such spaces admit formal R-models S which are admissible formal schemes; cf. Theorem 3.3.3. As usual, we denote the associated rigid space of an admissible formal R-scheme S by S_K. To be precise we denote all objects on rigid space by an subindex "K".

In the following we consider an Raynaud extension

$$1 \to T_K \to E_K \xrightarrow{q} B_K \to 1 \quad \widehat{=} \quad \phi' : M' \to B'_K$$

and its associated formal extension

$$1 \to \overline{T} \to \overline{E} \xrightarrow{\overline{q}} B \to 1 \quad \widehat{=} \quad \phi' : M' \to B'.$$

Proposition 6.2.5. *In the above situation let S be a quasi-compact admissible formal scheme with generic fiber S_K. Let L_K be a rigid analytic line bundle on $E_K \times S_K$.*

(a) *Then there exists an admissible formal blowing-up $S' \to S$ such that the restriction $L_K|_{\overline{E}_K \times S_K}$ extends to a formal line bundle $\overline{L}$ on $\overline{E} \times_S S'$.*
(b) *If L_K is a cubical rigid analytic line bundle on $E_K \times S_K$, then there exists an admissible formal blowing-up $S' \to S$ such that the cubical structure on L_K extends to a formal cubical line bundle on $\overline{L}$ on $\overline{E} \times S'$.*

Proof. (a) This follows from Lemma 6.2.4.

(b) By (a) we may assume that L_K extends to a formal line bundle $\overline{L}$ on $\overline{E} \times S$. Let r be the rigidificator of L_K along the unit section 0_S of $\overline{E} \times S$. Let $\mathcal{I}$ be the invertible sheaf on S such that $\mathcal{I} \cdot r = 0_S^* \mathcal{L}$, where $\mathcal{L}$ is the invertible sheaf associated to $\overline{L}$. Then $\mathcal{I} \otimes K$ is equal to $\mathcal{O}_{S_K}$ on the generic fiber. Now we replace $\mathcal{L}$ by the line bundle associated to $\mathcal{L} \otimes \mathcal{I}^{-1}$. Thus, we obtain a new R-model of L_K, where now the rigidificator r extends to a formal rigidificator along the unit section 0_S of $E_K \times S_K$.

The cubical structure $\tau : E_K^3 \times S_K \to \mathcal{D}_3(L_K)$ of L_K extends to a cubical structure of the new model $\overline{L}$. Indeed, τ is defined on the generic fiber of $\overline{E} \times S$ and on an S-dense open such scheme of $\overline{E} \times S$ because of the rigidificator; note that $\tau(0_S)$ equals the rigidificator. Then τ is section of the $\mathbb{G}_m$-torsor $\mathcal{D}_3(\overline{L})$, as follows from Lemma 3.4.17. □

Corollary 6.2.6. *In the situation of Proposition 6.2.5 let $\overline{T}_S \to S$ be a split formal torus and let L_K be a line bundle on $\overline{T}_{S_K}$. Then, there exists an admissible formal blowing-up $S' \to S$ such that L_K extends to a formal line bundle $\overline{L}$ on $\overline{T} \otimes_S S'$, and there exists an open covering $\{S_i; i \in I\}$ of S' such that $\overline{L}$ is trivial over $\overline{T} \otimes_S S_i$.*

Proof. Due to Proposition 6.2.5 the line bundle L_K extends to a formal line bundle $\overline{L}$ after a certain base change $S' \to S$. In addition we may assume that $S = \operatorname{Spf} A$ is affine. As in the proof of Lemma 6.2.4 we may assume that $\overline{L}$ is given by an ideal $\mathfrak{b}$ of $A\langle \zeta, \zeta^{-1} \rangle$ inducing a relative Cartier divisor. Then there exists a filtration by closed standard affine subschemes

$$S_0^n := S_0 \supset S_0^{n-1} \supset \cdots \supset S_0^{-1} = \emptyset$$

with $\dim S_0^\nu = \nu$ such that for $U_0^\nu := S_0^\nu - S_0^{\nu-1}$ the restriction

$$\mathfrak{b} \otimes_A \mathcal{O}_S\big(U_0^\nu\big)$$

is principal for $\nu = n, \dots, 0$. Indeed, the ring of regular functions of a torus over a field is factorial and a global generator on the fiber over a point $s \in S_0$ generates the ideal over an open dense subscheme of its closure $\overline{s}$. Denote by U^n the open subscheme of S associated to U_0^n. Then every lifting b_U of the generator of $\mathfrak{b} \otimes_A \mathcal{O}_S(U_0^n)$ generates $\mathfrak{b} \otimes_A \mathcal{O}_S(U^n)$. Since $U_0^n \subset S_0$ is standard affine; i.e., $U_0^n = (S_0)_f$ for a regular function $f \in A$, one can consider the admissible open subset

$$V_K^n(c) := \big\{x \in S_K; \big|f(x)\big| \geq c\big\}$$

of S_K for a constant $c \in |K|$ with $c < 1$. Then one can approximate the generator b_U by an element b_V of $\mathfrak{b} \otimes \mathcal{O}_{S_K}(V_K^n)$, which is a generator over U_K^n as well. By the maximum principle we obtain that, after replacing c by a suitable element $c' < 1$, the element b_V generates $\mathfrak{b} \otimes \mathcal{O}_{S_K}(V_K(c'))$ as well. Then we turn to the open subvariety

$$S_K^{n-1} := \big\{x \in S_K; \big|f(x)\big| \leq c'\big\}.$$

Blowing up the ideal (f, γ) with $\gamma \in R$ and $c' = |\gamma|$ we obtain a formal scheme $S' \to S$ such that S_K^{n-1} is induced by an open subscheme of S'. Over the inverse image V_0^{n-1} of S_0^{n-1} in S_0' every lifting of a generator of the ideal $\mathfrak{b} \otimes \mathcal{O}_{S'}(V_0^{n-1})$ is principal. Then one can proceed as before and obtain the asserted result after n steps. □

The next lemma shows that the descent of a cubical line bundle L_K on $E_K \times S_K$ to a line bundle on $B_K \times S_K$ depends on the trivialization of L_K over the torus $T_K \times S_K$.

Proposition 6.2.7. *In the situation of Proposition 6.2.5 let L_K be a cubical rigid analytic line bundle on $E_K \times S_K$ equipped with a trivialization s_T over $T_K \times S_K$ which is compatible with the cubical structure.*

(a) *Then there exists an admissible formal blowing-up $S' \to S$ such that (L_K, s_T) descends to a cubical formal line bundle N on $B \times S'$.*
    The line bundle $N_K := N \otimes K$ is uniquely determined by (L_K, s_T).
(b) *If q^*N_K is the trivial cubical torsor, the trivialization s_T is a character $m' \in M'$ and extends to a trivialization $s : E_K \to q^*N_K$. More precisely, $s \otimes e_{-m'}$ descends to a unique section $\sigma : B \times S' \to N \otimes P_{B \times \phi'(-m')}$, where the section $e_{m'} : E_K \to P_{B_K \times \phi'(m')}$ is the canonical one defined in Notation 6.1.7.*

Proof. (a) By Proposition 6.2.5 there exists an admissible formal blowing-up $S' \to S$ such that L_K extends to a formal cubical line bundle $\overline{L}$ on $\overline{E} \times S'$. Moreover, the section s_T induces a trivialization of the line bundle $\overline{L}_K := L_K|_{\overline{E}_K \times S_K}$ and extends to a trivialization $\overline{s}_T$ of the formal line bundle $\overline{L}$, as follows by a similar reasoning as in the proof of Proposition 6.2.5(b). Due to Theorem A.3.8 the couple $(\overline{L}, \overline{s}_T)$ descends to the formal cubical line bundle N on $B \times S'$. Thus, we have an isomorphism $\overline{\chi} : \overline{L} \xrightarrow{\sim} q^*N|_{\overline{E} \times S'}$ of cubical line bundles.

Now we will show that $\overline{\chi}$ extends to an isomorphism $\chi : q^*N_K \to L_K$ over E_K. Replacing L_K by $L_K \otimes q^*N^{-1}$, we have to show that any section $\overline{\sigma}$ of the $\mathbb{G}_m$-torsor L_K over $\overline{E}_K \times S_K$ extends to a section σ of the $\mathbb{G}_m$-torsor L_K over $E_K \times S_K$. The cubical structure induces the structure of a symmetric biextension on $\mathcal{D}_2(L_K)$ by Proposition A.3.5. In the following we write only E_K in stead of $E_K \times S_K$. Thus we obtain a commutative diagram

$$\begin{array}{ccccccc} L_K \times L_K & \longrightarrow & L_K & \qquad & (\ell_1, \ell_2) & \longmapsto & \ell_1 \star \ell_2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_K \times E_K & \longrightarrow & E_K & & (x_1, x_2) & \longmapsto & x_1 \cdot x_2. \end{array}$$

The sections $s_T : T_K \to L_K$ and $\overline{\sigma} : \overline{E} \to \overline{L}$ give rise to the diagram

$$
\begin{array}{ccc}
L_K|_{T_K} \times L_K|_{\overline{E}} & \longrightarrow & L_K \\
s_T \uparrow\downarrow\uparrow \overline{\sigma} & & \downarrow\uparrow \sigma \\
T_K \times \overline{E} & \longrightarrow & E_K
\end{array}
\qquad
\begin{array}{ccc}
(s(t), \overline{\sigma}(\overline{x})) & \longmapsto & s_T(t) \star \overline{\sigma}(\overline{x}) \\
\uparrow s_T \times \overline{\sigma} & & \uparrow \sigma \\
(t, \overline{x}) & \longmapsto & t \cdot \overline{x}
\end{array}
$$

which defines a trivialization of L_K. Since s_T is compatible with the cubical structure, the map $\sigma(z)$ for $z \in E_K$ is independent of the choice of $(t, \overline{x})$ with $z = t \cdot \overline{x}$. This section is compatible with the cubical structure of L_K, because the compatibility is satisfied over $\overline{E}_K$.

(b) If $q^* N_K$ is trivial, then s_T is a character m' and extends to a group homomorphism $s : \overline{E} \to N = P_{B \times \phi'(m')}$ via push-forward by Theorem A.3.9. Thus, we see that $s \otimes e_{-m'}$ descends to a section $\sigma : B \to N \otimes P_{B \times \phi'(-m')}$ uniquely, as follows from Theorem A.3.8. The uniqueness of N_K follows also from Theorem A.3.8 because of the fixed trivialization s_T. □

Corollary 6.2.8. *In the situation of Proposition 6.2.5 the category of cubical line bundles on $B_K \times S_K$ is via the pull-back with respect to q equivalent to the category of couples (L_K, s_T), where L_K is a cubical line bundle on E_K and s_T is a trivialization of $L_K|_{T_K \times S_K}$.*

The set of all possible descent data for L is a principal homogeneous space under the group of characters $M' = \mathrm{Hom}(T_K, \mathbb{G}_m)$.

Corollary 6.2.9. *In the situation of Proposition 6.2.5 let N_1, N_2 be cubical line bundles on $B \times S$. If $c : q^* N_1 \to q^* N_2$ is an isomorphism of cubical line bundles on $E_K \times S_K$, then $c|_{T_K \times S_K}$ is given by a unique character $m' \in M'$, and the isomorphism*

$$
c \otimes e_{-m'} : q^* N_1 \longrightarrow q^* N_2 \otimes q^* P_{B \times \phi'(-m')}
$$

descends to an isomorphism $N_1 \to N_2 \otimes P_{B \times \phi'(-m')}$.

Proof. The isomorphism c can be viewed as a trivialization of the cubical line bundle $q^*(N_2 \otimes N_1^{-1})$. By Proposition 6.2.7(b) the trivialization $c|_{T_K \times S_K}$ is given by a unique character $m' \in M'$ and, moreover, the section $c \otimes e_{-m'}$ descends to the canonical global section of $N_2 \otimes N_1^{-1} \otimes P_{B \times \phi'(-m')}$. □

Due to Lemma 2.7.4 every line bundle on an affine torus is trivial. There is only one cubical structure on the trivial line bundle, because every invertible function on T_K is a character times a constant by Proposition 1.3.4. Thus, every trivialization is compatible with the unique cubical structure. Therefore the assumption on the existence of s_T is always fulfilled in the case $S_K = \mathrm{Sp}\, K$.

Proposition 6.2.10. *In the situation of Proposition 6.2.5 assume, in addition, that the line bundle L_K is cubical. Then we have the following results:*

(a) *There exists an admissible formal blowing-up $S' \to S$ with an open covering $\{S^i; i \in I\}$ of S' and a formal cubical line bundle N_i on $B \times_S S^i$ for each $i \in I$ such that there exists an isomorphism*

$$\overline{\chi}_i : L_K|_{\overline{E}_K \times_S S^i_K} \xrightarrow{\sim} q^* N_i \otimes K|_{\overline{E}_K \times_S S^i_K}$$

of cubical line bundles; *cf. Definition* A.3.3. *Each pair* (N_i, χ_i) *is uniquely determined up to a tensor product with* $(P_{B\times\phi'(m')}, e_{m'})$; *cf. Notation* 6.1.7.

(b) *Let $M \subset E_K$ be a lattice of full rank. If L_K is equipped with an M-linearization, then the isomorphism $\overline{\chi}_i$ extends to an isomorphism*

$$\chi_i : L_K|_{E_K \times_S S^i_K} \xrightarrow{\sim} q^* N_i \otimes K|_{E_K \times_S S^i_K}$$

of cubical line bundles for every $i \in I$.

Proof. (a) By Corollary 6.2.6 there exists an admissible formal blowing-up $S' \to S$ with a covering $\{S^i; i \in I\}$ by open subschemes such that there exists a trivialization $\overline{\sigma}_i : \overline{T} \times S_i \to \overline{L}|_{\overline{T}\times S^i}$ of the cubical line bundle $\overline{L}|_{\overline{T}\times S^i}$. Then it follows from Theorem A.3.8 that for every $i \in I$ there exist a formal cubical line bundle N_i on $B \times_S S^i$ and an isomorphism $\overline{\chi}_i : L|_{\overline{E}\times_S S^i} \xrightarrow{\sim} q^* N_i|_{\overline{E}\times_S S^i}$ of cubical line bundles. The assertion about the uniqueness follows from Theorem A.3.8.

(b) This causes some trouble, since it is not clear that $L_K|_{T_K \times S^i_K}$ is trivial. If we had that, then the assertion would follow from Proposition 6.2.7. However, in the case, where S_K is a reduced point, then we are done by Lemma 2.7.4, because every trivialization of a rigidified line bundle on T_K is unique up to a character by Proposition 1.3.4. Moreover a trivialization of a cubical line bundle on a torus of units is also unique up to a character.

In the general case we may assume that $S^i = S$ and $N_i = N$. Moreover, we may assume that N has rigidificators $r(e_\rho) : S^i \to N$ along $q(e_\rho)$, for a basis $(e_1, \dots, e_r)$ of M. In the following we write $\overline{E}_K$ instead of $\overline{E}_K \times S_K$.

The M-linearization yields a section $c_m : E_K \to \tau_m^* L_K \otimes L_K^{-1}$. Since $\tau_m^* L_K|_{\overline{E}_K}$ has a formal model like $L_K|_{\overline{E}_K}$ and a trivialization over $\overline{T}_K$, we see by Theorem A.3.8 that $\tau_m^* L_K \otimes L_K^{-1}|_{\overline{E}_K} = \overline{q}^* P_{B_K\times\phi(\lambda(m))}$, where $\lambda : M_{S_K} \to M'_{S_K}$ is a map with $c_m(z) = \langle z, \lambda(m)\rangle$.

If $s \in S_K$ is point, then $L_K|_{T_K\times s}$ is trivial by Lemma 2.7.4, and hence the morphism $\overline{\chi}_s$ extends to a morphism χ_s as we have seen above. Thus, the M-linearization on $L_K|_{E_K\times s}$ induces an M-linearization on $q^*N|_{E_K\times s}$. As seen in Example 6.2.1, this is presented by a couple (r_s, λ_s). It is evident that $\lambda_s = \lambda|_{M\times s}$ for all points $s \in S_K$. In particular, $\phi' \circ \lambda = \varphi_N \circ \phi$ over the reduced space $S_{K,\mathrm{red}}$. Thus we can define a linearization (r, λ) on $q^* N_K$ over $E_K \times S_{K,\mathrm{red}}$ inductively by the rule

$$r(m_1 + m_2) = r(m_1) \otimes r(m_2) \otimes r(0)^{-1} \otimes \langle m_1, \lambda(m_2)\rangle$$

for all $m_1, m_2 \in M$ starting with the values of our basis $(e_1, \dots, e_r)$ of M. Likewise as in Example 6.2.1 we obtain an M-linearization of $q^* N_K$.

Then by Lemma 6.2.11 we see that $\overline{\chi}$ extends to a morphism χ over $E_K \times S_{K,\mathrm{red}}$. In particular, $L_K|_{T_K \times S_{K,\mathrm{red}}}$ is trivial. Then it is also trivial over $E_K \times S_K$, because a trivializing section over $T_K \times S_{K,\mathrm{red}}$ lifts to a trivializing section over $T_K \times S_K$, because $T_K \times S_K$ is quasi-Stein; cf. [51, 2.4]. Thus, we may assume that $L_K|_{T_K \times S_K}$ is the trivial $\mathbb{G}_m$-torsor. Then the trivialization over $\overline{T}_K \times S_K$ of the cubical $\mathbb{G}_m$-torsor L_K over $\overline{T}_K \times S_K$ extends to a section over $T_K \times S_K$. Finally again by Proposition 6.2.7 we obtain that $\overline{\chi}$ extends to an isomorphism over $E_K \times S_K$. □

Lemma 6.2.11. *In the situation of Proposition* 6.2.10(b) *every section $\overline{\sigma}$ over $\overline{E}_K \times S_K$ of the cubical $\mathbb{G}_m$-torsor L_K extends to a section over $E_K \times S_K$.*

Proof. To verify the assertion, it is enough to show that L_K is a trivial line bundle over $E_K \times S_K$. Indeed, every cubical structure on the trivial $\mathbb{G}_m$-torsor over $E_K \times S_K$ is given by a morphism $\tau : E_K^3 \times S_K \to \mathbb{G}_m$, which is necessarily a character. In fact, if S_K is reduced, then this follows from Proposition 1.3.4. In the case where S_K is not reduced, use the cocycle condition for τ; cf. Theorem A.3.9. Furthermore, again by the cocycle condition we see that there exists a character χ with $\mathcal{D}_3(\chi) = \tau$. In particular, every cubical structure on the trivial $\mathbb{G}_m$-torsor over $E_K \times S_K$ is isomorphic to the trivial cubical $\mathbb{G}_m$-torsor via a character. Thus, for our problem, we have just to show that L_K is trivial, because $\overline{\sigma}$ is a character.

We fix a basis $m'_1, \dots, m'_r$ of the character group M' and consider the associated product decomposition $E_K = E_{m'_1} \times_B \dots \times_B E_{m'_r}$. Let $|\cdot|_i$ be the absolute value on $E_{m'_i} := P_{B \times \phi'(m_i)}$. For $m \in M$ set

$$F_m := \left\{ z \in E_K \times S_K ;\, |m|_i^{-1} \le |z|_i \le |m|_i \text{ for } i = 1, \dots, r \right\}.$$

Then F_m is an admissible open subspace of E; it can be viewed as a polyannulus over B. We claim that the restriction of L_K to F_m is trivial for all $m \in M$. To justify this assertion, we proceed by induction and write $F_m = F'_m \times_B F''_m$, where

$$F'_m := \left\{ z \in E_{m'_1} \times_B \cdots \times_B E_{m'_r - 1} ;\, |m|_i^{-1} \le |z|_i \le |m|_i \text{ for } 1 \le i \le r-1 \right\},$$
$$F''_m := \left\{ z \in E_{m'_r} ;\, |m|_r^{-1} \le |z|_r \le |m|_r \right\}.$$

If $|m_d| \ne 1$, let

$$F_m^+ := \left\{ z \in E_{m'_r} ;\, |z|_d = |m|_d \right\},$$
$$F_m^- := \left\{ z \in E_{m'_r} ;\, |z|_d = |m|_d^{-1} \right\}$$

be the two connected components of the "boundary" of the relative annulus F''_m. We can fill the "wholes" of F''_m and hereby embed it into the $\mathbb{P}^1$-bundles P'' over B which is associated to $E_{m'_d}$. We know that L_K is trivial on $\overline{E}_K \times S_K$ and, due to the M-action, on all M-translates of $\overline{E}_K \times S_K$. Using the induction hypothesis, we may assume that L_K is trivial on $F'_m \times_B F_m^+$ and on $F'_m \times_B F_m^-$. Thus, we can extend $L_K|_{F'_m \times_B F''_m}$ to a line bundle L'' on $F'_m \times_B P'$. Since $\mathrm{Pic}_{\mathbb{P}^1_K/K}$ is the constant sheaf $\mathbb{Z}$, there is an $n \in \mathbb{Z}$ such that $L'' \cong \mathcal{O}_{F'_m \times_B P''}(n)$. Thus, we see that L'' and hence L_K are trivial on $F_m = F'_m \times_B F''_m$.

Now consider an exhausting sequence $F_1 \subset F_2 \subset \cdots$ of $E_K \times S_K$ by domains of type F_m. Since L_K is trivial over all the F_j, we can find a section f_j generating L_K over F_j. Then one can proceed as in the proof of Lemma 2.7.4 and show that L_K is trivial over $E_K \times S_K$. □

In the following we consider the situation (†) introduced in the beginning of this section. Let $M \subset E_K$ be a lattice of full rank.

Now we will consider M-linearizations of cubical line bundles on E_K; i.e., M-linearizations, which respect the cubical structures. Associated to the morphism $h : M \to E_K$ there is a canonical trivialization t of the biextension $(\phi \times \phi')^* P_{B_K \times B'_K}$ due to Proposition 6.1.8 which is given by

$$t : M \times M' \longrightarrow P_{B_K \times B'_K}, \ \left(m, m'\right) \longmapsto \left\langle m, m' \right\rangle := h(m)_{m'}.$$

The following statements of this section are also true in the general case $E_K \times S_K$, where S_K is a connected affinoid rigid space, and the proofs given for the base space $S_K = \operatorname{Sp} K$ work in the general case as well. Of course one has to replace $\operatorname{Spf} R$ by a formal R-model S or by an admissible blowing-up of S and an open covering of S; the latter is due to Proposition 6.2.10. In order to keep things simple we do not make this explicit in the following.

Proposition 6.2.12. *In the above situation let N be a cubical line bundle on B and $\varphi_N : B \to B', b \mapsto \tau_b^* N \otimes N^{-1}$, the induced group homomorphism. Then the following data are equivalent*:

(a) *M-linearizations on the cubical line bundle $q^* N$.*
(b) *Pairs (r, λ) consisting of a group homomorphism $\lambda : M \to M'$ satisfying $\varphi_N \circ \phi = \phi' \circ \lambda$ and of a trivialization $r : M \to \phi^* N \otimes K$ of the cubical line bundle $\phi^* N$ such that r is compatible with t; i.e., such that the diagram*

$$\begin{array}{ccc} (\phi \times \phi)^* \mathcal{D}_2(N) & \longrightarrow & (\phi \times \phi')^* P_{B_K \times B'_K} \\ \uparrow {\scriptstyle \mathcal{D}_2(r)} & & \uparrow {\scriptstyle t} \\ M \times M & \xrightarrow{\ \mathrm{id} \times \lambda\ } & M \times M' \end{array}$$

is commutative.

In the above situation, if an M-linearization c of $q^* N$ is given, then the trivialization $r : M \to q^* N$ is obtained by transporting the rigidificator of $q^* N$ by means of the M-action. In terms of sections, the isomorphism $c_m : q^* N \to \tau_m q^* N$ corresponding to the action of $m \in M$ on $q^* N$ can be presented by the mapping $f \longmapsto r(m) \otimes e_{\lambda(m)} \otimes r(0)^{-1} \otimes f$, as described in Example 6.2.1.

Proof. Let us start with an M-action $c = (c_m; m \in M)$ on the cubical line bundle q^*N. So c consists of a family of morphisms $c_m : q^*N \to \tau_m^* q^*N$ for $m \in M$, which are compatible with the cubical structure and satisfy certain compatibility conditions.

By Corollary 6.2.9 there exists a unique character $m' = -\lambda(m) \in M'$ such that

$$c_m \otimes e_{-\lambda(m)} : q^*N \longrightarrow \tau_m^* q^*N \otimes q^*P_{B\times -\phi'(\lambda(m))}$$

descends to an isomorphism of cubical line bundles

$$\gamma_m : N \longrightarrow \tau_{q(m)}^* N \otimes P_{B\times -\phi'\circ\lambda(m)}.$$

Clearly, the just defined map $\lambda : M \to M'$ is a lifting of φ_N; i.e., satisfies the condition $\phi' \circ \lambda = \varphi_N \circ \phi$. The compatibility condition of the c_m in the family shows that $\lambda : M \to M'$ is a group homomorphism.

Next we define $r : M \to \phi^*N$ via transporting the rigidificator $r(0)$ at 0 by the M-action. Then the isomorphism c_m can be written in the form

$$c_m := q^*\gamma_m \otimes e_{\lambda(m)} : q^*N \longrightarrow \tau_m^* q^*N, \;\; f \longmapsto r(m) \otimes e_{\lambda(m)} \otimes r(0)^{-1} \otimes f, \qquad (*)$$

which maps the rigidificator $r(0)$ to $r(m) = c_m(r(0))$; as explained in Example 6.2.1.

The compatibility condition of the M-linearization means that the diagram

$$\begin{array}{ccc} q^*N & \xrightarrow{\;c_{m_1+m_2}\;} & \tau_{m_1+m_2}^* q^*N \\ \Big\downarrow{\scriptstyle c_{m_1}} & & \Big\| \\ \tau_{m_1}^* q^*N & \xrightarrow{\;\tau_{m_1}^* c_{m_2}\;} & \tau_{m_1}^* \tau_{m_2}^* q^*N \end{array}$$

is commutative. By the formula $(*)$ one sees that this is equivalent to

$$r(m_1+m_2) \otimes e_{\lambda(m_1+m_2)} = \tau_{m_1}^* r(m_2) \otimes \tau_{m_1}^* e_{\lambda(m_2)} \otimes r(m_1) \otimes r(0)^{-1} \otimes e_{\lambda(m_1)}$$

for all $m_1, m_2 \in M$. This is equivalent to the rules

$$\lambda(m_1+m_2) = \lambda(m_1) + \lambda(m_2),$$
$$r(m_1+m_2) \otimes r(m_1)^{-1} \otimes r(m_2)^{-1} \otimes r(0) = \langle m_1, \lambda(m_2)\rangle$$

for all $m_1, m_2 \in M$. These conditions combined with $\varphi_N \circ \phi = \phi' \circ \lambda$ are equivalent to our condition (b).

Conversely, a pair (r, λ) gives rise to morphisms $c_m : q^*N \longrightarrow \tau_m^* q^*N$ of line bundles by the formula $(*)$ for $m \in M$. Indeed, by (b) we have $\varphi_N \circ \phi = \phi' \circ \lambda$. Thus, the line bundle $\tau_{\phi(m)}^* N \otimes N^{-1} \otimes P_{B\times -\phi'\circ\lambda(m)}$ is trivial. Then we obtain morphisms c_m by the formula $(*)$. Since $\lambda(m)$ is a character, one directly verifies that c_m respects the cubical structures. □

In the situation of Proposition 6.2.12 let (L_i, c_i) be two M-linearized cubical line bundles on E_K corresponding to pairs (N_i, r_i, λ_i) for $i = 1, 2$. Then it is easy to see that $(L_1 \otimes L_2, c_1 \otimes c_2)$ corresponds to $(N_1 \otimes N_2, r_1 \otimes r_2, \lambda_1 + \lambda_2)$.

An M-linearization of a cubical line bundle (L, c) on E_K is *translation invariant* if for every S-valued point x of E_K there exists an isomorphism $\psi : (L, c) \xrightarrow{\sim} \tau_x^*(L, c)$. It is called *trivial* if there exists an isomorphism of (L, c) to the trivial line bundle equipped with the trivial cubical structure.

Corollary 6.2.13. *In the situation of Proposition 6.2.12 let (r, λ) be an M-linearization of the cubical line bundle q^*N. Then the following conditions are equivalent*:

(a) *The M-linearization of q^*N is translation invariant.*
(b) *N is translation invariant and λ is trivial.*
(c) *N is translation invariant and the M-linearization of q^*N is the pull-back of an M-linearization of N.*

Proof. (a) → (b): The character $e_{\lambda(m)}$ is translation invariant if and only if $\lambda(m) = 0$. Indeed, look at the translation by points of the torus. Then we see from Corollary 6.2.9 that every morphism $c(x) : q^*N \to \tau_x^* q^*N$ of cubical structures with $c(x)|_{T_K} = 1$ descends to an isomorphism $c(x) : N \to \tau_{q(x)}^* N$. Thus, N is translation invariant.

(b) → (c): It follows from Corollary 6.2.9.

(c) → (a): If N is translation invariant, then N gives rise to a $\mathbb{G}_m$-extension of B. Every M-linearization on a translation invariant line bundle N is given by a multiplication $N \to \tau_{\phi(m)}^* N$ with $r(m) \otimes r(0)^{-1}$. The latter one is obviously translation invariant and its pull-back to E_K as well. □

Corollary 6.2.14. *In the situation of Proposition 6.2.5 the following conditions are equivalent*:

(a) *The M-linearization on q^*N is isomorphic to the trivial one.*
(b) *There is a character $m' \in M'$ such that there is a unique isomorphism $N \xrightarrow{\sim} P_{B \times \phi'(m')}$ sending $r(0)$ to 1 and $r(m) \mapsto \langle m, m' \rangle$ for $m \in M$.*

Proof. (a) → (b): From the assumption follows that q^*N is trivial. Thus, we see by Proposition 6.2.7 that we can identify $N = P_{B \times \phi'(m')}$ for a character $m' \in M'$ as rigidified line bundles, where $q^* P_{B_K \times \phi'(m')}$ is trivialized by the canonical section $e_{m'} : E_K \to q^* P_{B_K \times \phi'(m')}$. So the M-linearization is given by

$$r(m) \otimes r(0)^{-1} \longmapsto \tau_m^* e_{m'} \otimes e_{m'}^{-1} = \langle m, m' \rangle.$$

(b) → (a): The section $e_{m'} : E_K \to q^* P_{B_K \times \phi'(m')}$ induces an isomorphism from the trivial cubical $\mathbb{G}_{m,K}$-torsor to $q^* P_{B_K \times \phi'(m')}$ equipped with the cubical structure $(\langle \_, m' \rangle, 0)$. □

Proposition 6.2.15. *In the situation of Proposition 6.2.12 let $q' : E' \to B'$ be the torus extension associated to the group homomorphism $\phi := q \circ h$. The points of E'_K corresponds bijectively to the isomorphism classes of M-linearizations of cubical line bundles on E_K which are translation invariant.*

A point x' of E'_K gives rise to the cubical line bundle $P_{B\times q'(x')}$ on B; its M-linearization is given by the multiplication

$$x'_m : P_{B_K \times q'(x')} \longrightarrow \tau^*_{q(m)} P_{B_K \times q'(x')}, \; f \longmapsto f \otimes \langle m, x' \rangle,$$

with $\langle m, x' \rangle \in P_{\phi(m)\times q'(x')}$, where $q'(x') \in B'$ is viewed as a base space.

Proof. By the universal property of B'_K the points of B'_K are the translation invariant line bundles on B_K. Thus, a point x' of E'_K gives rise to a point $b' := q'(x')$ of B'_K which represents the translation invariant line bundle $P_{B_K \times b'}$. Regarding B'_K as the base space, the point x' is equivalent to a family of points $(x'_m \in P_{\phi(m)\times b'}; m \in M)$ satisfying $x'_{m_1+m_2} = x'_{m_1} \otimes x'_{m_2}$, where "$\otimes$" is the group law on $P_{B_K \times B'_K}$ with the base space B'_K. □

Our next topic is the study of global sections of q^*N. Here it is more convenient to work with invertible sheaves instead of line bundles. Therefore, we have to dualize our setting.

Notation 6.2.16. If N is a line bundle, then we denote the associated invertible sheaf $\mathcal{N}$ by the corresponding calligraphic letter; i.e., $\mathcal{N}$ associates to open subsets U the $\mathcal{O}(U)$-module

$$\mathcal{N}(U) := \mathrm{Hom}\big(N\big|_U, \mathbb{A}^1_U\big).$$

Thus, the canonical section

$$e_{m'} : E_K \longrightarrow P_{B_K \times \phi'(m')}$$

gives rise to a global section $e_{m'} \in \Gamma(E_K, q^*\mathcal{P}_{B_K \times \phi'(-m')})$; cf. Remark 1.7.2. Note the change in the sign. In particular, we have the formula

$$\tau^*_m e_{m'} = \langle m, m' \rangle \otimes e_{m'}.$$

A morphism $\alpha : N_1 \to N_2$ induces the morphism

$$\alpha^* : \mathcal{N}_2 \longrightarrow \mathcal{N}_1, \; \ell \longmapsto \alpha^*\ell := \ell \circ \alpha.$$

An M-linearization $c = (c_m; m \in M)$ on q^*N, which is identified with (r, λ) in the manner of Proposition 6.2.12, induces the morphism

$$c^*_m : \tau^*_m q^*\mathcal{N} \longrightarrow q^*\mathcal{N}, \; \ell \longmapsto \ell \otimes r(m)^{-1} \otimes r(0) \otimes e_{-\lambda(m)}.$$

Therefore, the M-linearization of $q^*\mathcal{N}$ is defined by $(c^*_m; m \in M)$, which is given by $(r^{-1}, -\lambda)$.

Lemma 6.2.17. *Let N be a cubical line bundle on B and let $\mathcal{N}$ be the associated invertible sheaf; cf. Notation 6.2.16. Then there is a canonical Fourier decomposition*

$$q_*q^*\mathcal{N} = \widehat{\bigoplus_{m'\in M'}} (\mathcal{N} \otimes \mathcal{P}_{B\times\phi'(m')}) \otimes e_{m'}$$

as a complete direct sum.

Proof. Every section $f \in \Gamma(U, q_*q^*\mathcal{N})$ over an open subset U of B can uniquely be represented as a convergent Fourier series

$$f = \sum_{m'\in M'} a_{m'} \otimes e_{m'},$$

with coefficients $a_{m'} \in \Gamma(U, \mathcal{N} \otimes \mathcal{P}_{B\times\phi'(m')})$. Indeed, the coefficient $a_{m'}$ is a section $a_{m'} : U \to N^{-1} \otimes P_{B\times\phi'(-m')}$. Thus, $a_{m'} \otimes e_{m'} : U \to N^{-1}$ is a section of N^{-1} and hence an element of $\Gamma(U, \mathcal{N})$.

There exists also a covering of B by open subsets $U_1, \ldots, U_n$ such that $N|_{U_i}$ is trivial and $E|_{U_i}$ is a free $\mathbb{G}^r_{m,K}$-torsor, then $f|_{U\cap U_i}$ can be represented as a convergent Laurent series, which is the restriction of the Fourier decomposition to $U \cap U_i$. Thus, we obtain the representation as asserted. □

Lemma 6.2.18. *Let N_1, N_2 be cubical line bundles on B with associated invertible sheaves $\mathcal{N}_1, \mathcal{N}_2$. Let $c : q^*N_1 \to q^*N_2$ be an isomorphism of cubical line bundles on E_K, where $c|_{T_K}$ is given by a character n'. Then the induced mapping of their associated invertible sheaves*

$$q_*(c) : q_*q^*\mathcal{N}_2 \longrightarrow q_*q^*\mathcal{N}_1$$

is a homogeneous morphism of their Fourier decompositions of degree n', whose components for $m' \in M'$ are given by

$$\begin{aligned}(\mathcal{N}_2 \otimes \mathcal{P}_{B\times\phi'(m'-n')}) \otimes e_{m'-n'} &\longrightarrow (\mathcal{N}_1 \otimes \mathcal{P}_{B\times\phi'(m')}) \otimes e_{m'},\\ \ell \otimes e_{m'-n'} &\longmapsto (\gamma \otimes \mathrm{id}_{\mathcal{P}_{B\times\phi'(m')}})^*\ell \otimes e_{m'},\end{aligned}$$

where $c \otimes e_{-n'}$ descends to $\gamma : N_1 \to N_2 \otimes P_{B\times\phi'(-n')}$; cf. Corollary 6.2.9.

Proof. By Corollary 6.2.9 the isomorphism $c \otimes e_{-n'} : q^*N_1 \to q^*N_2 \otimes q^*P_{B\times\phi'(n')}$ descends to an isomorphism

$$\gamma : N_1 \longrightarrow N_2 \otimes P_{B\times\phi'(n')}.$$

Then its dual is the mapping

$$\gamma^* : \mathcal{N}_2 \otimes \mathcal{P}_{B\times\phi'(-n')} \longrightarrow \mathcal{N}_1.$$

On the homogeneous summand of degree m' the morphism $q_*(c^*)$ equals

$$\gamma^*_{m'} := \gamma^* \otimes \mathrm{id}_{\mathcal{P}_{B\times\phi'(m')}} : \mathcal{N}_2 \otimes \mathcal{P}_{B\times\phi'(m'-n')} \otimes e_{m'-n'} \longrightarrow \mathcal{N}_1 \otimes \mathcal{P}_{B\times\phi'(m')} \otimes e_{m'}$$

which is the indicated morphism. □

Proposition 6.2.19. *Let N be a cubical line bundle on B equipped with an M-action $c = (c_m; m \in M) = (r, \lambda)$ in the manner of Proposition 6.2.12, which is given by the isomorphisms*

$$c_m : q^*N \longrightarrow \tau^*_m q^*N, \ \ f \longmapsto r(m) \otimes r(0)^{-1} \otimes f \otimes e_{\lambda(m)}.$$

Then each $c^*_m : \tau^*_m q^*\mathcal{N} \to q^*\mathcal{N}$ as a map of the associated invertible sheaves is homogeneous of degree $\lambda(m)$ and its components are given by

$$\begin{aligned}\tau^*_{q(m)}(\mathcal{N} \otimes \mathcal{P}_{B\times\phi'(m'-\lambda(m))}) \otimes e_{m'-\lambda(m)} &\longrightarrow (\mathcal{N} \otimes \mathcal{P}_{B\times\phi'(m')}) \otimes e_{m'},\\ \ell \otimes e_{m'-\lambda(m)} &\longmapsto \big(r(m)^{-1} \otimes r(0) \otimes \ell\big) \otimes e_{m'}.\end{aligned}$$

Let $f \in \Gamma(E_K, q^*\mathcal{N})$ be a global section with the Fourier decomposition

$$f = \sum_{m'\in M'} a_{m'} \otimes e_{m'}$$

and coefficients $a_{m'} \in \Gamma(B_K, \mathcal{N} \otimes \mathcal{P}_{B\times\phi'(m')})$. Then the section f is M-invariant if and only if its coefficients satisfy the relations

$$\tau^*_m a_{m'+\lambda(m)} = \big\langle m, -\lambda(m)\big\rangle \otimes \big\langle m, m'\big\rangle^{-1} \otimes r(m) \otimes r(0) \otimes a_{m'} \qquad (*)$$

for all $m \in M$ and $m' \in M'$.

Proof. The first assertion follows from Lemma 6.2.18 because of $c^*_m(a) = a \circ c_m$.

For the second assertion consider a global section $f : E_K \to q^*N^{-1}$. Then by using Notation 6.2.16 we obtain the following equations

$$\begin{aligned} f &= \sum_{m'\in M'} a_{m'} \otimes e_{m'},\\ \tau^*_m f &= \sum_{m'\in M'} \tau^*_m a_{m'} \otimes \tau^*_m e_{m'} = \sum_{m'\in M'} \tau^*_m a_{m'} \otimes \big\langle m, m'\big\rangle \otimes e_{m'},\\ c^*_m \tau^*_m f &= \sum_{m'\in M'} \big\langle m, m'\big\rangle \otimes r(m)^{-1} \otimes r(0) \otimes \tau^*_m a_{m'} \otimes e_{m'-\lambda(m)}.\end{aligned}$$

The M-invariance of f is equivalent to the relation $c^*_m \tau^*_m f = f$. Thus, we see that the coefficients have to fulfill the relations

$$\begin{aligned} a_{m'} &= \big\langle m, m' + \lambda(m)\big\rangle \otimes r(m)^{-1} \otimes r(0) \otimes \tau^*_m a_{m'+\lambda(m)}\\ &= \big\langle m, m'\big\rangle \otimes r(m)^{-1} \otimes r(0) \otimes \big\langle m, \lambda(m)\big\rangle \otimes \tau^*_m a_{m'+\lambda(m)},\end{aligned}$$

for all $m \in M$ and $m' \in M'$, and hence

$$\tau_m^* a_{m'+\lambda(m)} = \langle m, -\lambda(m)\rangle \otimes \langle m, m'\rangle^{-1} \otimes r(m) \otimes r(0)^{-1} \otimes a_{m'}$$

for all $m \in M$ and $m' \in M'$, as required. □

Corollary 6.2.20. *Let $\mathcal{N}$ be a cubical invertible sheaf on B and let (r, λ) be the M-linearization on $q^*\mathcal{N}$ in the manner of Proposition 6.2.12 or Notation 6.2.16. Then the following assertions are equivalent*:

(a) *λ is injective and there exists a global M-invariant section f of $q^*\mathcal{N}$ which is non-trivial.*
(b) *For all $m \in M - \{0\}$ is $|\langle m, \lambda(m)\rangle| < 1$ and there is an $m' \in M'$ and a non-vanishing global section $a_{m'}$ of $\mathcal{N} \otimes \mathcal{P}_{B\times\phi'(m')}$.*

Proof. This follows as in Theorem 2.7.12 by using the formula of Proposition 6.2.19. First, note the change of signs in the M-linearization data, when passing from N to $\mathcal{N}$; cf. Notation 6.2.16. Thus, the formula of Proposition 6.2.19 looks as in the proof of Theorem 2.7.12

$$\tau_m^* a_{m'-\lambda(m)} = \langle m, \lambda(m)\rangle \otimes \langle m, m'\rangle^{-1} \otimes r(m)^{-1} \otimes r(0) \otimes a_{m'}.$$

On the formal invertible sheaf $\mathcal{N}$ there is a well-defined absolute value; cf. Sect. 6.1, where we assume $|r(0)| = 1$. Since a translation does not alter the absolute value, one can proceed as in the proof of Theorem 2.7.12. Thus, iterating by m yields for the absolute values

$$|a_{m'-\lambda(i\cdot m)}| = \left|\langle m, \lambda(m)\rangle\right|^{i(i+1)/2} \cdot \left|r(m)\right|^{-i} \cdot \left|\langle m, m'\rangle\right|^{-i} \cdot |a_{m'}|.$$

Furthermore, one can prescribe an element $a_{m'} \in \Gamma(B, \mathcal{N} \otimes \mathcal{P}_{B\times\phi'(m')})$ in every residue class of $M'/\lambda(M)$ in order to obtain an M-invariant global section of q^*N. □

Remark 6.2.21. More generally, consider an invertible sheaf $\mathcal{N}$ on B and an M-linearization (r, λ) on $q^*\mathcal{N}$. Then we have the following formula

$$\dim_K \Gamma\left(E_K, q^*\mathcal{N}\right)^M = \sum_{m' \in (M'/\lambda(M))} \mu_{m'} \cdot \dim_K \Gamma(B, \mathcal{N} \otimes \mathcal{P}_{B\times\phi'(m')}),$$

where $\mu_{m'} = 1$ or 0. Only finitely many of the summands do not vanish. If the invertible sheaf $\mathcal{N}$ is ample, then we will see in Theorem 6.4.4 that

$$\dim_K \Gamma\left(E_K, q^*\mathcal{N}\right)^M = \#\left(M'/\lambda(M)\right) \cdot \dim_K \Gamma(B, \mathcal{N}).$$

6.3 Duality

In the following we will apply the results of Sect. 6.2 to study the representability of $\mathrm{Pic}_{A/K}$ for an abeloid variety A which admits a Raynaud representation; cf. Definition 6.1.5. We consider again the situation which was introduced at the beginning of Sect. 6.1, and our Convention 6.1.6. We start with two free abelian groups $M = \mathbb{Z}^r$ resp. $M' = \mathbb{Z}^r$ of rank r. Now we consider an extension of the generic fiber of a formal abelian R-scheme B by a torus $T = \mathbb{G}_m^r$ and consider $M' = \mathbb{X}(T)$ as the character group of T. Such a torus extension

$$0 \to T \to E \to B \to 0 \quad \widehat{=} \quad \phi' : M' \to B'$$

is equivalent to a group homomorphism ϕ' from M' to the dual B' of B; cf. Theorem A.2.8. Then, we introduce a lattice in E of rank r by choosing a group homomorphism $h : M \to E$. Thus, we end up with the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow h & \overset{\phi}{\searrow} & \\ T & \longrightarrow & E & \xrightarrow{q} & B \\ & & \downarrow p & & \\ & & A := E/M & & \end{array} \qquad \widehat{=} \qquad \phi' : M' \to B'.$$

Let $\phi := q \circ h : M \to B$ be the induced map and $p : E \to E/M$ the rigid analytic quotient. These data are equivalent to a commutative diagram

$$\begin{array}{ccc} & & P_{B\times B'} \\ & \overset{\langle\_,\_\rangle}{\nearrow} & \downarrow \\ M \times M' & \xrightarrow{\phi\times\phi'} & B \times B', \end{array}$$

where $\langle\_,\_\rangle$ is a bilinear form, which trivializes of the pull-back of the Poincaré bundle $P_{B\times B'}$, such that the absolute value $|\langle\_,\_\rangle|$ of the bilinear form is non-degenerate; cf. Proposition 6.1.8. This diagram gives rise to a situation which is dual to the one we started with:

$$\begin{array}{ccccc} & & M' & & \\ & & \downarrow h' & \overset{\phi'}{\searrow} & \\ T' & \longrightarrow & E' & \xrightarrow{q'} & B' \\ & & \downarrow p' & & \\ & & A' := E'/M' & & \end{array} \qquad \widehat{=} \qquad \phi : M \to B = B'',$$

where E' is associated to the map $\phi : M \to B = B''$. In such a situation the quotients $A = E/M$ and $A' = E'/M'$ exist as rigid analytic spaces and are abeloid varieties in Raynaud representation; cf. Proposition 6.1.4.

Our objective in this section is the representability of the functor $\mathrm{Pic}^{\tau}_{A/K}$ of translation invariant line bundles on A. As already mentioned at the beginning of Sect. 6.1 the functor $\mathrm{Pic}^{\tau}_{B/R}$ of formal line bundles on the smooth formal scheme B is representable by the smooth formal scheme B'. Even more is true:

Proposition 6.3.1. *Let B be a formal abelian variety and let B' its dual. Then, the associated rigid space B'_K represents the functor $\mathrm{Pic}^{\tau}_{B_K/K}$ of translation invariant line bundles on the associated rigid space B_K.*

Proof. This follows from Lemma 6.2.4 and the representability of $\mathrm{Pic}^{\tau}_{B/R}$ by B'. □

To show representability of $\mathrm{Pic}^{\tau}_{A/K}$ requires to consider rigidified line bundles of $A \times_K S$, where S is a connected affinoid space. Therefore, just from beginning we replace A by $A \times S$ and consider A as an S-space. This has the advantage that we can ignore every further base change. In fact, every consideration after base change reproduce the setting we started with. Recall that we carefully developed our preliminary results in Sect. 6.2 in such a general set up. Using the unit section $S \to A$ as a support for a rigidificator of line bundles on A, we can identify $\mathrm{Pic}_{A/S}$ with the rigidified Picard functor; cf. Definition 1.7.1 resp. [15, §8.1].

By the Theorem of the Cube 7.1.6 we can identify $\mathrm{Pic}_{A/S}$ with the functor

$$\mathrm{Cub}_{A/S} : (\text{Rigid } S\text{-Spaces}) \longrightarrow (\text{Sets}),$$

which associates to a rigid space $S' \to S$ over S the set of isomorphism classes of cubical line bundles on $A \times_S S'$; cf. Proposition A.3.4. Using the method of linearizations Sect. 1.7, to study $\mathrm{Cub}_{A/S}$ is equivalent to study the functor

$$\mathrm{Cub}^M_{E/S} : (\text{Rigid } S\text{-Spaces}) \longrightarrow (\text{Sets}),$$

which associates to a rigid analytic space $S' \to S$ over S the set of isomorphism classes of cubical line bundles on $E \times_S S'$ with M-linearizations; see Corollary 6.2.14. Indeed, there is a natural transformation

$$\mathrm{Cub}^M_{E/S} \xrightarrow{\sim} \mathrm{Cub}_{A/S}, \quad (L, c) \longmapsto L(c) := L/c,$$

cf. Proposition 1.7.14. Note that $\mathrm{Cub}^M_{E/S}$ is a sheaf. Locally on S with respect to the Grothendieck topology, a section of $\mathrm{Cub}^M_{E/S}$ can be described in terms of data built on B; cf. Proposition 6.2.10. We have the functor

$$\mathrm{Cub}^M_{E/B/S} : (\text{Rigid } S\text{-Spaces}) \longrightarrow (\text{Sets})$$

which associates to a rigid analytic space $S' \to S$ over S the set of isomorphism classes of couples (N, c), where N is a cubical line bundle on $B \times_S S'$ and c is an M-linearization on q^*N. An isomorphism from (N_1, c_1) to (N_2, c_2) is a isomorphism $\varphi : N_1 \to N_2$ of cubical line bundles such that $q^*\varphi$ is compatible with the linearizations. Thus, $\mathrm{Cub}^M_{E/B/S}$ is a sheaf of abelian groups with respect to the Grothendieck topology, where the group law is the tensor product. Due to Proposition 6.2.10 the morphism

$$\mathrm{Cub}^M_{E/B/S} \longrightarrow \mathrm{Cub}^M_{E/S} = \mathrm{Pic}_{A/S}, \quad (N, c) \longmapsto \big(q^*N, c\big) \cong q^*N(c),$$

is an epimorphism of sheaves with respect to the Grothendieck topology and its kernel is given by the character group $M' = \mathrm{Hom}(T, \mathbb{G}_m)$. Indeed, the character group M', viewed as a constant sheaf, injects into $\mathrm{Cub}^M_{E/B/S}$ via

$$M' = \mathrm{Hom}(T, \mathbb{G}_M) \longrightarrow \mathrm{Cub}^M_{E/B/S}, \quad m' \longmapsto (P_{B\times\phi'(m')}, \varepsilon_{m'}),$$

where $\varepsilon_{m'}$ is the M-linearization on $P_{B\times\phi'(m')}$

$$\varepsilon_{m',m} : P_{B\times\phi'(m')} \longrightarrow \tau^* P_{B\times\phi'(m')}, \; \ell \longmapsto \langle m, m' \rangle \otimes \ell.$$

The canonical section $e_{m'}$ yields a trivialization of $(q^* P_{B\times\phi'(m')}, \varepsilon_{m'})$, and hence the couple $(P_{B\times\phi'(m')}, \varepsilon_{m'})$ induces the trivial line bundle on A. Moreover, if a couple (N, c) induces the trivial line bundle on A, then there exists a character $m' \in M'$ such that $(N, c) = (P_{B\times\phi'(m')}, \varepsilon_{m'})$ due to Corollary 6.2.14. Thus, we obtain an exact sequence of sheaves

$$0 \to M' \to \mathrm{Cub}^M_{E/B/S} \to \mathrm{Cub}^M_{E/S} \to 0.$$

Next we want to have a closer look at the forgetful functor

$$\mathrm{Cub}^M_{E/B/S} \longrightarrow \mathrm{Cub}_{B/S} = \mathrm{Pic}_{B/S}, \quad (N, c) \longmapsto N.$$

The translation invariant M-linearizations of the trivial line bundle on B are represented by the torus $T' := \mathrm{Hom}(M, \mathbb{G}_m)$ which injects into $\mathrm{Cub}^M_{E/B/S}$ in a canonical way. Indeed, for a point $x \in T'$ the M-linearization on the trivial line bundle is defined by the multiplication with $x(m)$ for $m \in M$. Actually it is a closed and open subfunctor of the kernel.

The forgetful functor does not need to be an epimorphism of sheaves, since for a cubical line bundle N on B, which is not translation invariant, there does not necessarily exist an M-linearization on q^*N. Indeed, it can happen that $\varphi_N : B \to B'$ does not admit a lifting $\lambda : M \to M'$, as it is necessary for an M-linearization of q^*N. But on translation invariant line bundles $P_{B\times b'}$ there exist M-linearizations, because φ_N is trivial in this case.

Finally we have the following sequence of sheaves

$$0 \to T' = \mathrm{Hom}(M, \mathbb{G}_m) \to \mathrm{Cub}^M_{E/B/S} \to \mathrm{Pic}_{B/S} \to 0.$$

The sequence is exact if one restricts to translation invariant line bundles and M-linearizations, cf. Proposition 6.2.15. In Sect. 6.2 we have worked out the identification of the M-linearizations of q^*N. Thus, we have the following result:

Theorem 6.3.2. *The canonical morphisms*

$$\mathrm{Cub}^{M}_{E/B/S}/M' \xrightarrow{\sim} \mathrm{Cub}^{M}_{E/S} \xrightarrow{\sim} \mathrm{Pic}_{A/S}$$

are isomorphisms of sheaves with respect to the Grothendieck topology.

If S' is a connected rigid analytic S-spaces, then every element of $\mathrm{Cub}^{M}_{E/B/S}(S')$ can be presented by a triple (N, r, λ), where

N	*is a rigidified line bundle on $B \times_S S'$,*
$\lambda : M \to M'$	*is a group homomorphism,*
$r : M \to \phi^* N$	*is a trivialization,*

satisfying

$$\begin{aligned} &\phi' \circ \lambda = \varphi_N \circ \phi, \\ &r(m_1 + m_2) \otimes r(m_1)^{-1} \otimes r(m_2)^{-1} \otimes r(0)^{-1} = \langle m_1, \lambda(m_2)\rangle \end{aligned}$$

for all $m_1, m_2 \in M$, where $\mathcal{D}_2(N) = (\mathrm{id}_B \times \varphi_N)^* P_{B\times B'}$ are canonically identified in the last equality.

In particular, a couple (L, s_T) consisting of a rigidified line bundle L on A and a trivialization s_T of the cubical line bundle $p^*L|_{\overline{T}}$ corresponds to a triple (N, λ_L, r) in a unique way. Here the couple (N, r) depends on the section s_T whereas the homomorphism λ_L depends only on L, but not on the section.

Now we will concentrate on line bundles which are translation invariant. Due to Corollary 6.2.13 a triple (N, r, λ) is translation invariant if and only if N is translation invariant and λ is trivial. Then the condition on the section r is equivalent to the condition that r is an S'-valued point of E' by Proposition 6.2.15. Thus, the open and closed subfunctor of $\mathrm{Cub}^{M}_{E/B/S}$ representing translation invariant line bundles is represented by E'.

Theorem 6.3.3. *The quotient $A' := E'/M'$ is the dual of $A = E/M$.*

The Poincaré bundle $P_{A\times A'}$ is induced by the $(M \times M')$-linearization on the line bundle $(q \times q')^* P_{B\times B'}$ which is given by the couple (R, Λ), where

$$\begin{aligned} &\Lambda : M \times M' \longrightarrow M' \times M,\ \left(m, m'\right) \longmapsto \left(m', m\right), \\ &R : M \times M' \longrightarrow \left(q \times q'\right)^* P_{B\times B'},\ \left(m, m'\right) \longmapsto \langle m, m'\rangle. \end{aligned}$$

In terms of mappings, it is given by

$$C_{(m,m')}\left(z, z'\right) : P_{B\times B'} \longrightarrow P_{B\times B'},\ \omega \longmapsto \left(\langle z, m'\rangle \star \langle m, m'\rangle\right) \otimes \left(\langle m, z'\rangle \star \omega\right),$$

for $\omega \in P_{q(z)\times q'(z')}$ and $(z, z') \in E$. Here "$\star$" and "$\otimes$" are the group laws on $P_{B\times B'}$ when B' and/or B are viewed as parameter space.

Proof. Recall that the dual of $B \times_S B'$ is given by $B' \times_S B$. One easily shows that (R, Λ) satisfies the condition of Theorem 6.3.2. Thus, it gives rise to a line bundle $P_{A\times A'}$, which is rigidified along $A \times 0'$ and $0 \times A'$.

Now we want to show the universal property. After performing a base change, it suffices to consider line bundles on A. Thus, start with a rigidified line bundle L on A, which is invariant under translation. Since we can work locally with respect to the Grothendieck topology, we may assume that L is isomorphic to q^*N/c, where N is a rigidified line bundle on B and where c is an M-linearization on q^*N, cf. Proposition 6.2.10. Let (r, λ) be the couple associated to c. Due to Corollary 6.2.13 the line bundle N is translation invariant and $\lambda : M \to M'$ is the trivial one. Thus, the rigidified line bundle N is given by a point $b' : S \to B'$; i.e., there is a canonical isomorphism

$$N \xrightarrow{\sim} \left(\mathrm{id}_B \times b'\right)^* P_{B\times B'}$$

due to the universal property of $P_{B\times B'}$. The action c is induced from the left translation on N given by a section $r : M \to \phi^*N$; i.e., a group homomorphism $r : M \to P_{B\times b'}$ which is a lifting of $\phi : M \to B$; cf. Proposition 6.2.15. Such data bijectively correspond to an S-valued points z' of E', because E' is defined by the group homomorphism $\phi : M \to B = B''$; cf. Remark 6.1.3.

Now consider the pull-back of $((q \times q')^* P_{B\times B'}, C)$ under the maps

$$\left(\mathrm{id}_E \times z'\right) : E \times S \longrightarrow E \times_S E',$$
$$(\mathrm{id}_M, \lambda) : M \times M \longrightarrow M \times M'.$$

Since λ is trivial, the pull-back is isomorphic to the given data (q^*N, c). If we define $a' := p' \circ z'$, then we have that $(\mathrm{id}_A \times a')^* P_{A\times A'} \cong L$. □

Corollary 6.3.4. *The morphism $p : E \to A$ is the universal covering in the sense that every morphism $S' \to A$ from a rigid analytic space S' to A with $H^1(S', \mathbb{Z}) = 0$ to A factorizes through p.*

Proof. Since A is the dual of A' and E is the dual of E', it suffices to show that every morphism $S' \to A'$ factorizes through p'. After a base change we may assume $S' = S$. A morphism $\sigma : S \to A'$ is equivalent to a rigidified line bundle L on A which is translation invariant. As explained above, the canonical M-linearization on p^*L gives rise to local sections $\widehat{\sigma}_i : S \to E'$ lifting σ. The differences $\widehat{\sigma}_i \circ \widehat{\sigma}_j^{-1}$ give rise to a cocycle with values in M'. Since $H^1(S, M')$ vanishes, one can arrange the liftings $\widehat{\sigma}_i$ in such a way that they define a morphism $\widehat{\sigma} : S \to E'$. Thus, we see that $p' : E' \to A'$ is the universal covering. □

6.4 Algebraization

In this section we want to show that the universal covering $\widehat{A}$ of an abelian variety is induced by an algebraic R-scheme; i.e., $\widehat{A}$ is a algebraic torus extension of an abelian R-group scheme. In the following keep the notations of Sect. 6.3. At first we will study morphisms between uniformized abeloid varieties.

Proposition 6.4.1. *Let $A_i = E_i/M_i$ be uniformized abeloid varieties, where the Raynaud extension $q_i : E_i \to B_i$ corresponds to a morphism $\phi'_i : M'_i \to B'_i$ as in Proposition 6.1.9. Moreover, let M_i be a lattice in E_i, and set $\phi_i := q_i|_{M_i} : M_i \to B_i$ for $i = 1, 2$.*

(a) *Then there is a canonical bijection*

$$\mathrm{Hom}(A_1, A_2) \xrightarrow{\sim} \mathrm{Hom}\big((E_1, M_1), (E_2, M_2)\big), \quad \varphi_A \longmapsto \varphi_E = \big(\lambda', \varphi_B\big),$$

where φ_E is a lifting of φ_A sending M_1 to M_2.

Due to Proposition 6.1.9 a morphism $\varphi_E : E_1 \to E_2$ is equivalent to a pair (λ', φ_B) of morphisms $\lambda' : M'_2 \to M'_1$ and $\varphi_B : B_1 \to B_2$ with $\phi'_1 \circ \lambda' = \varphi'_B \circ \phi'_2$.

A morphism $\varphi_E : E_1 \to E_2$ maps M_1 to M_2 if and only if for all $m_1 \in M_1$ and $m'_2 \in M'_2$ holds $\langle \varphi_E(m_1), m'_2\rangle = pr(\langle m_1, \lambda'(m'_2)\rangle)$, where pr is the projection $P_{B_1 \times \phi'_1(\lambda'(m'_2))} \to P_{B_2 \times \phi'_2(m'_2)}$.

(b) *Using the notations of Theorem 6.3.3 the dual map*

$$\varphi'_A : A'_2 = E'_2/M'_2 \longrightarrow A'_1 = E'_1/M'_1$$

associated to $\varphi_A : A_1 \to A_2$ is equivalent to a morphism $\varphi'_E : E'_2 \to E'_1$ with $\varphi'_E(M'_2) \subset M'_1$, which is equivalent to a pair $(\lambda, \psi_{B'})$ of morphisms $\lambda : M_1 \to M_2$ and $\psi_{B'} : B'_2 \to B'_1$ with $\phi_2 \circ \lambda = \psi'_{B'} \circ \phi_1$.

The map $\psi_{B'}$ equals the dual map φ'_B of φ_B and $\lambda = \varphi_E|_{M_1}$ if one regards M_i as the lattice of E_i for $i = 1, 2$. Thus $\phi_2 \circ \varphi_E|_{M_1} = \varphi_B \circ \phi_1$.

(c) *Let $A_1 = A$ and $A_2 = A'$ be the dual of A. Identify M_1 with M, M_2 with M', M'_2 with M, and M'_1 with M'. Then we have:*

The morphism $\varphi_A : A \to A'$ from A to its dual A' is equivalent to a couple $(\lambda' : M \to M', \varphi_B : B \to B')$ satisfying $\phi' \circ \lambda' = \varphi'_B \circ \phi$.

If $\varphi_E : E \to E'$ is the lifting of φ_A, then the dual morphism φ'_A of φ_A is equivalent to a couple $(\varphi_E|_M, \varphi'_B)$ satisfying $\phi' \circ \varphi_E|_M = \varphi'_B \circ \phi$.

The map φ_A is symmetric in the sense that $\varphi'_A = \varphi_A$ if and only if $\varphi_E|_M = \lambda'$ and $\varphi_B = \varphi'_B$. The condition $\varphi_E|_M = \lambda'$ is equivalent to

$$\big\langle m_1, \lambda(m_2)\big\rangle = \xi\big(\big\langle m_2, \lambda(m_1)\big\rangle\big) \quad \text{for all } m_1, m_2 \in M.$$

Proof. (a) The restriction of φ_A to $\overline{E}_1$ yields a morphism $\varphi_{\overline{E}} : \overline{E}_1 \to \overline{E}_2$ due to Proposition 5.4.6. The restriction of $\varphi_{\overline{E}}$ to the formal torus $\overline{T}_1$ gives rise to a morphism between $\overline{T}_1$ and $\overline{T}_2$ which is equivalent to a morphism λ' between their character groups, so it extends to a morphism from T_1 to T_2 and hence $\varphi_{\overline{E}}$ extends to a lifting $\varphi_E : E_1 \to E_2$ of φ_A. In particular, φ_E maps M_1 into M_2. Conversely, every morphism $\varphi_E : E_1 \to E_2$ sending M_1 to M_2 gives rise to a morphism $\varphi_A : A_1 \to A_2$ by taking quotients. Thus, we obtain the asserted bijection. Due to Proposition 6.1.9

the map φ_E is equivalent to a pair of maps $\varphi_B : B_1 \to B_2$ and $\lambda' : M_2' \to M_1'$ with $\phi_1' \circ \lambda' = \varphi_B' \circ \phi_2'$. The last asserted condition follows from Proposition 6.1.9(b).

(b) Due to (a) the dual map φ_A' is equivalent to a couple of morphisms $(\lambda, \psi_{B'})$, where $\lambda : M_1 \to M_2$ is a homomorphism of the character groups of the E_i''s, and where $\psi_{B'} : B_2' \to B_1'$ is a morphism of the dual abelian varieties B_i' of B_i satisfying $\phi_2 \circ \lambda = \psi_{B'}' \circ \phi_1$.

Every translation invariant line bundle L on A corresponds to a triple $(N, r, 0)$ in the manner of Theorem 6.3.2, then $\varphi_A^* L$ corresponds to the triple $(\varphi_B^* N, \varphi_E|_{M_1}^* r, 0)$. Thus, we see $\psi_{B'} = \varphi_B'$ and $\lambda = \varphi_E|_{M_1}$.

(c) Follows from (b) and Proposition 6.1.12. □

There is also an analog of Remark 2.7.9 concerning the morphism φ_L.

Proposition 6.4.2. *Let $A := E/M$ be a uniformized abeloid variety and L a line bundle on A. According to Proposition 6.4.1 the morphism*

$$\varphi_L : A \longrightarrow A', \ a \longmapsto \tau_a^* L \otimes L^{-1},$$

corresponds to a couple (λ', φ_B). According to Theorem 6.3.2 the line bundle L corresponds to a triple (N, r, λ_L). Then we have that $\varphi_B = \varphi_N$ and $\lambda' = \lambda_L$.

Proof. The M-linearization associated to (N, λ_L, r) is given by

$$c_m : q^* N \longrightarrow \tau_m^* q^* N, \ f \longmapsto f \otimes r(m) \otimes r(0)^{-1} \otimes e_{\lambda_L(m)}.$$

If $z \in E$, then the M-linearization of $\tau_z^*(N, \lambda_L, r)$ has the form

$$\tau_z^* c_m : \tau_z^* q^* N \longrightarrow \tau_z^* \tau_m^* q^* N, \ f \longmapsto f \otimes r(m) \otimes r(0)^{-1} \otimes \tau_z^* e_{\lambda_L(m)}.$$

Since $\tau_z^* q^* N = q^* \tau_{q(z)}^* N$ and $\tau_z^* e_{\lambda_L(m)} = \langle z, \lambda_L(m) \rangle \cdot e_{\lambda_L(m)}$, we see that

$$\tau_z^* c_m \otimes c_m^{-1} : q^*\big(\tau_{q(z)}^* N \otimes N^{-1}\big) \longrightarrow \tau_m^* q^*\big(\tau_{q(z)}^* N \otimes N^{-1}\big), \ f \longmapsto f \otimes \big\langle z, \lambda_L(m) \big\rangle.$$

Thus, we obtain that $\varphi_B = \varphi_N$ and $\lambda' = \lambda_L$. □

Corollary 6.4.3. *As in Proposition 6.4.1 consider two uniformized abeloid varieties $A_i = E_i/M_i$ for $i = 1, 2$ and a homomorphism $\varphi_A : A_1 \to A_2$ which corresponds to a couple (λ', φ_B). Let L_2 be a line bundle on A_2 which corresponds to data (N_2, r_2, λ_2) in the sense of Proposition 6.2.12. Then the pull-back $\varphi_A^* L$ corresponds to a triple (N_1, r_1, λ_1) which is given in the following way:*

(i) $N_1 := \varphi_B^* N_2$,
(ii) $r_1 := r_2 \circ \varphi_E|_{M_1} : M_1 \longrightarrow \phi_1^* N_1, m_1 \longmapsto r_2(\varphi_E(m_1))$,
(iii) $\lambda_1 := \lambda' \circ \lambda_2 \circ \varphi_E|_{M_1} : M_1 \to M_1'$.

Proof. As already discussed in the proof of Proposition 6.4.1(b), the line bundle $\varphi_A^* L_2$ is associated to the triple $(\varphi_B^* N_2, \varphi_E|_{M_1}^* r, \lambda_1)$. In particular, one obtains that $N_1 = \varphi_B^* N_2$ and $r_1 := r_2 \circ \varphi_E|_{M_1}$. So it remains to show $\lambda_1 = \lambda' \circ \lambda_2 \circ \varphi_E|_{M_1}$. Since $\phi_2' \circ \lambda_2 = \varphi_{N_2} \circ \phi_2$, we obtain by Proposition 6.4.1 the commutative diagram

$$\begin{array}{ccccccc}
M_1 & \xrightarrow{\varphi_E|_{M_1}} & M_2 & \xrightarrow{\lambda_2} & M_2' & \xrightarrow{\lambda'} & M_1' \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_2'} & & \downarrow{\scriptstyle \phi_1'} \\
B_1 & \xrightarrow{\varphi_B} & B_2 & \xrightarrow{\varphi_{N_2}'} & B_2' & \xrightarrow{\varphi_B'} & B_1'.
\end{array}$$

To determine λ_1 consider the composition $\varphi_{A'} \circ \varphi_{L_2} \circ \varphi_A$, which is equal to $\varphi_{\varphi_{A_1}^* L_2}$. This implies the equality $\lambda_1 = \lambda' \circ \lambda_2 \circ \varphi_E|_{M_1}$ for the homomorphisms of their character groups by Propositions 6.4.1 and 6.4.2. □

Theorem 6.4.4. *Let A be a uniformized abeloid variety over a non-Archimedean field K. Let $\mathcal{L}$ an rigidified invertible sheaf on A and let $(\mathcal{N}, r, \lambda)$ be the associated data via Theorem 6.3.2. Then the following conditions are equivalent*:

(a) *$\mathcal{L}$ is ample in the sense of Definition 1.7.3.*
(b.1) *$\mathcal{N}$ is ample in the sense of Definition 1.7.3 and*
(b.2) *$\langle \cdot, \lambda(\cdot) \rangle$ is positive definite on $M \times M$.*

If the conditions are satisfied, we have the following formula

$$\dim H^0(A, \mathcal{L}) = \#\big(M'/\lambda(M)\big) \cdot \dim H^0(B, \mathcal{N}).$$

Proof. Consider the mapping

$$\varphi_{\mathcal{L}} : A \longrightarrow A',\ x \longmapsto \tau_x^* \mathcal{L} \otimes \mathcal{L}^{-1}.$$

Due to Proposition 6.4.1(c) the lifting $\varphi_E : E \to E'$ of $\varphi_{\mathcal{L}}$ corresponds to a couple (λ', φ_B). Then we have that $\varphi_B = \varphi_{\mathcal{N}}$ and $\lambda' = \lambda$ by Proposition 6.4.2.

By Lemma 7.1.9 an invertible sheaf $\mathcal{I}$ on A or B is ample if and only if $\mathcal{I}$ has a non-trivial global section and $\varphi_{\mathcal{I}}$ is an isogeny. Thus, the equivalence of (a) and (b) becomes a question about global sections of $\mathcal{L}$ and $\mathcal{N}$ on the one hand, about $\varphi_{\mathcal{L}}$ and $(\lambda, \varphi_{\mathcal{N}})$ on the other hand. Obviously, $\varphi_{\mathcal{L}}$ is an isogeny if and only if $(\lambda, \varphi_{\mathcal{N}})$ is an isogeny; i.e., λ is injective and $\varphi_{\mathcal{N}}$ is an isogeny. The existence of a non-trivial global section of $\mathcal{L}$ is equivalent to the positivity of the form $\langle \cdot, \lambda(\cdot) \rangle$ and the existence of a non-trivial global section of $\mathcal{N}$ due to Corollary 6.2.20. Note that an ample sheaf on an abelian variety has always a non zero section, because the Euler characteristic of L given by $\chi(L) = \dim \Gamma(A, \mathcal{L})$ which is positive; cf. [74, §16 and §17].

The formula for the dimensions follows from the proof of Remark 6.2.21. □

Definition 6.4.5. Let E be a Raynaud extension and $M \subset E$ a lattice of full rank and character group M'. A *polarization* of (E, M) consists of

(i) a group homomorphism $\lambda_M : M \to M'$,
(ii) a polarization $\varphi_B : B \to B'$

such that the following conditions are satisfied:

(a) $\phi' \circ \lambda_M = \varphi_B \circ \phi$,
(b) $\langle m_1, \lambda_M(m_2)\rangle = \xi(\langle m_2, \lambda_M(m_1)\rangle)$ for all $m_1, m_2 \in M$,
(c) $\langle \cdot, \lambda_M(\cdot)\rangle$ is symmetric and positive definite on $M \times M$,

where ξ is the flipping of the factors in the sense of Remark 6.1.10.

If φ_B is a polarization, then $\varphi_B = \varphi'_B$ is symmetric and the morphism $\varphi_B \otimes_K \overline{K}$ is of the form φ_N for an ample invertible sheaf $\mathcal{N}$ on $B \otimes_K \overline{K}$, where $\overline{K}$ is an algebraic closure of K; cf. [72, Chap. VI, §2, Def. 6.2].

Theorem 6.4.6. *In the situation of Definition 6.4.5 let $A = E/M$ be the quotient. Then there is a one-to-one correspondence between the set of polarizations φ_A of A and the set of polarizations (λ_M, φ_B) of (E, M).*

The degrees are related by the formula $\deg \varphi_A = (\deg \lambda_M)^2 \cdot \deg \varphi_B$.

Proof. Let us start with a polarization $\varphi_A : A \to A'$. The morphism φ_A has a lifting $\varphi_E : E \to E'$ with $\varphi_E(M) \subset M'$. Due to Proposition 6.4.1, the morphism φ_E is equivalent to a couple (λ_M, φ_B) satisfying $\phi' \circ \lambda_M = \varphi_B \circ \phi$. To verify the conditions of Definition 6.4.5, we may assume that $\varphi_A = \varphi_{\mathcal{L}}$ is given by an ample invertible sheaf $\mathcal{L}$. Now $\mathcal{L}$ corresponds to a triple $(\mathcal{N}, r, \lambda)$ in the manner of Theorem 6.3.2. By Proposition 6.4.2 the couple (λ_M, φ_B) equals the couple $(\lambda, \varphi_{\mathcal{N}})$. Thus, we see by Theorem 6.4.4 that the invertible sheaf $\mathcal{N}$ is ample and hence φ_B is a polarization and that the condition (c) is fulfilled. Since $\varphi_{\mathcal{L}}$ is symmetric as explained in Remark 6.1.10, the condition (b) follows from Proposition 6.1.12.

Conversely, a couple (λ, φ_B) with $\phi' \circ \lambda_M = \varphi'_B \circ \phi$ induces a morphism $\varphi_E : E \to E'$. The symmetry condition (b) of Definition 6.4.5 implies $\varphi_E(M) \subset M'$ by Proposition 6.1.12. Thus, φ_E induces a morphism $\varphi_A : A \to A'$ on the quotients. Now assume that $\varphi_B = \varphi_{\mathcal{N}}$ with an ample invertible sheaf $\mathcal{N}$ on B. Then the map φ_A is induced by an invertible sheaf $\mathcal{L}$ on A if there is a trivialization $r : M \to q^*\mathcal{N}$ which is compatible with λ_M in the sense of Theorem 6.3.2. One defines the map $r : M \to q^*\mathcal{N}$ by choosing $r(m_i) \in \mathcal{N}_{q(m_i)}$ for a basis $m_1, \ldots, m_r$ of M and by extending it as prescribed by the formula in Theorem 6.3.2. In fact, use the form and the canonical isomorphism $\mathcal{D}_2(\mathcal{N}) = (\mathrm{id}_B \times \varphi^*_{\mathcal{N}})\mathcal{P}_{B \times B'}$. Then, it follows from Theorem 6.4.4 that the invertible sheaf associated to the data $(\mathcal{N}, r, \lambda_M)$ is ample, and hence λ_A is a polarization.

The formula for the degrees follows from Theorem 6.4.4. Indeed, φ_A is given by an invertible sheaf $\mathcal{L}$ on A corresponding to a triple $(\mathcal{N}, r, \lambda_M)$. Then φ_B is given by $\varphi_{\mathcal{N}}$ and φ_A corresponds to $(\lambda_M, \varphi_{\mathcal{N}})$. Now we have that $\dim \Gamma(B, \mathcal{N})^2 = \deg \varphi_{\mathcal{N}}$ and $\dim \Gamma(A, \mathcal{L})^2 = \deg \varphi_A$ due to [74, Sect. 16]. Thus, the formula follows from Theorem 6.4.4. □

Proposition 6.4.7. *In the situation of Theorem 6.4.6 we have the following*:

(a) *If* $\varphi_A = (\lambda_M, \varphi_B)$ *is a polarization on* A, *then* $\mathcal{L} := (\mathrm{id}_A, \varphi_A)^* \mathcal{P}_{A\times A'}$ *is ample, and hence* A *is the analytification of an abelian variety. The morphism* $\varphi_{\mathcal{L}}$ *is associated to* $(2\lambda_M, \varphi_{\mathcal{N}})$ *with* $\mathcal{N} := (\mathrm{id}_B, \varphi_B)^* \mathcal{P}_{B\times B'}$.
(b) *If* $\varphi_A = \varphi_{\mathcal{F}}$ *in* (a) *with an ample sheaf* $\mathcal{F}$, *then* $\mathcal{L} = \mathcal{F} \otimes [-1]_A^* \mathcal{F}$ *in* (a).

Proof. (b) Since $\mathcal{F}$ is ample, A is an abelian variety by Proposition 7.1.10. Then the assertion follows from [72, Chap. VI, §2, Prop. 6.10].

(a) This can be checked after base change to an algebraic closure of K, and then it follows from (b), because φ_A is of the form $\varphi_{\mathcal{F}}$. Moreover, the morphism $\varphi_{\mathcal{L}}$ is associated to the couple $(2\lambda_M, \varphi_{\mathcal{N}})$, where $\mathcal{N} := (\mathrm{id}_B, \varphi_B)^* \mathcal{P}_{B\times B'}$ by Proposition 6.4.2. □

Theorem 6.4.8. *In the situation of Theorem 6.4.6, if* A *is an abelian variety, then* B *is an abelian variety with good reduction and* E *is an algebraic torus extension of* B.

Proof. If A is an abelian variety, then there exists an ample invertible sheaf $\mathcal{L}$ on A. By Theorem 6.3.2 the invertible sheaf $\mathcal{L}$ is associated to a triple $(\mathcal{N}, r, \lambda)$. By Theorem 6.4.4 the invertible sheaf $\mathcal{N}$ is ample, and hence B is an abelian variety. Since E is associated to an r-tuple of formal line bundles $E_1, \ldots, E_r$, the extension $E \to B$ is algebraic if and only if the line bundles $E_1, \ldots, E_r$ are algebraic. The latter is true due to the GAGA Theorem 1.6.11. □

Corollary 6.4.9. *Let* X_K *be a smooth projective curve with semi-stable reduction. Then its Jacobian* $J_K := \mathrm{Jac}(X_K)$ *admits a Raynaud representation* $J_K = \widehat{J}_K/M$, *where* $M \subset \widehat{J}_K$ *is a lattice and* $\widehat{J}_K$ *is an algebraic torus extension of an abelian variety with good reduction.*

In the next section we will discuss the canonical polarization of $\mathrm{Jac}(X_K)$.

6.5 Polarization of Jacobians

In Sect. 2.8 we constructed the Jacobian T_K/M of a Mumford curve by using automorphy factors. This approach intrinsically creates a polarization of Definition 2.8.1; i.e., a morphism $\lambda : M \to M'$ from the lattice M of T_K to the character group M' of T_K. The associated pairing $\langle m_1, \lambda(m_2)\rangle$ for $m_1, m_2 \in M$ is just the evaluation of the characters at the points of the lattice. Therefore, this polarization is called the canonical one in Definition 2.8.1.

Due to Corollary 2.9.16 of Riemann's vanishing theorem the canonical polarization coincides with the theta polarization. Since automorphy factors are closely related to linearizations, a similar approach for the Jacobian J_K of a curve X_K with not necessarily split reduction $\widetilde{X}$ should work as well.

For simplicity we assume in the following that the base field K is algebraically closed and that X_K is a connected smooth projective curve with a semi-stable reduction $\widetilde{X}$, whose irreducible components are smooth. To start with we reassemble our results on the Jacobian J_K of X_K in order to point out the features used in the following.

In Proposition 5.2.3 there is an explicit presentation of $\widetilde{J} := \operatorname{Jac}\widetilde{X}$ as a torus extension

$$1 \to \widetilde{T} \to \widetilde{J} \to \widetilde{B} \to 1 \quad \widehat{=} \quad \widetilde{\phi}' : H_1(\widetilde{X}, \mathbb{Z}) \to \widetilde{B}', \tag{$\widetilde{\dagger}$}$$

where $H_1(\widetilde{X}, \mathbb{Z}) := H_1(\varGamma(\widetilde{X}), \mathbb{Z})$ is the first homology group of the graph $\varGamma(\widetilde{X})$ of coincidence of the irreducible components of $\widetilde{X}$. Here $\widetilde{B} = \operatorname{Jac}\widetilde{X}'$ is the Jacobian of the normalization $\widetilde{X}' \to \widetilde{X}$ of $\widetilde{X}$ and $\widetilde{B}'$ is the dual of $\widetilde{B}$. In particular, $H_1(\widetilde{X}, \mathbb{Z})$ is regarded as the character group of $\widetilde{J}$.

Let $\tilde{p}_X : \widehat{\widetilde{X}} \to \widetilde{X}$ be the universal covering of $\widetilde{X}$, which resolves the circuits in the configuration of the irreducible components of $\widetilde{X}$ and let $\varGamma := \operatorname{Deck}(\widehat{\widetilde{X}}/\widetilde{X})$ be its deck transformation group. Moreover, we fix a *smooth point* $\widehat{\tilde{x}}_0 \in \widehat{\widetilde{X}}$ with image $\tilde{x}_0 \in \widetilde{X}_0$. Then there is a canonical morphism $\varGamma \to H_1(\widetilde{X}, \mathbb{Z})$ given by

$$\varGamma \longrightarrow H_1(\widetilde{X}, \mathbb{Z}), \quad \gamma \longmapsto c_\gamma := \overline{\widehat{\tilde{x}}_0, \gamma(\widehat{\tilde{x}}_0)},$$

where c_γ is the image of the path from $\widehat{\tilde{x}}_0$ to $\gamma(\widehat{\tilde{x}}_0)$. The map is surjective and represents $H_1(\widetilde{X}, \mathbb{Z})$ as the maximal abelian quotient of $\varGamma$. We denote the residue class map by

$$\tilde{h}' : H := \varGamma_{\mathrm{ab}} := \varGamma/[\varGamma, \varGamma] \xrightarrow{\sim} H_1(\widetilde{X}, \mathbb{Z}), \; \overline{\gamma} \longmapsto c_\gamma,$$

and hence we obtain the geometric interpretation of $\widetilde{\phi}'$

$$\widetilde{\phi}' : H_1(\widetilde{X}, \mathbb{Z}) \longrightarrow \widetilde{B}', \; c \longmapsto -\varphi_{\widetilde{\Theta}}(c),$$

where $\varphi_{\widetilde{\Theta}} : \widetilde{B} \to \widetilde{B}'$ is the theta polarization on $\widetilde{B}$; cf. Proposition 5.2.3. Later on we will see that $\varphi_{\widetilde{\Theta}}$ is the reduction of the theta polarization φ_Θ on $\overline{J}_K$. In this way $H_1(\widetilde{X}, \mathbb{Z})$ is viewed as the character group $\widetilde{M}'$ of $\widetilde{T}$.

In Sect. 5.3 we defined an admissible open subgroup $\overline{J}_K$ of $J_K := \operatorname{Jac} X_K$ which admits a smooth formal R-model $\overline{J}$ with reduction $\widetilde{J}$. The formal scheme $\overline{J}$ is a formal torus extension

$$1 \to \overline{T} \to \overline{J} \to B \to 1 \quad \widehat{=} \quad \phi' : M' \to B', \tag{$\overline{\dagger}$}$$

where $M' = \operatorname{Hom}(\overline{T}, \overline{\mathbb{G}}_{m,R})$ is the character group of the formal torus $\overline{T}$ and B' is dual of the formal abelian R-scheme B. Due to Corollary 5.5.6 the canonical morphism $M' \to \widetilde{M}'$ is bijective; i.e., every character $\widetilde{J} \to P_{\widetilde{B} \times \widetilde{\phi}'(m')}$ uniquely lifts to a character $\overline{J} \to P_{B \times \phi'(m')}$. The extension ($\overline{\dagger}$) is a lifting of the extension ($\widetilde{\dagger}$).

Thus, the diagram

$$\begin{array}{ccc} M' := \mathrm{Hom}(\overline{T}, \overline{G}_{m,K}) & \xrightarrow{\quad \phi' \quad} & B' \\ \Big\| & & \Big\downarrow \\ \widetilde{M}' := \mathrm{Hom}(\widetilde{T}, \mathbb{G}_{m,k}) & \xrightarrow{\quad \widetilde{\phi}' \quad} & \widetilde{B}' \end{array}$$

is commutative. In particular, we obtain an isomorphism

$$h' : H := H_1(\widetilde{X}, \mathbb{Z}) \xrightarrow{\sim} M' := \mathrm{Hom}(\overline{T}, \overline{\mathbb{G}}_{m,K}).$$

Lemma 6.5.1. *The isomorphism $\tilde{h}' : H \longrightarrow \widetilde{M}'$ factorizes through*

$$h' : H := H_1(\widetilde{X}, \mathbb{Z}) \xrightarrow{\sim} M' := \mathrm{Hom}(\overline{T}, \overline{\mathbb{G}}_{m,K}).$$

In particular, the reduction of $\phi'(h'(\gamma))$ coincides with $-\varphi_{\widetilde{\Theta}}([c_\gamma])$; cf. Proposition 5.2.3.

The T_K-extension

$$1 \to T_K \to \widehat{J}_K \to B_K \to 1 \quad \widehat{=} \quad \phi' : M' \to B'_K, \tag{†}$$

of B_K fits into a commutative diagram

$$\begin{array}{ccc} \overline{T}_K & \hookrightarrow & T_K \\ \big\downarrow & & \big\downarrow \\ \overline{J}_K & \hookrightarrow & \widehat{J}_K . \end{array}$$

Thus, the group of characters can be identified in the following way

$$M' := \mathrm{Hom}(\overline{T}_K, \overline{\mathbb{G}}_{m,K}) = \mathrm{Hom}_B(\overline{J}_K, P_{B\times B'}) = \mathrm{Hom}_B(\widehat{J}_K, P_{B\times B'}),$$

where $P_{B\times B'}$ is the Poincarѐ bundle on $B \times B'$. The second and third identification are defined by the push-forward; cf. Sect. A.2.

For the set of K-valued points of $\widehat{J}_K$ there is a canonical identification

$$\widehat{J}_K \xrightarrow{\sim} \mathrm{Hom}_{B'}\big(M', P_{B\times B'}\big), \quad t \longmapsto \big[m' \longmapsto m'(t) \in P_{B\times\phi'(m')}\big],$$

where M' is regarded as a B'-space via ϕ'; cf. Notation 6.1.7.

In Definition 4.6.4 we introduced the universal covering $p_X : \widehat{X}_K \to X_K$. The reduction of $\widehat{X}_K$ is given by $\widehat{\widetilde{X}}$ and its deck transformation group is canonically identified with Γ. Thus, we have the commutative diagram

$$\begin{array}{ccc} \widehat{X}_K & \xrightarrow{\ \widehat{\rho}\ } & \widehat{\widetilde{X}} \\ \Big\downarrow{\scriptstyle p_X} & & \Big\downarrow{\scriptstyle \tilde{p}_X} \\ X_K & \xrightarrow{\ \rho\ } & \widetilde{X} \end{array}$$

and Γ acts on it equivariantly. Furthermore, we fix a point $\widehat{x}_0 \in \widehat{X}$ with $\widehat{\rho}(\widehat{x}_0) = \widehat{\tilde{x}}_0$ and set $x_0 = p_X(\widehat{x}_0)$. Due to Corollary 6.3.4 there is a canonical commutative diagram

$$\begin{array}{ccc} \widehat{X}_K & \xrightarrow{\ \widehat{\iota}\ } & \widehat{J}_K \\ \Big\downarrow{\scriptstyle p_X} & & \Big\downarrow{\scriptstyle p_J} \\ X_K & \xrightarrow{\ \iota\ } & J_K \end{array} \qquad \begin{array}{ccc} \widehat{x} & \longmapsto & \widehat{\iota}(\widehat{x}) \\ \Big\downarrow & & \Big\downarrow \\ p_X(\widehat{x}) & \longmapsto & [p_X(\widehat{x}) - x_0] \end{array}$$

depending on the base points. The map $p_J : \widehat{J}_K \to J_K$ is surjective and its kernel is a lattice; cf. Theorem 5.5.11.

Lemma 6.5.2. *In the above situation we have for every* $\gamma \in \Gamma$:

(i) $\widehat{\iota}(\gamma(z)) \cdot \widehat{\iota}(z)^{-1} = \widehat{\iota}(\gamma(z_0)) \cdot \widehat{\iota}(z_0)^{-1}$ *for all* $z, z_0 \in \widehat{X}_K$.
(ii) $\widehat{\iota}(\gamma(z)) = \widehat{\iota}(z) \cdot m_\gamma$ *with* $m_\gamma := \widehat{\iota}(\gamma(\widehat{x}_0))$ *for all* $z \in \widehat{X}_K$.
(iii) *The map* $h : H \to \widehat{J}_K, \gamma \longmapsto m_\gamma := \widehat{\iota}(\gamma(\widehat{x}_0))$, *is a homomorphism.*

Proof. (i) The images of both sides of the equation under p_J are the unit element of J_K. Since the fibers of $p_J : \widehat{J}_K \to J_K$ are discrete and $\widehat{X}_K$ is connected, the assertion is clear.

(ii) Consider the image under p_J. Then it follows from (i)

$$\begin{aligned} p_J \circ \widehat{\iota} \circ \gamma(z) &= \big[p_X\big(\gamma(z)\big) - p_X(\widehat{x}_0)\big] \\ &= \big[p_X\big(\gamma(z)\big) - p_X(z)\big] \cdot \big[p_X(z) - p_X(\widehat{x}_0)\big] \\ &= \big[p_X\big(\gamma(\widehat{x}_0)\big) - p_X(\widehat{x}_0)\big] \cdot \big[p_X(z) - p_X(\widehat{x}_0)\big] \\ &= p_J\big(\widehat{\iota}\big(\gamma(\widehat{x}_0)\big)\big) \cdot p_J\big(\widehat{\iota}(z)\big). \end{aligned}$$

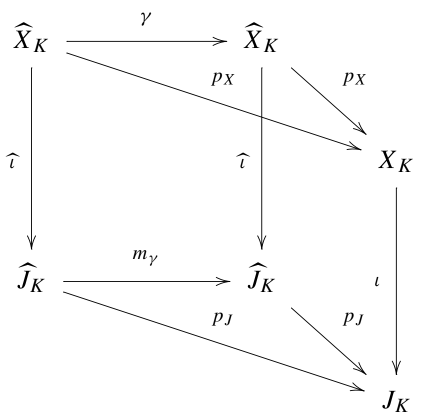
Thus, we obtain $\widehat{\iota}(\gamma(z)) = \widehat{\iota}(z) \cdot \widehat{\iota}(\gamma(\widehat{x}_0)) \cdot m$ for some $m \in \operatorname{Ker} p_J$. Since the fibers of $p_J : \widehat{J}_K \to J_K$ are discrete and $\widehat{X}_K$ is connected, the last equation is valid for all $z \in \widehat{X}_K$ and hence m is independent of z. Inserting $z = \widehat{x}_0$ shows that $m = \widehat{\iota}(\widehat{x}_0) = 0$, and hence $\widehat{\iota}(\gamma(z)) = \widehat{\iota}(z) \cdot m_\gamma$.

(iii) This follows from (ii), because

$$m_{\alpha\beta} = \widehat{\iota}\big(\alpha\beta(\widehat{x}_0)\big) = \widehat{\iota}\big(\beta(\widehat{x}_0)\big) \cdot m_\alpha = m_\beta \cdot m_\alpha.$$

Since $\widehat{J}_K$ is commutative, the map $\Gamma \to \widehat{J}_K$ factorizes through its maximal abelian quotient. Thus, it induces a homomorphism from H to $\widehat{J}_K$. □

Lemma 6.5.2(ii) yields the following commutative diagram



(∗)

for every deck transformation $\gamma \in \Gamma$, where the lower horizontal map m_γ is the multiplication by m_γ. Thus, we obtain the group homomorphisms

$$\begin{array}{ccccccc} \Gamma & \longrightarrow & H_1(\widetilde{X}, \mathbb{Z}) & \xrightarrow{h} & \widehat{J}_K & \xrightarrow{q} & B_K \\ \gamma & \longrightarrow & \overline{\gamma} & \longrightarrow & m_\gamma := \widehat{\iota}(\gamma(\widehat{x}_0)) & \longrightarrow & q(m_\gamma). \end{array}$$

The map h is injective. Indeed, otherwise the rank of $h(H)$ is less than the dimension of the torus part T_K of $\widehat{J}_K$ and hence the variety $\widehat{J}_K/h(H)$ is not proper. So the Abel-Jacobi map $(\widehat{X}/\Gamma)^{(g)} \to \widehat{J}_K/h(H) \to J_K$ could not be surjective, because $(\widehat{X}/\Gamma)^{(g)} = X^{(g)}$ is proper.

Lemma 6.5.3. *In the above situation we have an isomorphism*

$$h : H \xrightarrow{\sim} M := \operatorname{Ker} p_J \subset \widehat{J}_K.$$

Proof. The map h maps H into $\operatorname{Ker} p_J$, since for every $\gamma \in \Gamma$ the point

$$p_J\big(\widehat{\iota}(\gamma(\widehat{x}_0))\big) = \big[p_X\big(\gamma(\widehat{x}_0)\big)\big] - p_X(\widehat{x}_0) = [x_0 - x_0]$$

is the unit element of J_K. Since h is injective, the rank of $h(M)$ is equal to the rank of H. As the rank of H is equal to the torus part of $\widehat{J}_K$, it is equal to the rank of M. Thus, the index of $h(H)$ in M is finite, and hence $h(H)$ is a lattice of full rank in $\widehat{J}_K$. Furthermore, we obtain a surjective morphism

$$\varphi : \widehat{J}_K/h(H) \longrightarrow \widehat{J}_K/\ker p_J$$

of connected proper rigid analytic groups. By reason of dimensions the map φ is finite. Moreover, φ is étale. Thus, the fibers of φ consists of finitely many reduced points. There is a commutative diagram departing from the symmetric product

$$\begin{array}{ccccc} \widehat{X}^{(g)} & & \xrightarrow{\quad p_X \quad} & & X^{(g)} \\ & \searrow & & \swarrow & & \searrow \psi \\ & & \widehat{J}_K/h(H) & \xrightarrow{\quad \varphi \quad} & & \widehat{J}_K/M. \end{array}$$

The fibers of ψ are connected, as they are projective spaces; cf. [68, 3.14]. Therefore, the fibers of φ are connected. So φ is an isomorphism. □

By Proposition 6.1.8 the lattice $M \subset \widehat{J}_K$ gives rise to a bilinear form

$$\langle \_, \_ \rangle : M \times M' \longrightarrow P_{B\times B'},\ \left(m, m'\right) \longmapsto \left\langle m, m' \right\rangle.$$

Due to Theorem 6.3.3 the dual J'_K of J_K is the quotient $\widehat{J}'_K/M'$ of the affine torus extension

$$T'_K \to \widehat{J}'_K \to B'_K \quad \widehat{=} \quad \phi' : M \to B = B''_K$$

by the lattice M', where M' is canonically embedded into $\widehat{J}'_K$ via the map

$$M' \longrightarrow \widehat{J}'_K, m' \longmapsto \begin{bmatrix} m' : M \to P_{B\times B'} \\ \quad m \longmapsto \langle m, m' \rangle \end{bmatrix}.$$

The character m_γ of T'_K induced by $\widehat{\iota}(\gamma(\widehat{x}_0))$ sends $t' \in T'_K$ to the multiplier of the M-linearization of $\mathbb{A}^1_{\widehat{J}_K}$ over the action of m_γ on $\widehat{J}_K$.

Next, let us recall the theta polarization of J_K. Associated to the morphism $\iota : X_K \longrightarrow J_K, x \longmapsto [x - x_0]$, for every $n \in \mathbb{N}$ we have the morphism

$$\iota^{(n)} : X^{(n)} \longrightarrow J_K,\ x_1 + \cdots + x_n \longmapsto [x_1 - x_0] \otimes \cdots \otimes [x_n - x_0].$$

Let $g \geq 1$ be the genus of X_K. Then $\iota^{(g)}$ is birational and the image Θ of $\iota^{(g-1)}$ is an effective divisor on J_K. We call Θ the *theta divisor*.

The divisor $\Theta \subset J_K$ gives rise to the morphism

$$\varphi_\Theta : J_K \longrightarrow J'_K,\ a \longmapsto \left[\tau_a^* \mathcal{O}_{J_K}(\Theta) \otimes \mathcal{O}_{J_K}(-\Theta)\right].$$

Due to Theorem 5.1.6(e) the map $-\varphi_\Theta$ is the inverse of the autoduality map

$$\varphi' : J'_K \longrightarrow J_K,\ a' \longmapsto \left[\iota^* P_{J\times a'}\right].$$

These maps give rise to a commutative diagram of isomorphisms

$$\begin{array}{ccccc} H^1(J'_K,\mathbb{Z}) & \xrightarrow{\sim} & H^1(J_K,\mathbb{Z}) & \xrightarrow{\sim} & H^1(X,\mathbb{Z}) \\ \| & & & & \| \\ \mathrm{Hom}(\mathbb{G}_{m,K},J''_K) & = & = & = & \mathrm{Hom}(\mathbb{G}_{m,K},J_K) \end{array}$$

In the upper row the morphisms are defined via the pull-backs by $\varphi^*_{-\Theta}$ and ι^*, where the first is bijective. The identification in the lower row is due to the identification of J_K and its bi-dual. The vertical identifications are induced by the universal properties of J''_K and J_K, because every cocycle n in $H^1(J'_K,\mathbb{Z})$ and in $H^1(X,\mathbb{Z})$ gives rise to line bundle on $J'_K \times \mathbb{G}_{m,K}$ and on $X_K \times \mathbb{G}_{m,K}$, respectively; cf. Corollary 5.5.6.

The map φ_Θ admits a lifting

$$\widehat{\varphi}_\Theta : \widehat{J}_K \longrightarrow \widehat{J}'_K$$

of their uniformizations. By Proposition 6.1.9 the map $\widehat{\varphi}_\Theta$ is equivalent to a couple $(\lambda_\Theta,\varphi_B)$, where

$$\lambda_\Theta : M = \mathbb{X}\big(\widehat{J}'_K\big) \longrightarrow M' = \mathbb{X}(\widehat{J}_K)$$

is a homomorphism of their character groups and $\varphi_B : B \to B'$ is a morphism from B to its dual B' such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\lambda_\Theta} & M' \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} \\ B = B'' & \xrightarrow{\varphi'_B=\varphi_B} & B' \end{array}$$

is commutative. Viewing M as the lattice of $\widehat{J}_K$ and M' as the lattice of J'_K, one knows $\widehat{\varphi}_\Theta|_M = \lambda_\Theta$ by Proposition 6.4.1(c). Due to Corollary 6.1.11 the image $\widehat{\varphi}_\Theta(m_1)$ of $m_1 \in M$ is given by the point

$$\big\langle m, \widehat{\varphi}_\Theta(m_1)\big\rangle = \xi\left(\big\langle m_1, \lambda_\Theta(m)\big\rangle\right) \quad \text{for } m \in M$$

where ξ is the symmetry on the pull-back $(\mathrm{id}_B \times \varphi_B)^* P_{B\times B'}$ over $B \times B$; i.e., the canonical morphism $\xi : P_{b_1\times\varphi_B(b_2)} \xrightarrow{\sim} P_{b_2\times\varphi_B(b_1)}$ for all points $b_1, b_2 \in B$. Especially, we obtain the following formula.

Remark 6.5.4. In the above situation we have the following formula

$$\big\langle m_1, \lambda_\Theta(m_2)\big\rangle = \xi\left(\big\langle m_2, \lambda_\Theta(m_1)\big\rangle\right) \quad \text{for all } m_1, m_2 \in M.$$

Let us summarize the results which we obtained so far; cf. Corollary 6.4.9.

Theorem 6.5.5. *In the above situation we have the following results*:

(o) *The cycle group* $H_1(\widetilde{X}, \mathbb{Z})$ *admits two canonical isomorphism*; *on the one hand* $h : H \xrightarrow{\sim} M \subset \widehat{J}_K$ *to the lattice of* $\widehat{J}_K$ *and on the other hand* $h' : H \xrightarrow{\sim} M' = \mathrm{Hom}_B(\widehat{J}_K, P_{B\times B'})$ *to the character group of* $\widehat{J}_K$.

In particular, M' *can also be regarded as the lattice of* $\widehat{J}'_K$.

(i) *The quotient* $\widehat{J}_K/M = J_K$ *is the Raynaud representation of* J_K.

(ii) *The quotient* $\widehat{J}'_K/M'$ *is the Raynaud representation of the dual* J'_K.

(iii) *The abelian part* B *is an abelian* R*-scheme and the torus extension* $\widehat{J}_K$ *is the analytification of an affine torus extension of* $B \otimes_R K$.

One can also write down the universal sheaf on $X_K \times J_K$, but this requires more information on the Poincaré bundle on $J_K \times J'_K$ and on the auto-duality of Jacobians as well.

Remark 6.5.6. In the situation of Theorem 6.5.5 every closed point of $\widehat{J}_K$ represents an isomorphism class of a pair (L_K, c), where L_K is a line bundle on $\widehat{X}_K$, which admits a formal extension on the formal model $\widehat{X}$ of $\widehat{X}_K$ with degree 0 on each irreducible component of the reduction of $\widehat{X}$ and where c is a Γ-linearization on L_K.

Such a pair (L_K, c) induces the trivial line bundle on X_K if and only if there exists an $\gamma \in H$ such that $L_K \cong \widehat{\iota}^* q^* P_{B_K \times q' \circ \lambda_\Theta(h(\gamma))}$ and c is the canonical Γ-linearization of its trivialization by $\varepsilon_{\lambda_\Theta(h(\gamma))} := \widehat{\iota}^* e_{\lambda_\Theta(h(\gamma))}$; i.e., the equality $c(\alpha) = \langle h(\alpha), \lambda_\Theta(h(\gamma))\rangle$ for $\alpha \in \Gamma$.

Proof. Due to Corollary 5.1.7(c) every line bundle L_K of degree zero on X_K is a pull-back $\iota^* P_{J_K \times a'}$ for a unique point $a' \in J'_K$. Moreover, $P_{J_K\times a'}$ is induced by an M-linearization (r, λ) on $q^* P_{B\times b'}$ for a unique $b' := q'(a') \in B'$ due to Theorem 6.3.2. Since $P_{J_K\times a'}$ is translation invariant, we have that $\lambda = 0$, and hence $r : M \to P_{B_K\times b'}$ is a group homomorphism. Therefore, the pull-back $\widehat{\iota}^* q^* P_{B_K\times b'}$ is a formal line bundle on $\widehat{X}_K$ and its reduction has degree zero on every irreducible component of $\widehat{\widehat{X}}$. The Γ-linearization c is induced by the M-linearization r via the residue map and $h : H \to M$; note that the absolute value $|r(m)|$ can be different from 1.

The Γ-linearization induces the trivial line bundle on X_K if and only if b' lies in $\phi'(M')$ and c is given by some $m' \in M'$. Then one can write $m' = \lambda_\Theta(h(\overline{\gamma}))$ for a unique $\overline{\gamma} \in H$. □

In the following we will compute the absolute value $|\langle h(\alpha), h'(\beta)\rangle|$ of the pairing in terms of data of the semi-stable reduction of the curve X_K. Therefore we fix the situation for the following.

Notation 6.5.7. Let X_K be a connected smooth projective curve of genus $g \geq 1$. Let $\rho : X_K \to \widetilde{X}$ be a semi-stable reduction such that its irreducible components are smooth. Let $\widetilde{X}' \to \widetilde{X}$ be the normalization of $\widetilde{X}$ and $\tilde{z}_1, \ldots, \tilde{z}_N$ the double points of $\widetilde{X}$. Let r be the rank of the torus part of $\widetilde{J}$ and assume that the first points $\tilde{z}_1, \ldots, \tilde{z}_r$ are chosen in such a way that $\widetilde{X} - \{\tilde{z}_1, \ldots, \tilde{z}_r\}$ is a tree-like configuration of irreducible components. Every cycle $c \in H_1(\widetilde{X}, \mathbb{Z})$ can be written in the form

$$c = m_1\tilde{z}_1 + \cdots + m_N\tilde{z}_N$$

with integers $m_1, \ldots, m_N \in \mathbb{Z}$ as in Proposition 5.2.3. Let $Z_i := \rho^{-1}(\tilde{z}_i)$ be the open annulus above $\tilde{z}_i$ for $i = 1, \ldots, N$. Fix an orientation on the graph $\Gamma(\widetilde{X})$ of coincidence of the irreducible components of $\widetilde{X}$; cf. Definition 5.2.2. Let $\tilde{z}' \in \widetilde{X}_\mu$ be the point on the source and $\tilde{z}'' \in \widetilde{X}_\nu$ be the point on the target of the edge associated to the double point $\tilde{z}$ of $\widetilde{X}$. Let $\zeta_i : Z_i \to \mathbb{G}_{m,K}$ be a coordinate function respecting the orientation; i.e., its absolute value function $|\zeta_i|$ increases by moving through Z_i from $\tilde{z}_i'$ to $\tilde{z}_i''$. Let $\varepsilon(\tilde{z}_i)$ be the height of the annulus Z_i for $i = 1, \ldots, N$.

Lemma 6.5.8. *The morphism $\varphi_\Theta|_{\overline{J}_K} : \overline{J}_K \to \overline{J}'_K$ induces a morphism $\widetilde{\varphi}_\Theta : \widetilde{J} \to \widetilde{J}'$ of their reductions, and hence a morphism $\widetilde{\varphi}_B : \widetilde{B} \to \widetilde{B}'$ of their abelian parts after dividing out the tori. Moreover, $\widetilde{\varphi}_B$ is the canonical morphism $\varphi_{\widetilde{\Theta}} : \widetilde{B} \to \widetilde{B}'$ associated to the theta divisor $\widetilde{\Theta}$ of $\widetilde{B}$; cf. Proposition 5.2.3.*

Proof. Go back to the construction of $\overline{J}_K$ in Sect. 5.3. In the following we adapt the notation of Lemma 5.3.1. There are introduced subdomains $A_\rho \subset Z_\rho \subset X_K$ with base points a_ρ^0 for $\rho = 1, \ldots, r$ which are isomorphic to annuli of height 1. There are subdomains $W_j \subset X_K^{(g_j)}$ with base points $w_j^0 := w_{j,1}^0 + \cdots + w_{j,g_j}^0$, where $w_{j,i}^0 \in X_K$ for $j = 1, \ldots, s$ and $i = 1, \ldots, g_j$, where g_j is the genus of the component $\widetilde{X}_j$.

Then put $A := A_1 \times \cdots \times A_r$ and $W := W_1 \times \cdots \times W_s$. These have smooth formal R-models, which we denoted by the same symbols. Then the product $(A \times W)_K$ is embedded into $\overline{J}_K$ under the Abel-Jacobi map defined with respect to the chosen base points. The group $\overline{J}_K$ was constructed by gluing translates of $(A \times W)_K$.

Next we consider the translate of the theta divisor $\Theta := \iota^{(g-1)}(X_K^{(g-1)})$

$$\Theta_{a,w} := \tau_\kappa(\Theta) \quad \text{with } \kappa := \bigotimes_{\rho=1}^{r}[a_\rho^0 - x_0] \otimes \bigotimes_{j=1}^{s}\bigotimes_{i=1}^{g_j}[w_{j,i}^0 - x_0].$$

Then $\Theta_{a,w} \cap (A \times W)_K$ is the set of all points $(a, w) \in (A \times W)_K \subset \overline{J}_K$ which can be represented in the form

$$\left(\bigotimes_{\rho=1}^{r}[a_\rho - a_\rho^0] \otimes \bigotimes_{j=1}^{s}\bigotimes_{i=1}^{g_j}[w_{j,i} - w_{j,i}^0]\right)^\vee,$$

where in the tensor product one term $[\ldots]$ collapses to 1. The intersection $\Theta_{a,w} \cap (A \times W)_K$ is a Cartier divisor of $(A \times W)_K$. Since $(A \times W)_K$ admits a smooth formal R-model $A \times W$, the divisor $\Theta_{a,w} \cap (A \times W)_K$ extends to a relative

Cartier divisor on the R-model $A \times W$ in the evident way. Moreover, $\Theta_{a,w} \cap \overline{J}_K$ is a Cartier divisor on $\overline{J}_K$ and extends to a relative Cartier divisor $\overline{\Theta}_{a,w}$ on $\overline{J}$ due to Lemma 6.2.4. Since $A \times W$ is immersed into $\overline{J}$ as an formal open subvariety of $\overline{J}$, the divisor $\overline{\Theta}_{a,w} \cap (A \times W)$ coincides with the constructed one on $A \times W$.

Thus, we have a well-defined reduction $\widetilde{\Theta}_{a,w} \subset \widetilde{J}$, which splits into a sum

$$\widetilde{\Theta}_{a,w} = \widetilde{E}_1 + \cdots + \widetilde{E}_r + \widetilde{D}_W,$$

where $\widetilde{E}_\varrho$ is the divisor induced by $\widetilde{W} \times \prod_{\rho \neq \varrho} \widetilde{A}_\rho \times \{a_\varrho^0\}$ and $\widetilde{D}_W$ is the part induced by the symmetric products on the components of genus $g_j \geq 1$. The invertible sheaves $\mathcal{O}_{\widetilde{J}}(\widetilde{E}_\rho)$ are pull-backs from line bundles $P_{\widetilde{B}\times[\tilde{c}'_\rho]}$ on $\widetilde{B}$ which define the torus extension

$$1 \to \widetilde{T} \to \widetilde{J} \to \widetilde{B} \to 1 \quad \widehat{=} \quad H_1(\widetilde{X}, \mathbb{Z}) \to \widetilde{B}', \quad \tilde{c}_\rho \mapsto \big[\tilde{c}'_\rho\big] := -\varphi_{\widetilde{\Theta}}\big([\tilde{c}_\rho]\big),$$

where $\tilde{c}_1, \ldots, \tilde{c}_r$ is the canonical basis of $H_1(\widetilde{X}, \mathbb{Z})$ associated to the double points $\tilde{z}_1, \ldots, \tilde{z}_r$; cf. Notation 6.5.7. The line bundle $[\tilde{c}_\rho]$ on $\widetilde{X}$ is induced by the simple cycle passing through $\tilde{z}_\rho$ for $\rho = 1, \ldots, r$ in the sense of Proposition 5.2.3.

The map φ_Θ is equal to the map $\varphi_{\Theta_{a,w}}$, since Θ and $\Theta_{a,w}$ differ by a translation. The map $\varphi_\Theta : J_K \to J'_K$ induces commutative diagrams

$$\begin{array}{ccc} \overline{J} & \xrightarrow{\ \overline{\varphi}_\Theta\ } & \overline{J}' \\ \big\downarrow{\scriptstyle \overline{q}} & & \big\downarrow{\scriptstyle \overline{q}'} \\ B & \xrightarrow{\ \varphi_B\ } & B' \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{J} & \xrightarrow{\ \widetilde{\varphi}_\Theta\ } & \widetilde{J}' \\ \big\downarrow{\scriptstyle \tilde{q}} & & \big\downarrow{\scriptstyle \tilde{q}'} \\ \widetilde{B} & \xrightarrow{\ \widetilde{\varphi}_B\ } & \widetilde{B}'. \end{array}$$

More precisely, the morphism $\overline{\varphi}_\Theta$ is associated to the couple $(\lambda_\Theta, \varphi_B)$, where $\lambda_\Theta : M \to M'$ is a group homomorphism satisfying $\phi' \circ \lambda_\Theta = \varphi'_B \circ \phi$, and $\widetilde{\varphi}_\Theta$ is associated to $(\lambda_\Theta, \widetilde{\varphi}_B)$. Due to Proposition 6.4.2 we have $\varphi_B = \varphi_{\mathcal{N}}$ for an invertible sheaf $\mathcal{N}$ on B such that $\overline{q}^*\mathcal{N} = \mathcal{O}_{\overline{J}}(\Theta_{a,w})$.

For the reduction one obtains that $\widetilde{\varphi}_\Theta$ is associated to the couple $(\lambda_\Theta, \varphi_{\widetilde{\mathcal{N}}})$. Moreover, the pull-back $\tilde{q}^*\widetilde{\mathcal{N}}$ is isomorphic to $\mathcal{O}_{\widetilde{J}}(\widetilde{\Theta}_{a,w})$. The invertible sheaf $\mathcal{O}_{\widetilde{J}}(\widetilde{E}_\rho)$ is the pull-back of the invertible sheaf $\mathcal{P}_{\widetilde{B}\times[\tilde{c}'_\rho]}$, which is translation invariant. Thus, we see that $\varphi_{\widetilde{B}}$ is associated to the invertible sheaf $\mathcal{O}_{\widetilde{B}}(\widetilde{D}_W)$. The latter coincides with the theta divisor $\widetilde{\Theta}$ of $\widetilde{B}$. □

Lemma 6.5.9. *Keep the notations of Notation 6.5.7. If $\gamma \in \Gamma = \mathrm{Deck}(\widehat{X}/X)$ is a deck transformation of $\widehat{X} \to X$, then denote by c_γ in $H_1(\widetilde{X}, \mathbb{Z})$ the homology class of the associated cycle*

$$\overline{\widehat{\tilde{x}}_0, \gamma(\widehat{\tilde{x}}_0)} = m_1\tilde{z}_1 + \cdots + m_N\tilde{z}_N$$

with integers $m_i \in \mathbb{Z}$ which is image of the path from $\widehat{\tilde{x}}_0$ to $\gamma(\widehat{\tilde{x}}_0)$. Let $Z_i := \rho^{-1}(\tilde{z}_i)$ be the annulus of height $\varepsilon(\tilde{z}_i)$ above $\tilde{z}_i$ viewed as a subset of $\widehat{X}_K$. Consider the *morphism*

$$f_\gamma := h'(\gamma) \circ \widehat{\iota} \colon \widehat{X} \longrightarrow \widehat{J}_K \longrightarrow P_{B \times \phi'(h'(\gamma))}.$$

Then we have the following results:

(a) $f_\gamma|_{Z_i} = h'(\gamma)((\varphi_i \circ \zeta_i) \cdot \overline{u}_i \cdot f_i)$, *where* $\varphi_i : \mathbb{G}_{m,K} \to \widehat{J}_K$ *is a group homomorphism*, $\zeta_i : Z_i \to \mathbb{G}_{m,K}$ *is a coordinate function*, $\overline{u}_i$ *maps to the formal fiber* $\overline{J}_+(e)$ *at the unit element and* $f_i \in \widehat{J}_K$ *is a* K*-valued point.*

(b) *The slope of* $|f_\gamma|$ *along the path from* $\widehat{x}_0$ *to* $\gamma(\widehat{x}_0)$ *is non-constant only when passing the annuli above double points of the reduction of* $\widehat{X}$ *which belong to the path from* $\widehat{x}_0$ *to* $\gamma(\widehat{x}_0)$.

Over such a double point $\tilde{x}$ *the slope of* $|f_\gamma|$ *is log-linear of order* ± 1 *and its total amount over such a point is* $\varepsilon(\tilde{z}_i)^{\pm 1}$.

The slope of $|f_\gamma|$ *along the path from* $\widehat{x}_0$ *to* $\gamma(\widehat{x}_0)$ *it is strictly increasing or strictly decreasing.*

The numbers m_i *indicate how often* (*respecting the orientation*) *the path passes a double point* $\tilde{x}$ *above a given double point* $\tilde{z}_i$ *of* $\widetilde{X}$.

Proof. We may assumed that c_γ is a simple path; i.e., $m_i \in \{0, \pm 1\}$, without back-trackings and meets only one of the points $(\tilde{z}_1, \dots, \tilde{z}_r)$. Put $c'_\gamma := -\varphi_{\widetilde{\Theta}}([c_\gamma])$.

(a) By Proposition 5.4.8 the map $\widehat{\iota}|_{Z_i}$ can be represented in the form $(\varphi_i \circ \zeta_i) \cdot \overline{u}_i \cdot f_i$. The map $\overline{u}_i$ maps to $\overline{J}_+(e)$, because its reduction is a map from a rational variety to an abelian variety and the torus part is captured by $\varphi_i \circ \zeta_i$.

(b) We view f_γ as a section of the $\mathbb{G}_{m,K}$-torsor $\widehat{\iota}^* q^* P_{B \times \phi'(h'(\gamma))}$. Since $P_{B \times \phi'(h'(\gamma))}$ is a translation invariant formal line bundle, the pull-back of its reduction to every irreducible component of $\widetilde{X}$ has degree 0. Therefore, every section of the reduction has as many zeros as poles on every irreducible component of $\widetilde{X}$. Since f_γ has neither poles nor zeros, after adjusting the norm of f_γ on the irreducible component, the reduction $\tilde{f}_\gamma$ of f_γ can have zeros or poles only at the points $\tilde{z}'_i$ or $\tilde{z}''_i$. Thus, if $\tilde{f}_\gamma$ has a zero on a component, then it is the only one and of order 1.

Indeed, we may subdivide Z_i by a concentric annulus $\overline{Z}_i$ of height 1. The reduction of $f_\gamma|_{\overline{Z}_i}$ behaves like the pull-back of the tautological section of $\tilde{q}^* P_{\widetilde{B} \times c'_\gamma}$ by Proposition 5.2.3. Since c_γ is a simple circuit which passes through exactly one $\tilde{z}_\rho$ of the double points $\tilde{z}_1, \dots, \tilde{z}_r$, we obtain that for $\overline{Z}_i = \overline{Z}_\rho$ the morphism $\overline{Z}_\rho \hookrightarrow \overline{J}_K$ is a closed immersion. Since the reduction $\widetilde{Z}_\rho$ is rational, the induced map to $\widetilde{B}$ is constant to a point $\tilde{b}$. Therefore, the reduction of the restriction f_γ onto $\overline{Z}_\rho$ behaves like the tautological section $\widetilde{Z}_\rho \to P_{\tilde{b} \times c'_\rho}$; cf. Proposition 5.2.3. This implies that $\tilde{f}_\gamma$ has a simple pole at $\tilde{z}'_\rho$ and a simple zero at $\tilde{z}''_\rho$ and no other zeros or poles. For $\rho' \neq \rho$ the reduction of the restriction f_γ onto $\overline{Z}_{\rho'}$ is constant. Since a zero of a section of line bundle of degree zero on a component $\widetilde{X}_i$ implies also a pole on $\widetilde{X}_i$, we see that the slope of $|f_\gamma|$ behaves as asserted. The absolute value of the total amount follows from Corollary 4.3.3. The indeterminacy in the sign is due to the freedom of choosing the orientation of the graph. Along the path from $\widehat{x}_0$ to $\gamma(\widehat{x}_0)$ it is strictly increasing or strictly decreasing, because the path follows the consecutive sequence of zeros and poles. □

Corollary 6.5.10. *Keep the notations of Notation 6.5.7. For $\alpha, \beta \in \Gamma$ consider the associated cycles*

$$c_\alpha = m_1\tilde{z}_1 + \cdots + m_N\tilde{z}_N, \qquad c_\beta = n_1\tilde{z}_1 + \cdots + n_N\tilde{z}_N.$$

Then the absolute value of the pairing $\langle h(\alpha), h'(\beta)\rangle$ is given by

$$\left|\langle h(\alpha), h'(\beta)\rangle\right| = \prod_{i=1}^{N} \varepsilon(\tilde{z}_i)^{m_i \cdot n_i}.$$

In particular, the pairing $|\langle m_1, \lambda(m_2)\rangle|$ for $m_1, m_2 \in M$ with $\lambda := h' \circ h^{-1}$ is positive definite.

Proof. The absolute value $|\langle h(\alpha), h'(\beta)\rangle|$ is exactly the total amount of the slope of f_β along the path $\overline{\widehat{x}_0, \alpha(\widehat{x}_0)}$. Note that $h(\alpha) = \widehat{\iota}(\alpha(\widehat{x}_0))$; cf. Lemma 6.5.2. Thus, the assertion follows from Lemma 6.5.9. □

Next we want to compare λ and λ_Θ. Let $c_1, \ldots, c_r \in H$ be the simple cycles and $\gamma_1, \ldots, \gamma_r \in \Gamma$ such that the path $\overline{\widehat{x}_0, \gamma_\rho(\widehat{x}_0)}$ induces c_ρ for $\rho = 1, \ldots, r$. Then c_ρ gives rise to a character m'_ρ on T_K and $\widehat{\iota} \circ \gamma_\rho(\widehat{x}_0)$ to a character m_ρ of T'_K. Then $(m_1, \ldots, m_r)$ and $(m'_1, \ldots, m'_r)$ are basis of M and M', respectively. The map $\widehat{\varphi}_\Theta : T_K \to T'_K$ corresponds to $\lambda_\Theta : M \to M'$. Then we have that $\lambda_\Theta(m_\rho) = m'_\rho$. Indeed, consider the autoduality map $\widehat{\varphi}' : \widehat{J}'_K \to \widehat{J}_K$. This map sends the line bundle $(\mathbb{A}^1_{\widehat{J}_K}, t')$ with M-action $t' \in T'_K$ to a line bundle $(\mathbb{A}^1_{\widehat{X}_K}, t)$ with Γ-action. The Γ-action is given by $(t_1, \ldots, t_r)$ with respect to the basis $(m'_1, \ldots, m'_r)$ of M' and $t_\rho := m_\rho(t')^{-1}$. The inverse appears here, because of the relation between line bundles and invertible sheaves. So we obtain $\lambda = \lambda_\Theta$ by Theorem 5.1.6(e).

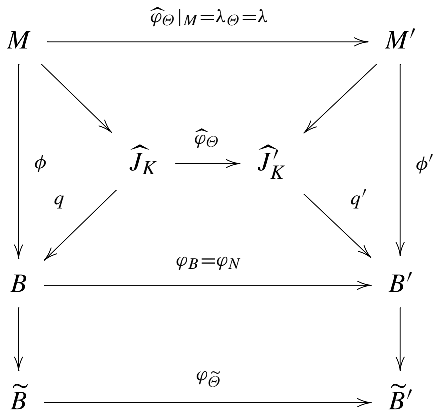
Theorem 6.5.11. *In the situation of Notation 6.5.7 let X_K be a connected smooth projective curve with semi-stable reduction $\widetilde{X}$ and assume that the irreducible components $\widetilde{X}_1, \ldots, \widetilde{X}_n$ of $\widetilde{X}$ are smooth. Let Θ be the theta divisor of $J_K := \mathrm{Jac}(X_K)$; i.e., the image of $X_K^{(g-1)}$ in J_K under $\iota^{(g-1)}$ with respect to some K-rational point $x_0 \in X_K$. The line bundle $L := \mathrm{Hom}(\mathcal{O}_{J_K}(\Theta), \mathcal{O}_{J_K})$ is associated to a triple (N, r, λ_Θ) in the sense of Theorem 6.3.2. Then we have the following results*:

(i) $\varphi_\Theta := \varphi_L : J_K \to J'_K$ *is equivalent to* $(\lambda_\Theta, \varphi_B) = (\lambda_\Theta, \varphi_N)$.
(ii) $\varphi_N : B \to B'$ *reduces to the theta polarization of* $\widetilde{B} = \prod_{i=1}^n \mathrm{Jac}(\widetilde{X}_i)$.
(iii) $\lambda_\Theta = \lambda$ *where* $\lambda = h' \circ h^{-1}$.

The morphism $\varphi_\Theta : J_K \to J'_K$ is a polarization of abelian varieties and $(\lambda_\Theta, \varphi_B)$ is a polarization in the sense of Definition 6.4.5.

Proof of Theorem 6.5.11. The assertion (i) follows from Propositions 6.1.9 and 6.4.2. The assertion (ii) follows from Lemma 6.5.8 and (iii) was explained above. □

The summary of this section is that the following diagram is commutative



6.6 Parameterizing Degenerating Abelian Varieties

Let us return to the subject announced in the preface concerning the uniformization and construction of abelian varieties over a non-Archimedean field K. First, we summarize the main results of this chapter.

In Theorem 6.4.8 we saw that an abelian variety A over K having semi-abelian reduction can be uniformized in the sense that A is a quotient of a semi-abelian R-group scheme E by a lattice M of periods. Thus, combining this with the results of Theorem 6.4.6 and Proposition 6.4.1, we obtain an equivalence between the categories (ppAbVar) and (ppDegData).

Here (ppAbVar) is the category whose objects are principally polarized abelian varieties (A, φ_A) over a non-Archimedean field K having semi-abelian reduction over the valuation ring R of K and whose morphisms are morphisms of abelian varieties respecting the polarizations.

Here (ppDegData) is the category of principally polarized degeneration data whose objects are pairs $((E, M), (\varphi_B, \lambda_M))$, where E is a semi-abelian R-group scheme with abelian generic fiber and where M is a lattice in the generic fiber $E_\eta = E \otimes_R K$. The pair (φ_B, λ_M) is a principal polarization in the sense of Definition 6.4.5. The morphisms are the morphism of R-group schemes respecting lattices and polarization.

The compatibility of this equivalence with the construction of their duals is shown in Theorem 6.3.3. We recapitulate the results by the following theorem.

Theorem 6.6.1. *The* Raynaud uniformization *constitutes a functor*

$$\mathrm{RC} : (\mathrm{ppAbVar}) \longrightarrow (\mathrm{ppDegData}),\ \ (A, \varphi_A) \longmapsto \big((E, M), (\varphi_B, \lambda_M)\big),$$

which is an equivalence of categories. The inverse functor

$$\mathrm{MC} : (\mathrm{ppDegData}) \longrightarrow (\mathrm{ppAbVar}),\ \ \big((E, M), (\varphi_B, \lambda_M)\big) \longmapsto (E/M, \varphi_{E/M}),$$

is usually called Mumford construction. *By passing to duals, one obtains*

$$\begin{array}{ccc} (\mathrm{ppAbVar}) & \longrightarrow & (\mathrm{ppAbVar}) \\ {\scriptstyle \mathrm{RC}}\downarrow & & \downarrow{\scriptstyle \mathrm{RC}} \\ (\mathrm{ppDegData}) & \longrightarrow & (\mathrm{ppDegData}) \end{array}$$

$$\begin{array}{ccc} (A, \varphi_A) & \longmapsto & (A', \varphi'_A) \\ \big\downarrow & & \big\downarrow \\ ((E, M), (\varphi_B, \lambda_M)) & \longmapsto & ((E', M'), (\varphi'_B, \varphi_E|_M)) \end{array}$$

Anticipating the main result of Corollary 7.6.2 of the next chapter, the last theorem is also true without polarization data. Hereby, one has to replace (ppAbVar) by the category of abeloid varieties defined over algebraically closed non-Archimedean field K and whose morphisms are the morphism of abeloid varieties. The category (ppDegData) is replaced by the category of pairs (E, M), where E is a Raynaud extension of a formal abelian R-scheme B and a lattice $M \subset E$ of periods. The morphism are morphisms of rigid analytic group varieties respecting lattices.

Finally let us briefly discuss the approach of Chai and Faltings to the uniformization and construction of principally polarized abelian varieties as contained in [27, Chaps. 2 and 3] and relate it to the method presented in this book. They work more generally over a normal Noetherian ring R which is complete with respect to an ideal $\mathfrak{a}$, whereas the rigid analytic methods allow only to work over a valuation ring or an admissible formal scheme over a valuation ring. Using the notions of this book, the program of [27, Chaps. 2 and 3] is to establish a correspondence

$$\big\{\text{classes of } [A, \varphi_A]\big\} \quad \Longleftrightarrow \quad \big\{\text{classes of } \big[(E, M), (\lambda_M, \varphi_B)\big]\big\}.$$

On the left-hand side, A is a semi-abelian scheme over R of relative dimension g with abelian fiber A_η and φ_A is a principal polarization of A_η. The special fiber $A_s := A \otimes_R R/\mathfrak{a}$ is an extension of an abelian $R/\mathfrak{a}$-scheme B_s of dimension $g - r$ by a split torus T_s of rank r.

On the right-hand side, E is an extension of an abelian R-scheme B over R of relative dimension $g-r$ by a split torus of constant rank r and M is a constant R-group scheme $\mathbb{Z}_R^r$ such that its generic fiber M_η is a lattice in E_η. The pair (λ_M, φ_B) is a principal polarization of (E, M) which corresponds to our Definition 6.4.5, where principal means that λ_M and φ_B are isomorphisms. The positivity can be rephrased by asking $\langle m, \lambda_M(m)\rangle \in \mathfrak{a}P_{B\times B'}$ for all $m \in M - \{0\}$.

Using the above correspondence, it is plausible how to construct formal charts of a toroidal compactification of the moduli stack of principally polarized abelian varieties. Indeed, the data on the right-hand side can be parameterized by a family of data varying over the moduli stack of principally polarized abelian varieties of dimension $g-r$; cf. [27, Chap. 4.3]. Thus, these data variety are parameterized by

$$\begin{aligned}
M \text{ symmetric lattice in } E &\mathrel{\widehat{=}} r(r+1)/2 \text{ points in } q^*(\mathrm{id}_B, \varphi_B)^* P_{B\times B}\\
E \text{ universal } \mathbb{G}_m^r\text{-extension of } B &\mathrel{\widehat{=}} r \text{ points in the dual of } B\\
B \text{ universal abelian scheme over } &\mathrel{\widehat{=}} \text{relative dimension } (g-r) \text{ over}\\
A_{g-r} \text{ mod. space of p.p. ab. var. } &\mathrel{\widehat{=}} \text{dimension } (g-r)(g-r+1)/2 \text{ over } \mathbb{Z}
\end{aligned}$$

where $q: E \to B$ is the canonical morphism, $\varphi_B : B \to B'$ is the polarization and $P_{B\times B'}$ is the Poincaré bundle. Of course, in the first row we mean points in an open cone in $q^*(\mathrm{id}_B, \varphi_B)^* P_{B\times B}$. The dimension count

$$\frac{(g-r)(g-r+1)}{2} + r(g-r) + \frac{r(r+1)}{2} = \frac{g(g+1)}{2}$$

shows that the parameter space has the right dimension. In the absence of a polarization, as discussed above for a valuation ring R, one has to ignore the polarization and, in particular, the symmetry of the periods. Therefore, the universal lattices are parameterized by r points in an open subset of E. Moreover, the universal abeloid varieties with good reduction depends on $(g-r)^2$ modules. Thus, the dimension count yields

$$(g-r)^2 + r(g-r) + rg = g^2.$$

Let us start with the description of how Chai and Faltings proceed with the map from the left to the right. As usual denote by M' the character group of T_s. Due to the lifting of tori, one associates to A the Raynaud extension on the formal level

$$0 \to \overline{T} \to \overline{A} \to B \to 0, \qquad (\overline{\dagger})$$

where $\overline{A}$ is the formal completion of A with respect to the special fiber and where B is a formal abelian R-scheme defined by the quotient of $\overline{A}$ by $\overline{T}$. Let L_η be an ample symmetric line bundle on A_η.

By a result of Raynaud [81, Théorème XI, 1.13] L_η extends to a line bundle L on A, which is relatively ample over $S := \operatorname{Spf} R$. Using the descent of cubical sheaves, the pull-back $\overline{L}$ of L to $\overline{A}$ descends to a relatively ample line bundle on B.

This shows that B is an abelian R-group scheme. Thus, one gets in fact an algebraic Raynaud extension over R

$$0 \to T \to E \to B \longrightarrow 0 \quad \widehat{=} \quad \phi' : M' \to B', \tag{$\dagger$}$$

whose formal completion along the special fiber is the given ($\overline{\dagger}$), where M' is the character group of T and where ϕ' is a homomorphism from M' to the dual B' of B.

In our case the corresponding Raynaud extension is the analytification of E_η, which was constructed as a push-out via the open analytic embedding of $\overline{T}_\eta$ into the torus T_η. Note that Chai and Faltings do not have the lattice $M_\eta \hookrightarrow E_\eta$ at this stage. Thus, in order to define the dual extension, they cannot use the induced map $\phi : M_\eta \to B_\eta$ as we do. Instead they use the dual A'_η of A_η. Since A'_η is a quotient of A_η with respect to a finite group scheme, one can show that A'_η extends to a semi-abelian group R-scheme A'. As above they arrive at a second extension

$$0 \to T' \to E' \to B' \to 0 \quad \widehat{=} \quad \phi : M \to B''. \tag{$\dagger'$}$$

They show that B' is, in fact, the dual of B and hence B'' is isomorphic to B. Furthermore, the polarization φ_A on the generic fiber uniquely extends to a morphism from A to A' and hence determines a pair (λ_M, φ_B).

The key point in [27, Chap. 2] is to find the lattice $M_\eta \hookrightarrow E_\eta$. In our approach the lattice comes in via the uniformization of A_η, as explained in Theorem 5.6.5. Indeed, we show that the map $\overline{A}_\eta \to A_\eta$ extends to a surjective map $E_\eta \to A_\eta$ and obtain the lattice as the kernel of this map. Then we prove the formula ($*$) of Proposition 6.2.19 for M_η-invariant sections on the pull-back p^*L_η for line bundles L_η on A_η by using the M_η-action and the descent for cubical line bundles from $\overline{E}$ to B.

Knowing the formula ($*$) of Proposition 6.2.19 for M_η-invariant sections, it is clear that the lattice is encoded in the Fourier coefficients of the formal expansion on every non-trivial section of the line bundle L_η on A_η. Indeed, as one can see from the formula, one can recover the part $\langle m, m' \rangle$ for every pair $(m, m') \in M \times M'$. This is equivalent to the embedding $M_\eta \hookrightarrow E_\eta$; cf. Proposition 6.1.8. We remind the reader that for stating the formula of Proposition 6.2.19 for M_η-invariant sections it is only necessary to know the line bundle N_η and the map $\lambda : M \to M'$ satisfying the condition $\varphi_N \circ \phi = \phi' \circ \lambda$, which can be deduced from the map φ_L as explained in Proposition 6.4.2. Also it is not necessary to explicitly mention the M_η-action on N_η, because one is only interested in M_η-invariant sections. Indeed, the set of such sections coincides with the set of global sections of L_η; see Remark 6.2.21. Therefore, one can replace $\Gamma(E, q^*N)^M$ by $\Gamma(A_\eta, L_\eta)$.

The approach in [27] is based on this fact. Indeed, take any ample line bundle L on A and let $\overline{L}$ be the pull-back of L to $\overline{A}$. By the descent of cubical line bundles, $\overline{L}$ descends to a line bundle N on B. Now consider a non-trivial section f of L and look at the Fourier expansion of the pull-back of $f|_{\overline{A}}$. Since they derive the maps $\lambda : M \to M'$ and $\varphi_N : B \to B'$ from $\varphi_L : A \to A'$, they can state the formula of Proposition 6.2.19 for sections of L. In order to verify it, it suffices to establish it only after a base change by a "large" discrete valuation ring dominating the given ring R. The remaining proof in [27, Chap. 2] consists in showing the following facts, where R is now a discrete valuation ring:

(a) There exists a Fourier coefficients $0 \neq a_{m'} \in \Gamma(B, N \otimes P_{B\times\phi'(m')})$.
(b) $a_{m'}$ and $\tau_m^* a_{m'+\lambda(m)}$ are proportional on the generic fiber.

These facts are shown by using addition formulas for theta series.

Now let us say a few words of how to obtain the map from the right to the left, usually, referred to as Mumford construction of degenerating abelian varieties. The construction is somehow contained in our results of Theorems 6.4.4 and 6.4.6 for special cases, where the base consists of a complete discrete valuation ring. One starts with a Raynaud extension

$$0 \to T \to E \xrightarrow{q} B \to 0 \quad \widehat{=} \quad \phi' : M' \to B'$$

of an abelian R-scheme B by a split torus T corresponding to a homomorphism from the character group $M' = \mathbb{Z}^r$ of T' to the dual B' of B. Let M be the constant group R-group scheme $\mathbb{Z}^r$ and consider an injective homomorphism $\widehat{\phi} : M_\eta \to E_\eta$ on the generic fiber. The latter will play the role of a "lattice" on the generic fiber E_η; the discreteness of M_η will be imposed later by the requirement of the existence of a polarization. Furthermore, assume that there is a polarization (λ_M, φ_B) on (E, M) similarly as we do. The existence of a polarization is as good as an ample line bundle N on B and a map $\lambda : M \to M'$ such that λ is compatible with $\varphi_N : B \to B'$ in the sense $\varphi_N \circ \phi_\eta = \phi'_\eta \circ \lambda$, where $\phi_\eta : M_\eta \to B_\eta$ is the homomorphism induced by $\widehat{\phi}$ and λ satisfies a symmetry condition. Thus, one obtains an M_η-linearization of $q^* N_\eta$ on E_η similarly as we do in the proof of Theorem 6.4.6.

The key point now is the construction of the quotient E/M. This is done by using formal geometry. One has to extend all objects to objects over R, which have to satisfy certain completeness conditions. The key is the notion of a relatively complete model (P, L) of E associated to (N, λ).

Firstly, P is a B-scheme of locally finite type. This should be thought of as an R-model of E_η such that all K'-valued points of E_η specialize on it, where K' is the field of fractions of a discrete valuation ring R' dominating R with center over the "closed point" $V(\mathfrak{a})$. This model is as close as possible to E. One asks that E is an open subscheme of P and that the M-action on P coincides on the generic fiber with the M_η-action induced by the E_η-action via the inclusion $M_\eta \hookrightarrow E_\eta$ on P.

Secondly, L is an ample line bundle on P extending the line bundle $q^* N_\eta$. It is of the form $L(P/B) \otimes p^* N$, where the line bundle $L(P/B)$ is trivial on the generic fiber E_η and is relatively ample with respect to $P \to B$; here $p : P \to B$ is the structural morphism. Moreover, one requires that the E_η-linearization on the trivial line bundle $\mathcal{O}_{E_\eta} = L_\eta \otimes p^* N_\eta^{-1}$ given by the translation extends to an E-linearization of $L(P/B) = L \otimes p^* N^{-1}$.

Such relatively complete models exist and they are used to construct the quotient, which is obtained by the smooth part of P/M. Note that a priori P/M is a formal scheme over R, but the M-linearization on the ample line bundle L gives rise to an ample line bundle L/M over P/M. Therefore, P/M is algebraic and its smooth locus part E/M is, in fact, an algebraic R-group scheme, which has to be viewed as the quotient. The completeness condition forces the quotient to be proper over the open part of Spec R, where the schematic closure of the lattice $\widehat{\phi}(M_\eta)$ in E has the full rank.

Now let us compare this approach with the construction of this book. The completion $\overline{P}$ of P with respect to its closed fiber is a formal analytic structure on E_η which is compatible with the M-action on E_η. The formal completion $\overline{L}$ of L is a formal line bundle with M-action. Since it is impossible to extend the M_η-linearization given by λ and r to the formal line bundle q^*N_η on $\overline{P}$, one has to introduce the line bundle $L(P/B)$, which is trivial on the generic fiber; its role is to produce certain norms in order to make the extension of the M_η-linearization on L_η possible.

In our approach we perform the quotient E_η/M_η as a rigid analytic space. Since we do not keep track of the R-model structure, already given on E, the construction of the quotient is easy in our case and the construction of the relatively complete model has disappeared. In order to show the algebraicity of the quotient, we have to produce an ample line bundle on E_η/M_η; this is the point, where we use the polarization data. If we want to obtain the good R-model structure of E_η/M_η, we have to define a formal model $\overline{P}$ of E_η with M_η-action and $\overline{E}_\eta$-action, where $\overline{E}_\eta$ is the formal completion of E with respect to it special fiber.

Such a model can be obtained in the following way: Take a basic cell Λ of the lattice $\widehat{\phi}(M_\eta)$ and U_Λ be the associated relative polyannulus associated to Λ. Then glue the translates of U_Λ under the M_η-action along the neighboring faces. So we get a formal structure $\overline{P}$ on E_η such that M acts on $\overline{P}$. Then $\overline{P}/M$ yields a formal model of E_η/M_η such that $\overline{E}$ survived as a formal open part of $\overline{P}/M$. Thus, we obtain a good formal model of A_η. For getting the good algebraic R-model it would also require an ample line bundle on $\overline{P}/M$ or, equivalently, an ample line bundle on $\overline{P}$ with M_η-linearization. The latter is not so easy to produce and would lead to the consideration of a relatively complete model as used in Mumford's construction.

Let us conclude with some remarks. If one is only interested in the toroidal compactification of the moduli of principally polarized abelian varieties, it is not necessary to study analytic tori or abeloid varieties of type E/M, because one only uses the functorial equivalence of the above correspondence. Therefore, it is enough to work just from the beginning with polarizations and to do the minimum. Hereby one avoids the consideration of rigid analytic tori or their counterparts in the mixed case which are not algebraic. Thus, as we see from the method of Chai and Faltings, it is not necessary to go back to the geometry which is behind the construction. Hopefully, our approach could disclose the geometric ideas behind the constructions.

Chapter 7
Abeloid Varieties

Every connected compact complex Lie group of dimension g can be presented as a quotient $\mathbb{C}^g/\Lambda$ of the affine vector group $\mathbb{C}^g$ by a lattice Λ of rank $2g$. From the multiplicative point of view, it can be presented as a quotient $\mathbb{G}_{m,\mathbb{C}}^g/M$ of the affine torus $\mathbb{G}_{m,\mathbb{C}}^g$ by a multiplicative lattice M of rank g. In the rigid analytic case the situation is more complicated because of the phenomena of good and multiplicative reduction, which in general occur in a twisted form. For example look at the rigid analytic uniformization of abelian varieties in Theorem 5.6.5.

The fundamental example of a proper rigid analytic group A_K is the analytic quotient $A_K = E_K/M_K$ in Raynaud representation; cf. Definition 6.1.5, where E_K is an extension of a proper rigid analytic group B_K with good reduction by an affine torus T_K, where M_K is a lattice in E_K of rank equal to $\dim T_K$; cf. Proposition 6.1.4. The main result of this chapter is that every smooth rigid analytic group, which is proper and connected, is of the form E_K/M_K after a suitable extension of the base field. This is a generalization of Grothendieck's Stable Reduction Theorem [42, I, Exp. IX, 3.5] as well as of the rigid analytic uniformization of abelian varieties.

The proof requires advanced techniques; it mainly relies on the stable reduction theorem for smooth curve fibrations which are not necessarily proper. In Sect. 7.5 we compactify such a curve fibration by using the Relative Reduced Fiber Theorem 3.4.8 and approximation techniques provided in Sect. 3.6. Then we can apply the moduli space of marked stable curves. Therefore, one can cover the given group A_K by a finite family of smooth curve fibrations with semi-stable reduction.

In a second step one deduces from such a covering the largest open subgroup $\overline{A}_K$ which admits a smooth formal R-model $\overline{A}$ by well-known techniques on group generation dating back to A. Weil; cf. Sect. 7.2. The formal group $\overline{A}$ is a formal torus extension of a formal abelian R-scheme B. The prolongation of the embedding $\overline{T} \hookrightarrow \overline{A}$ of the formal torus to a group homomorphism $T_K \to A_K$ of the associated affine torus T_K follows by the approximation theorem and a discussion on the convergence of group homomorphisms; cf. Sect. 7.3.

Thus, the group homomorphism $\overline{A}_K \to A_K$ extends to a group homomorphism from the push-out $\widehat{A}_K := T_K \amalg_{\overline{T}} \overline{A}$ to A_K. The surjectivity of the map $\widehat{A}_K \to A_K$ is shown by an analysis of the map from the curve fibration to A_K. In fact, the

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whole torus part is induced by the double points in the reduction of the stable curve fibration; cf. Sect. 7.4.

So far we are concerned only with the case, where the base field is algebraically closed. But it is not difficult to see that the whole approach can be done after a suitable finite separable field extension if one starts with a non-Archimedean field which is not algebraically closed.

If the non-Archimedean field in question has a discrete valuation, there is a notion of a formal Néron model. Then our result implies a semi-abelian reduction theorem for such Néron models. As a further application one can deduce that every abeloid variety has a dual; i.e., the Picard functor of translation invariant line bundles on A_K is representable by an abeloid variety.

7.1 Basic Facts on Abeloid Varieties

In this section we gather some basic results on abeloid varieties which are analogous to facts on abelian varieties; cf. [74, pp. 43–44]. In the following we assume that K is a non-Archimedean field.

Definition 7.1.1. An *abeloid variety* is a rigid analytic group variety, which is proper, smooth and connected. Usually we will write its group law "+" additively and the inverse map by "−".

We will need the theorem of the cube for invertible sheaves on abeloid varieties A. Since A is proper over Sp K, we can apply to A the cohomology theory for proper rigid analytic spaces as provided by Kiehl [50]; cf. Theorem 1.6.4. In particular, we have the semi-continuity theorem for coherent sheaves; cf. [52, §5]. Then one can conclude as in [74, Sect. 10] that the rigidity lemma and hence the theorem of the cube are valid for invertible sheaves on A or on the generic fiber of formal abelian R-schemes B. In the following we mean by a rigid analytic variety a rigid analytic space, which is geometrically reduced.

Lemma 7.1.2 (Rigidity lemma). *Let X be an irreducible proper rigid analytic variety with a rational point x_0, let Y and Z be any rigid analytic varieties, where Y is irreducible. If $f : X \times Y \to Z$ is a morphism such that for a rational point $y_0 \in Y$ the set $f(X \times \{y_0\}) = \{z_0\}$ is a single point, then there exists a morphism $g : Y \to Z$ such that $f = g \circ p_2$, where $p_2 : X \times Y \to Y$ is the projection.*

Proof. Define $g : Y \to Z$ by $g(y) = f(x_0, y)$. To show $f = g \circ p_2$ it suffices to verify that these morphisms coincide on some non-empty open subset of $X \times Y$, since $X \times Y$ is irreducible and reduced. Let U be an admissible affinoid neighborhood of z_0 in Z. Since X is proper, there exists an affinoid neighborhood V of y_0 in Y such that $f(X \times V) \subset U$. For each $y \in Y$ the proper variety $X \times \{y\}$

is mapped by f into the affinoid variety U. Since the image is proper by Corollary 1.6.5, it is a single point of U. So, if $x \in X, y \in V$, then we have that $f(x, y) = f(x_0, y) = g \circ p_2(x, y)$. □

The rigidity Lemma 7.1.2 implies the following facts.

Corollary 7.1.3. *If X and Y are abeloid varieties and $f : X \to Y$ is any morphism, then $f(x) = h(x) + a$, where h is a homomorphism from X to Y and $a := f(e_X) \in Y$, where e_X is the unit element of X.*

Proof. Replacing f by $f - f(e_X)$, we may assume $f(e_X) = e_Y$. Then f is a homomorphism. Indeed, consider the morphism $F : X \times X \to X$ defined by $F(x_1, x_2) = f(x_1 + x_2) - f(x_2) - f(x_1)$. Now we have that $F(X \times \{e_X\}) = F(\{e_X\} \times X) = \{e_Y\}$. Thus, we see by Lemma 7.1.2 that $F = e_Y$ is constant on $X \times X$; i.e., f is a homomorphism. □

Corollary 7.1.4. *Every abeloid variety is commutative.*

Proof. The inversion map $X \to X, x \mapsto x^{-1}$, is a group homomorphism by Corollary 7.1.3; i.e., X is commutative. □

Proposition 7.1.5. *Assume that K is algebraically closed. Let $f : X \to Y$ be a morphism of rigid analytic varieties, where X is an abeloid variety. For each $x \in X$ denote by F_x the connected component of $f^{-1}(f(x))$ which contains x. Then there is a closed connected subgroup F of X such that $F_x = x + F$.*

Proof. We follow the proof [74, p. 88]. Consider the morphism

$$\phi : X \times F_x \longrightarrow Y, \ \phi(z, u) = f(z + u).$$

Since F_x is proper and connected and $\phi(\{e\} \times F_x) = f(x)$, the rigidity lemma implies $\phi(z, u) = \phi(z, x)$ for all $z \in X, u \in F_x$. In particular, $f(z - x + F_x) = f(z)$ for all $z \in X$. Since $z - x + F_x$ is connected, $z - x + F_x$ is contained in F_z. Reversing the positions of $x, z \in X$, we actually have $F_z = z - x + F_x$ for all $z, x \in F_x$. In particular, if $F = F_e$, then $F_z = z + F$ for all $z \in F$, so it only remains to show that F is a subgroup of X. Let $u \in F$. Then $F_{-u} = -u + F$ and hence $e \in F_{-u}$. Therefore, $F_{-u} = F_e = F$. Thus, we see $F - u = F$ for each $u \in F$, so F is a subgroup of X. □

Theorem 7.1.6. *Let $f : A \to S$ be a proper smooth rigid analytic group space over a rigid analytic S with connected fibers.*

(a) [Theorem of the Cube] *Let L be a rigidified line bundle on A. Then $\mathcal{D}_3 L$ is canonically trivial as a rigidified line bundle on A^3 and the trivialization gives rise to a cubical structure on L; cf. Sect. A.3.*

(b) [Theorem of the Square] *For every line bundle L on A and all S-valued points x and y of A, there is a line bundle N on S such that*

$$\left(\tau_x^* L \otimes L^{-1}\right) \otimes \left(\tau_y^* L \otimes L^{-1}\right) \cong \left(\tau_{x+y}^* L \otimes L^{-1}\right) \otimes f^* N,$$

where $\tau_a : A \to A$ is the left translation by an S-valued point a of A.

Proof. This follows from Lemma 7.1.2 in the same manner as shown in [74, p. 59] if S is a variety. In the case, where S is not geometrically reduced, one can follow the proof in [74, Chap. 3, §10]; cf. Proposition A.3.4. □

Corollary 7.1.7. *Let A be an abeloid variety and let $\mathcal{L}$ be an invertible sheaf on A. If $\mathcal{L}$ admits a non-trivial section, then $\mathcal{L}^{\otimes 2}$ is generated by its global sections.*

Proof. Since cohomology commutes with flat base change, we may assume that K is algebraically closed. For every closed point a there is an isomorphism

$$\rho : \tau_a^* \mathcal{L} \otimes \tau_{-a}^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{\otimes 2}$$

due to Theorem 7.1.6(b). Since there is a non-trivial global section f of $\mathcal{L}$, the sections $\rho(\tau_a^* f \otimes \tau_{-a}^* f)$ for $a \in A(K)$ generate $\mathcal{L}^{\otimes 2}$. For showing this, consider the dense open subset $U := \{z \in A;\ f(z) \neq 0\}$. Then for a closed point $x \in A$ consider the intersection $(-x + U) \cap (x - U)$, which is not empty, as U is dense in A. Thus, there exists a point $a \in (-x + U) \cap (x - U)$. Then $\tau_a^* f(x) = f(x + a) \neq 0$ and $\tau_{-a}^* f(x) = f(x - a) \neq 0$, and hence $\rho(\tau_a^* f \otimes \tau_{-a}^* f)(x) \neq 0$. Thus, we see that $\mathcal{L}^{\otimes 2}$ is generated by its global sections. □

Proposition 7.1.8. *Let A be an abeloid variety of dimension g. If $\mathcal{M}(A)$ is the field of its meromorphic functions, then we have the results*:

(a) *The degree of transcendency of $\mathcal{M}(A)$ over K is less or equal to g.*
(b) *The degree of transcendency of $\mathcal{M}(A)$ over K is equal to g if and only if A is an abelian variety.*

Proof. We may assume that K is algebraically closed.

(a) This follows as in the complex analytic case by using blowing-ups, the proper mapping Theorem 1.6.4 and GAGA in Theorem 1.6.11; cf. [37, Chap. 10, §6.4].

(b) Let $f_1, \ldots, f_g$ be a basis of transcendency of $\mathcal{M}(A)$ and let $D \subset A$ be the pole divisor of $f_1 \cdot \ldots \cdot f_g$. Let $\mathcal{L} := \mathcal{O}_A(2D)$. Due to Corollary 7.1.7 the invertible sheaf $\mathcal{L}$ is generated by its global sections. If $h_0, \ldots, h_N$ is a basis of $\Gamma(A, \mathcal{L})$, then this gives rise to a proper morphism

$$h := (h_0, \ldots, h_N) : A \longrightarrow \mathbb{P}_K^N,$$

because A is proper. The image of h is a closed algebraic subset due to the closed image theorem in Corollary 1.6.5. By Corollary 1.6.12 we see that $h(A)$ is projective algebraic. It has dimension $g = \dim A$, because the functions $f_1, \dots, f_g$ would be algebraically dependent otherwise. So the general fibers of h have dimension 0. Thus, there exists a dense open set $U \subset h(A)$ such that

$$h|_{h^{-1}(U)} : h^{-1}(U) \longrightarrow U$$

is finite due to the semi-continuity of the fiber dimension [52, 3.6]. Then, we can use translations as in the proof of Corollary 7.1.7 to move the subset $h^{-1}(U)$ around in A, in order to show that $\mathcal{L}^{\otimes 2}$ has a basis $\tilde{h}$ such that $\tilde{h}$ gives rise to a morphism $\tilde{h} : A \to \mathbb{P}_K^{\tilde{N}}$ with finite fibers everywhere. This implies that A is algebraic. Indeed, $\tilde{h}$ is finite, as A is proper. The "if-part" is well-known, since an algebraic variety of dimension g has always g independent rational functions. □

Lemma 7.1.9. *Let A be an abeloid variety. Let $\mathcal{L}$ be an invertible sheaf on A. Assume that there is a non-trivial section of $\mathcal{L}$ and that there are only finitely many closed points $x \in A$ such that $\tau_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_A$ is trivial. Then $\mathcal{L}$ is ample in the sense of Definition 1.7.3.*

Proof. We may assume that K is algebraically closed. Due to Corollary 7.1.7 the invertible sheaf $\mathcal{L}^{\otimes 2}$ is generated by global sections. A basis $\theta_0, \dots, \theta_n$ of the global sections of $\Gamma(A, \mathcal{L}^{\otimes 2})$ gives rise to a morphism

$$\theta : A \longrightarrow \mathbb{P}_K^n, \; x \longmapsto \big(\theta_0(x), \dots, \theta_n(x)\big).$$

Next we want to show that the fibers of θ are finite. To verify this, we have to make use of the assumption that the group

$$H := \big\{x \in A(K); \tau_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_A\big\}$$

is finite. Since $\mathcal{L}$ has a section, $\mathcal{L}$ is equal to $\mathcal{O}_A(D)$, where D is an effective divisor on A. By the Theorem of the Square 7.1.6 the divisor

$$D_a := \tau_a D + \tau_{-a} D \sim 2D$$

is linearly equivalent to $2D$ for every $a \in A(K)$. Thus, there exists a meromorphic function θ_a on A with

$$\operatorname{div} \theta_a = D_a - 2D.$$

Now consider the fiber $\Theta := \theta^{-1}(\theta(e))$ of the unit element e of A and let F be the irreducible reduced component of Θ with $e \in F$. Since θ_a is a global section of $\mathcal{L}^{\otimes 2}$, it can be represented as a linear combination

$$\theta_a = \lambda_0\theta_0 + \dots + \lambda_n\theta_n \quad \text{with } \lambda_i \in K.$$

Since F is contained in the fiber of $\theta(e)$, there is a non-vanishing holomorphic function $\lambda : F \to \mathbb{G}_{m,K}$ such that

$$\big(\theta_0(x), \dots, \theta_n(x)\big) = \lambda(x) \cdot \big(\theta_0(e), \dots, \theta_n(e)\big)$$

for all $x \in F$. Since F is proper and connected, $\lambda(x)$ is a constant $c \in K$. Thus, we see

$$\theta_a(x) = (\lambda_0, \ldots, \lambda_n) \cdot \big(\theta_0(x), \ldots, \theta_n(x)\big)^t = c \cdot \theta_a(e)$$

for all $x \in F$. Thus, $F \subset D_a$ in the case $\theta_a|_F = 0$ or $F \cap D_a = \emptyset$ otherwise. We may assume $D \neq [-1]D$; otherwise replace D by a translate $\tau_z D$ for some $z \in A$. Consider a point $a \in D$ with $a \notin [-1]D$. Now look at the function θ_a as defined above with

$$\operatorname{div} \theta_a = \tau_a D + \tau_{-a} D - 2D.$$

Then F is a closed subgroup of A by Proposition 7.1.5. Since $e = \tau_{-a}(a) \in \tau_{-a} D$ and $e \notin \tau_a D$, the component F of the fiber Θ satisfies $F \subset \tau_{-a} D$ by what we have shown above. Thus, we have $F + a \subset D$. The latter is true for all a in a dense open subset of D. Therefore, it follows $F + D \subset D$ by continuity. Thus, we see that F is a subset of H and hence finite by assumption. Furthermore, it is shown in Proposition 7.1.5 that the connected component of every fiber $\theta^{-1}(\theta(x))$ which contains x is of the form $x + F$. This shows that all the fibers of θ are finite. Since θ is proper, it is finite and hence, $\mathcal{L}$ is ample. □

Proposition 7.1.10. *Let A be an abeloid variety. Then the following conditions are equivalent*:

(a) *There exists an ample line bundle $\mathcal{L}$ on A.*
(b) *A is the analytification of an abelian variety.*

Proof. (a) → (b): Since $\mathcal{L}^{\otimes n}$ for an $n \geq 1$ gives rise to a closed embedding of A into a projective space, A is the analytification of an algebraic group variety by the GAGA Theorem 1.6.11, and hence A is an abelian variety.

(b) → (a): Since an abelian variety is projective, there exists an ample sheaf on A. □

7.2 Generation of Subgroups by Smooth Covers

We start with a result on the extension of formally rational maps to group varieties in the style of the classical theorem of A. Weil. In the following we assume that K is a non-Archimedean algebraically closed field.

Theorem 7.2.1. *Let S be an admissible formal R-scheme assumed to be normal with generic fiber S_K and geometrically reduced special fiber $S \otimes_R k$.*

Let $G_K \to S_K$ be a quasi-compact rigid analytic S_K-group variety. Let $\overline{G} \to S$ be a separated smooth admissible formal S-group scheme with connected fibers such that its generic fiber $\overline{G}_K$ is an admissible open subvariety of G_K. Let $p : X \to S$ be a separated morphism of admissible formal R-schemes, which is smooth and surjective. Let $U \subset X$ be an R-dense open subscheme.

Let $u_K : X_K \to G_K$ be an S_K-morphism such that $u_K(U_K) \subset \overline{G}_K$. Assume that the morphism $\overline{u}_K := u_K|_{U_K} : U_K \to \overline{G}_K$ extends to a morphism $\overline{u} : U \to \overline{G}$. Then consider the formal rational morphism

$$v : X \times_S X \dashrightarrow \overline{G}, \ (x_1, x_2) \longmapsto \overline{u}(x_1) \cdot \overline{u}(x_2)^{-1},$$

where "·" denotes the group law on G_K.

(a) *If U is S-dense in X, then u_K extends to a morphism $u : X \to \overline{G}$.*
(b) *If the geometric fibers of X/S are connected, v is defined everywhere.*

Before we start the proof, let us consider a special case.

Lemma 7.2.2. *Keep the situation of Theorem 7.2.1. Let $e : S \to \overline{G}$ be the unit element of $\overline{G}$ and $\overline{G}_+(e)$ the subset of $\overline{G}_K$ consisting of all points which specialize into the unit section $\tilde{e}$ of $\overline{G}_0$. Then $v_K(x_1, x_2) \in \overline{G}_+(e)$ for every S-valued point $(x_1, x_2) \in (X \times_S X)_K$ if the points x_1 and x_2 have the same reduction $\tilde{x}$ in $X_0 \to S_0$.*

Proof. Consider a closed point $\tilde{x} \in X_0$ with image $\tilde{s} \in S_0$. It suffices to show the assertion in a formal neighborhood of $\tilde{s}$. Let $H \subset \overline{G}$ be an affine formal open subset of $\overline{G}$ which contains $\tilde{e}$. We may assume that u maps U to H, eventually after shrinking U. Since X is smooth over S, eventually after shrinking X there exists a system of coordinates $\xi_1, \ldots, \xi_d$ such that $d\xi_1, \ldots, d\xi_d$ generate $\Omega^1_{X/S}$ at $\tilde{x}$, where d is the relative dimension of X/S. Since the problem is local, we may assume that X is affine and that $d\xi_1, \ldots, d\xi_d$ generate $\Omega^1_{X/S}$ over all of X. Then put $\zeta_i := p_1^*\xi_i - p_2^*\xi_i$ for $i = 1, \ldots, d$, where $p_i : X \times_S X \to X$ is the i-th projection for $i = 1, 2$.

There exists a formal open subscheme $Y \subset X \times_S X$ such that the locus $V(\zeta_1, \ldots, \zeta_d) \cap Y$ coincides with the diagonal $\Delta_{X/S}$ of $X \times_S X$.

For an element $c \in |K^\times|$ with $|c| < 1$ set

$$\Delta(c) := \left\{ y \in Y_K ; \left|\zeta_i(y)\right| \leq c \text{ for } i = 1, \ldots, d \right\}.$$

Now we assert that $v_K(\Delta(c)) \subset H_K$ for every $c \in |K^\times|$ with $|c| < 1$.

This follows from the extension properties of holomorphic functions. Indeed, since H_K is open and v_K is continuous, there exists an $\varepsilon \in |K^\times|$ with $0 < \varepsilon \leq c$ such that $v_K(\Delta(\varepsilon)) \subset H_K$. Since u_K maps U_K to H_K, the map v_K maps $\Delta(\varepsilon) \cup W_K$ into H_K, where

$$W_K := \Delta(c) \cap (U \times U)_K = \Delta(c) \cap (X \times U)_K.$$

Since H_K is affinoid, it suffices to show that every holomorphic function on $W_K \cup \Delta(\varepsilon)$ is a restriction of a holomorphic function on $\Delta(c)$.

This follows similarly as exercised in Proposition 5.4.5. Indeed, the map

$$(\zeta, p_2) : \Delta(c) \longrightarrow \mathbb{B}^d_K(c) \times_K X_K$$

is an isomorphism for every $c \in |K^\times|$ with $c < 1$, because $(\zeta_1, \ldots, \zeta_d)$ is a system of coordinates on the formal fiber $(X \times_S s)_+(x)$ for every $x \in X_K$, where $s \in S$ is the image of $x \in X_K$. By [89, III-20, Thm. 2] there exists a Noether normalization $\tilde{\varphi} : \tilde{X} \to \mathbb{A}_k^n$ such that $\tilde{\varphi}(\tilde{X} - \tilde{U}) \subset V(\tilde{\eta}_n)$, where $\tilde{\eta}_1, \ldots, \tilde{\eta}_n$ are the coordinate functions on $\mathbb{A}_k^n$. Let $\varphi : X_K \to \mathbb{B}^n$ be a lifting of $\tilde{\varphi}$. Thus, we obtain a finite morphism

$$(\zeta, \varphi \circ p_2) : \Delta(c) \longrightarrow \mathbb{B}_K^d(c) \times_K \mathbb{B}_K^n.$$

The components of a point $(x_1, x_2) \in \Delta(c)$ reduce to the same point $\tilde{x}_1 = \tilde{x}_2$ in $\tilde{X}$. Therefore, $|(\zeta, \varphi)^* \eta_n(y)| = 1$ for all $y \in W_K$. We can view v_K as an N-tuple of functions with $N := d + n$.

We have to verify that a function f on $\Delta(\varepsilon) \cup W_K$ is defined on $\Delta(c)$. This follows from the fact that every holomorphic function on the union

$$V_K := \left(\mathbb{B}_K^d(\varepsilon) \times \mathbb{B}_K^n\right) \cup \left\{z \in \mathbb{B}_K^d(c) \times \mathbb{B}_K^n;\, \left|\eta_n(z)\right| = 1\right\}$$

is a restriction of a holomorphic function on $\mathbb{B}_K^d(c) \times \mathbb{B}_K^n$ by Proposition 5.4.5. Choose a maximal linearly independent system $(b_1, \ldots, b_r)$ in the A-module $B := \mathcal{O}_{X_K \times_S X_K}(\Delta(c))$ over $A = \mathcal{O}_{\mathbb{B}_K^{d+n}}(\mathbb{B}_K^d(c) \times \mathbb{B}_K^n)$. There exists an element $a \in A - \{0\}$ with $aB \subset Ab_1 + \cdots + Ab_r$. We can represent $af = a_1 b_1 + \cdots + a_r b_r$ uniquely, where a_i is holomorphic function on V_K, and hence $a_i \in A$ for all $i = 1, \ldots, n$ due to Proposition 5.4.5. Thus, we see that f belongs to the field of fractions of B. Now consider the characteristic polynomial of f over the field of fraction of A. By the same reasoning as above we see that its coefficients belong to A. Since B is normal, we obtain that f belongs to B. Thus, we obtain that v_K maps $\Delta(c)$ to H_K.

Since $H_K \subset \overline{G}_K$ was an arbitrarily open neighborhood of e, we can avoid every point $g \in \overline{G}$ which does not belong to the formal fiber at e. Thus, we see $v_K(\Delta_K) \subset \overline{G}_+(e)$, where Δ_K is the set of points of $(X \times_S X)_K$ which specialize into the diagonal of $(X \times_S X)_0$. □

Proof of Theorem 7.2.1. (a) We may assume that S and X are affine admissible formal R-schemes. Set $Y := X \times_S X$. As above let

$$\Delta_K \subset Y_K := (X \times_S X)_K$$

be the set of points of Y_K which specialize into the diagonal of $(X \times_S X)_0$. Consider the map

$$v_K : Y_K \longrightarrow G_K,\ (x_1, x_2) \longmapsto u_K(x_1) \cdot u_K(x_2)^{-1}.$$

Let $\overline{H} \subset \overline{G}$ be an affine open formal neighborhood of the unit section. In the following we denote by the subindex "0" the objects obtained by base change with $R \to R/R\pi$, for all formal objects.

We know that v_K maps the subset Δ_K into $\overline{H}_K$ by Lemma 7.2.2. Due to Theorem 3.3.4 there exists an R-model G of G_K such that $\overline{H}_K$ is induced by an open affine subscheme H of G. Let $A_0 := G_0 - H_0$ be the complement of the reduction H_0 in G_0. Choose an admissible formal blowing-up $\varphi : Z \to Y$ such that v_K

is induced by a formal morphism $w : Z \to G$; cf. Theorem 3.3.4. Since the points of Δ_K are mapped to $\overline{H}_K$ under v_K, the pre-image $\varphi^{-1}(\Delta_0)$ of the diagonal Δ_0 is disjoint from $w^{-1}(A_0)$. Thus, $\varphi(w^{-1}(A_0))$ is closed, since φ_0 is proper, and disjoint from Δ_0.

Let $V \subset Y$ be the open subscheme associated to $Y_0 - \varphi(w^{-1}(A_0))$. Therefore V_K is mapped to H_K. Then v_k extends to a formal morphism $v : V \to \overline{H}$. Indeed, $\overline{H}$ is affine and $\mathcal{O}_{\overline{H}}(\overline{H})$ and $\mathcal{O}_Y(V)$ are the rings of power bounded functions on H_K and V_K, respectively, because their special fibers are geometrically reduced; cf. Proposition 3.4.1.

Thus, we see that $v : Y \dashrightarrow \overline{H}$ is defined on a formally open neighborhood of Δ_K. Now let $V \subset Y$ be the largest open subscheme, where v_K is defined as a formal morphism. Since $U \subset X$ is S-dense, $V \cap (X \times_S U) \neq \emptyset$. Thus, for every $x_1 \in X_K$ there exists a point $x_2 \in X_K$ such that $(x_1, x_2) \in V_K$ and $x_2 \in U_K$. Since

$$u_K(x_1) = v_K(x_1, x_2) \cdot u_K(x_2) \in \overline{G}_K,$$

we see that u_K maps X_K to $\overline{G}_K$. Then, $u_K : X_K \to \overline{G}_K$ extends to a formal morphism $u : X \to \overline{G}$, as seen by a descent argument as in [15, 2.5/5]; cf. the end of the proof of Lemma 5.4.7.

(b) We have already seen that v_K is defined in an open neighborhood $U \subset Y$ of the diagonal Δ. Since the fibers of Y/S are connected and hence irreducible, U is S-dense in Y. Then it follows from (a) that v_K extends to a formal S-morphism $v : Y \to \overline{G}$. □

Theorem 7.2.3. *Let G_K be a smooth commutative rigid analytic group. Let X be a smooth admissible formal R-scheme, which is separated, quasi-compact and connected.*

Let $o \in X_{\mathrm{rig}}$ be a rational point and let $u_K : X_{\mathrm{rig}} \to G_K$ be a rigid analytic flat *morphism sending the point o to the unit element of G_K.*

Then X generates a smallest open subgroup H_K of G_K via u_K, which admits a smooth formal R-model H such that u_K factorizes through H_K.

In particular, u_K extends to a formal morphism $u : X \to H$. If d is the dimension of G_K, the multiplication map $u^{2d} : X^{2d} \to H$ is surjective.

If X_0 is a rational variety, the reduction of H is a linear algebraic group.

If X is a formal ball $\mathbb{B}^d_R$, then H has unipotent reduction and the underlying variety of H_K is isomorphic to a ball.

Remark 7.2.4. Actually the assumption on the flatness of u_K in Theorem 7.2.3 can be avoided. In the proof given below the flatness is used to know that the image of u_K is an admissible open subdomain of G_K due to Corollary 3.3.8(d). If we replace Corollary 3.3.8(d) by Proposition 3.3.9 which is much deeper, one has enough information on the image of u_K and one can proceed in a similar way as we will do.

In a first step we show that there exists a dense open subscheme $U \subset X^{2d}$ such that the image of $u_K^{2d}(U_K)$ is an admissible open affinoid subvariety Z_K with smooth R-model Z and the group law of G_K induces a birational formal group law on Z.

Definition 7.2.5. Let Z be a separated smooth admissible formal R-scheme assumed to be geometrically connected. A *birational formal group law* is a formal rational morphism

$$m : Z \times_R Z \dashrightarrow Z,\ (x, y) \longmapsto xy,$$

i.e., m is defined on a dense open subscheme, such that the maps

$$\Phi : Z \times_R Z \longrightarrow Z \times_R Z,\ (x, y) \longmapsto (x, xy),$$
$$\Psi : Z \times_R Z \longrightarrow Z \times_R Z,\ (x, y) \longmapsto (xy, y),$$

are birational in the formal sense; i.e., Φ and Ψ are isomorphisms on dense open parts, and m is associative; i.e., $m(xy, z) = m(x, yz)$, whenever both sides are defined.

Proposition 7.2.6. *Let Z be a separated admissible formal R-scheme, which is smooth and geometrically connected. Let m be a birational formal group law on Z. Then there exists a smooth formal R-group scheme $\langle Z\rangle$ of topologically finite type with group law $\overline{m}$ together with an R-dense open subscheme $Z' \subset Z$ and an open immersion $Z' \hookrightarrow \langle Z\rangle$ such that*

(i) *the image of Z' is R-dense in $\langle Z\rangle$ and*
(ii) *$\overline{m}$ restricts to m on $Z' \times Z'$.*

Proof. First, we know that m induces a k-birational group law on the reduction $\tilde{Z}$. This gives rise to an associated smooth group scheme $\langle\tilde{Z}\rangle$ with similar properties as asserted; cf. [15, 5.1/5]. The group $\langle\tilde{Z}\rangle$ is obtained by gluing copies of a dense open part $\tilde{Z}'$ of $\tilde{Z}$ by translation. This construction lifts to every formal level and hence yields our result. □

Now we come to the *proof of Theorem 7.2.3*. We keep the notation of Theorem 7.2.3.

Lemma 7.2.7. *In the situation of Theorem 7.2.3 assume, in addition, that X is symmetric; i.e., there is a formal birational map $\xi : X \dashrightarrow X$ with the property $u_K(\xi(x)) = u_K(x)^{-1}$, whenever ξ is defined at $x \in X_K$.*

Then there exists an admissible open affinoid subvariety $Z_K \subset G_K$ with smooth formal R-model Z contained in the image of $u_K^{2d} : X_K^{2d} \to G_K$ such that $(u_K^{2d})^{-1}(Z_K)$ is formally dense open in X_K^{2d} and such that the group law of G_K induces to a birational formal group law on Z.

Proof. We are allowed to replace X by a dense open subscheme by Theorem 7.2.1. By Corollary 3.3.8(d) the image is a finite union of open affinoid subvarieties of G_K. Then, by Proposition 3.1.12 we may assume that X_K is affinoid and the image $Z_K \subset G_K$ of u_K is an open affinoid subvariety.

The map $u_K : X_K \to Z_K$ induces a surjective map of their canonical reductions. Since X_K has a smooth irreducible canonical reduction, the reduction of Z_K is irreducible as well. After a further shrinking we may assume that Z_K is the generic fiber of a smooth formal R-scheme Z. Indeed, the special fiber Z_k of Z is geometrically reduced, because it is dominated by the special fiber X_k which is geometrically reduced. Thus, we can replace X_K by Z_K and may assume that u_K is an open immersion, whenever we start with a flat morphism $u_K : X_K \to G_K$.

For every integer $n \geq 1$ we have the morphism

$$u_K^n : X_K^n \longrightarrow G_K,\ (x_1, \ldots, x_n) \longmapsto u_K(x_1) \cdot \ldots \cdot u_K(x_n).$$

Let d be the dimension of G_K and consider the map

$$v_K := u_K^{2d} : X_K^{2d} \longrightarrow G_K.$$

As discussed above, v_K induces a rational dominant formal map

$$v : X^{2d} \dashrightarrow Z,$$

where Z is a smooth formal R-scheme whose generic fiber is an open subvariety of G_K. Let $U(2d) \subset X^{2d}$ be a dense open subscheme such that the map v_K extends to a surjective morphism $v : U(2d) \to Z$.

For every $n \in \{1, \ldots, 2d-1\}$ consider the projection

$$p_{n+1,\ldots,2d} : X^{2d} \longrightarrow X^{2d-n}$$

onto the last components and set

$$W(2d-n) := p_{n+1,\ldots,2d}\big(U(2d)\big) \subset X^{2d-n}.$$

For every $y = (y_{n+1}, \ldots, y_{2d}) \in W(2d-n)$ the map

$$X^n \dashrightarrow Z, \quad x \longmapsto v(x, y),$$

is formally rational. In particular, we have now $p(W(2d-n)) \subset W(2d-m)$ for all $n \leq m$, where $p : X^{2d-n} \longrightarrow X^{2d-m}$ is the projection onto the last components. Then one can look at the schematic closure $\widetilde{Z(n,y)}$ of the image under the rational mapping

$$X^n \dashrightarrow X^{2d} \dashrightarrow Z \to \widetilde{Z}, \quad x \longmapsto \widetilde{v(x,y)},$$

for every $y := y(n) \in W(2d-n)$. These subschemes are contained in each other

$$Z\big(\widetilde{n, y(n)}\big) \subset Z\big(\widetilde{n+1, y(n+1)}\big)$$

where $y(n+1) := p(y(n))$. By reason of dimensions there exists an integer $n \leq d$ with

$$Z\big(\widetilde{n, y(n)}\big) = Z\big(\widetilde{n+1, y(n+1)}\big)$$

for all $y(n) \in W(2d-n)$ in the formal dense open subscheme $W(2d-n)$ of X^{2d-n}. Thus we obtain that

$$Z\big(\widetilde{n, y(n)}\big) = Z\big(n+\widetilde{2, y(n}+2)\big) = \widetilde{Z(2d)}.$$

It suffices to show the first equality. Every general point in $Z(n+\widetilde{2, y(n}+2))$ can be written in the form

$$\tilde{z}_1\tilde{z}_2\ldots\tilde{z}_n\tilde{z}_{n+1}\tilde{z}_{n+2}\tilde{y}_{n+3}\ldots\tilde{y}_{2d} = \tilde{z}'_1\tilde{z}'_2\ldots\tilde{z}'_n\tilde{z}'_{n+1}\tilde{y}_{n+2}\tilde{y}_{n+3}\ldots\tilde{y}_{2d}$$

where $\tilde{z}_i$ is the reduction of a point $u_K(x_i)$ for $x_i \in X$ and similarly $\tilde{z}'_i$. The latter product is an element of $Z(n+\widetilde{1, y(n}+1))$. This implies that

$$Z \times Z \dashrightarrow Z, \quad (z_1, z_2) \longmapsto z_1 \cdot z_2,$$

is birational. The associativity of the birational law is clear, since it is a restriction of a group law. Since X is symmetric, Z is symmetric in the formal birational sense. Therefore, the maps Φ and Ψ of Definition 7.2.5 are birational in the formal sense. □

Proof of Theorem 7.2.3. Due to Theorem 7.2.1 we are allowed to replace X by an R-dense open subscheme. Now we replace X by $X \times X$ and u_K by v_K defined by $v_K(x_1, x_2) := u_K(x_1)u_K(x_2)^{-1}$. Thus, v_K is also flat and its image is symmetric in the formal birational sense.

By Lemma 7.2.7 there exists an open admissible subvariety Z_K of G_K with smooth connected formal model Z such that the group law on G_K restricts to a birational formal group law on Z and $v_K : X_K \to G_K$ restricts to a formal rational map $X_K \dashrightarrow Z_K$. Due to Proposition 7.2.6 the birational group law gives rise to an admissible open subgroup H_K of G_K which admits a smooth formal model H and Z can be viewed as a dense open part of H. The map $v_K : X_K \to G_K$ extends to a formal morphism $v : X \to H$ due to Theorem 7.2.1. Therefore, we obtain $u_K(x_1)u_K(x_2)^{-1} \in H_K$ for all $x_1, x_2 \in X_K$. Since the image of the distinguished point $u_K(o)$ is the unit element, we see that $u_K(x) \in H_K$, and hence that u_K maps X_K to H_K.

The assertion concerning the surjectivity of $u^{2d} : X^{2d} \to \overline{G}$ is clear by reasons of dimension. If X is a rational variety, then $\overline{G}$ has linear reduction due to the structure theorem of Chevalley; cf. [15, 9.2/1]. If X_K is a ball, the assertion follows from the theorem of Lazard [23, IV, §4, no. 4, 4.1]. □

Definition 7.2.8. In the situation of Theorem 7.2.3 we denote by $\langle X, u\rangle$ the formal R-group scheme, whose generic fiber is an open subgroup of G_K generated by the morphism $u_K : X_K \to G_K$. In our notation we do not indicate the chosen base point $o \in X_K$, since the group does not depend on it. Although we have not introduced a base point explicitly, we will refer to $\langle X, u_K\rangle$ as the *subgroup generated by the smooth cover* $u_K : X_K \to G_K$.

Theorem 7.2.1 has an important consequence.

Let S be an admissible formal R-scheme with geometrically reduced fibers such that its generic fiber S_K is normal. Let $X \to S$ be a flat admissible formal morphism with geometrically reduced equidimensional fibers of dimension $d \geq 1$. Let

$$\begin{aligned} &\overline{X} \quad \text{be the smooth part of } X/R, \\ &Y \quad \text{be the smooth part of } X/S. \end{aligned}$$

Then $\overline{X}$ is R-dense in X and Y is S-dense in X.

Let G_K be a smooth rigid group and let

$$u_K : X_K \longrightarrow G_K$$

be a flat rigid analytic morphism. Then $\overline{X}$ gives rise to a smooth formal R-group scheme $\langle \overline{X}, u_K \rangle$ via u_K by Theorem 7.2.3 such that $\langle \overline{X}, u_K \rangle_{\mathrm{rig}}$ is an open admissible rigid analytic subgroup of G_K.

Corollary 7.2.9. *In the above situation of above we have that*

$$u_K(x_1) \cdot u_K(x_2)^{-1} \in \langle \overline{X}, u_K \rangle$$

for all points $x_1, x_2 \in X_K$ if x_1, x_2 belong to the same connectedness component of $Y \times_S s$ for any R-valued point s of S.

Proof. Set $H = \langle \overline{X}, u_K \rangle$. After an étale base change $S' \to S$ we may assume that Y is marked by S-valued points $\sigma_i : S \to Y$ for $i = 1, \ldots, r$ such that

$$Y = Y(\sigma_1) \cup \cdots \cup Y(\sigma_r),$$

where $Y(\sigma_i)$ is the union of the connected components of the fibers of Y/S which meet σ_i; cf. [39, IV$_3$, 15.6.5]. Now consider a component $Z = Y(\sigma_i)$; in particular, Z has geometrically connected fibers over S. Now look at the rigid analytic S_{rig}-group $G_K \times_K S_K$ and consider the morphism

$$v_K : Z_K \longrightarrow G_K \times_K S_K; \; x \mapsto u_K(x) \cdot u_K\big(\sigma_i\big(f(x)\big)\big)^{-1}$$

induced by u_K, where $f : X \to S$ is the structural map. In this situation we can apply Theorem 7.2.1(b), because $\overline{X} \cap Z$ is R-dense in Z. Thus, we obtain the extension $v : Z \longrightarrow H \times_R S$. The projection to H yields the claim. □

7.3 Extension of Formal Tori

In the following let G_K be a smooth commutative rigid analytic group.

Proposition 7.3.1. *Let $\overline{\varphi} : \overline{\mathbb{G}}_{m,K} \to G_K$ be a homomorphism of rigid analytic groups such that $\overline{\varphi}$ factorizes through an admissible open subgroup H_K of G_K, which admits a smooth formal R-model H.*

If the image of $\overline{\varphi}$ is relatively compact in G_K; cf. Definition 3.6.1, then $\overline{\varphi}$ extends to a homomorphism $\varphi : \mathbb{G}_{m,K} \to G_K$ in a unique way.

Proof. The assertion concerning the uniqueness is evident. So we may assume that K is algebraically closed. The reduction $\widetilde{H}$ of H is an extension of an abelian variety $\widetilde{B}$ by a linear group $\widetilde{L}$. The linear commutative group $\widetilde{L}$ is a product of a torus $\widetilde{T}$ and a unipotent group $\widetilde{U}$.

Let $\tilde{\xi}_1, \ldots, \tilde{\xi}_r$ be a system of coordinates of $\widetilde{B}$ at the unit element. Let ξ_i be a lifting of $q^*\tilde{\xi}_i$, where $q : \widetilde{H} \to \widetilde{B}$ is the canonical map. For an element $c \in |K^\times|$ with $|c| < 1$ consider the open subset

$$L(c) := \left\{x \in H_K; \left|\xi_1(x)\right| \leq c, \ldots, \left|\xi_r(x)\right| \leq c\right\}.$$

For a suitable $c < 1$ the map $\overline{\varphi}$ maps $\overline{\mathbb{G}}_{m,K}$ to $L(c)$ and $L(c)$ admits a smooth R-model with linear reduction. Thus, we may replace H_K by $L(c)$, and hence we may assume that H has linear reduction. The reduction $\widetilde{H}$ is a product $\mathbb{G}^t_{m,k} \times \mathbb{G}^s_{a,k}$. So the formal fiber of the rigid analytic space H_K at the unit element is isomorphic to an $(s+t)$-dimensional open ball.

Since H_K is relatively compact in G_K, we see by Theorem 3.6.7 that we can approximate $\overline{\varphi}$ by a morphism

$$\overline{\alpha} : \overline{\mathbb{G}}_{m,K} \longrightarrow H_K$$

with reduction $\overline{\alpha}_0 = \overline{\varphi}_0$ such that $\overline{\alpha}$ extends to a morphism

$$\alpha : \mathbb{G}_{m,K}\left(\varepsilon^2\right) \longrightarrow G_K$$

for some $\pi \in K^\times$ with $\varepsilon := |\pi| < 1$, where

$$\mathbb{G}_{m,K}(\varepsilon) := \left\{t \in \mathbb{G}_{m,K}; \varepsilon \leq |t| \leq \varepsilon^{-1}\right\}.$$

Now consider the morphism

$$\overline{u} := \overline{\alpha} \cdot \overline{\varphi}^{-1} : \overline{\mathbb{G}}_{m,K} \longrightarrow H_K.$$

We can choose the approximation such that $\overline{u}$ factorizes through the formal fiber of H at the unit element. Thus, there exists a closed ball $\mathbb{B}^{s+t}_K$ in this formal fiber such that $\overline{u}$ factorizes through it. The ball $\mathbb{B}^{s+t}_K$ generates an admissible open subgroup U_K with a smooth formal model U which has unipotent reduction; cf. Theorem 7.2.3. As a rigid analytic space U_K is isomorphic to a ball $\mathbb{B}^d_K$ with $d = s+t$. In particular, in our situation the reduction of $\overline{u} : \overline{\mathbb{G}}_{m,K} \to U$ is constant and equals the unit element.

Next consider the morphism

$$w : \mathbb{G}_{m,K}(\varepsilon) \times \mathbb{G}_{m,K}(\varepsilon) \longrightarrow G_K,$$

where w is defined by

$$w(x_1, x_2) = \alpha\bigl(x_1^{-1}x_2^{-1}\bigr) \cdot \alpha(x_1) \cdot \alpha(x_2)$$

for $x_1, x_2 \in \mathbb{G}_{m,K}(\varepsilon)$. Denote by $\overline{w}$ the restriction of w to $\overline{\mathbb{G}}_{m,K} \times \overline{\mathbb{G}}_{m,K}$. Since $\overline{\varphi}$ is a homomorphism of groups, we have

$$\overline{w}(\xi_1, \xi_2) = \overline{u}\bigl(\xi_1^{-1}\xi_2^{-1}\bigr) \cdot \overline{u}(\xi_1) \cdot \overline{u}(\xi_2).$$

Thus, we see that $\overline{w}$ factorizes through the formal fiber of U at the unit element. After enlarging ε, we may assume that w factorizes through U_K. Since $U_K \cong \mathbb{B}_K^d$ is isomorphic to a ball with the unit element as center, we can write the map $\overline{u}$ in terms of a d-tuple of Laurent series

$$\overline{u}(\xi) = \sum_{\nu \in \mathbb{Z}} \overline{u}_\nu \xi^\nu \in \bigl(R\langle\xi, \xi^{-1}\rangle\bigr)^d,$$

where ξ is a coordinate on $\overline{\mathbb{G}}_{m,K}$.

We want to show that $\overline{u}$ converges on $\mathbb{G}_{m,K}(\varepsilon^2)$. It suffices to show

$$|\overline{u}_\nu| \le \varepsilon^{2|\nu|} \quad \text{for all } \nu \in \mathbb{Z}. \tag{7.1}$$

For this we need a consideration of formal group laws. Let

$$F : U \times U \longrightarrow U$$

be the formal group law on U. Then F is of the following type

$$F(X, Y) = X + Y + q(X, Y) \in R\langle X, Y\rangle^d,$$

where $q(X, Y) \in R\langle X, Y\rangle^d$ is a d-tuple power series in $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$, where every monomial with non-zero coefficient is at least of quadratic total order. Indeed, it is divisible at least by a product X_iY_j for some $i, j \in \{1, \dots, d\}$. For the composition of three factors we obtain

$$\begin{aligned} F\bigl(X, F(Y, Z)\bigr) &= F\bigl(X, Y + Z + q(Y, Z)\bigr) \\ &= X + \bigl(Y + Z + q(Y, Z)\bigr) + q\bigl(X, Y + Z + q(Y, Z)\bigr) \\ &= X + Y + Z + Q(X, Y, Z), \end{aligned}$$

where $Q(X, Y, Z)$ is a d-tuple of power series in X, Y, Z, where every monomial is at least of quadratic order. Thus, we obtain the following:

For $x, y, z \in \pi^\lambda U := \mathbb{B}_K^d(|\pi|^\lambda)$ there exist $u \in U$ and $w \in U$ such that

$$x \cdot y = x + y + \pi^{2\lambda} u, \tag{7.2}$$

$$x \cdot y \cdot z = x + y + z + \pi^{2\lambda} w, \tag{7.3}$$

where "$\cdot$" denotes the group law on U and where "$+$" denotes the usual group law on $\mathbb{B}_K^d$ given be the addition of components.

Now we apply these rules to the d-tuple of Laurent series

$$w(\xi_1,\xi_2)=\sum_{\mu,\nu\in\mathbb{Z}} w_{\mu,\nu}\xi_1^{\mu}\xi_2^{\nu}\in\big(R\langle\pi\xi_1,\pi/\xi_1,\pi\xi_2,\pi/\xi_2\rangle\big)^d.$$

Because of $\varepsilon=|\pi|$, the absolute values of the coefficients satisfy

$$|w_{\mu,\nu}|\le\varepsilon^{|\mu|+|\nu|}\quad\text{for all }\mu,\nu\in\mathbb{Z}. \tag{7.4}$$

Furthermore, if $\overline{u}(\overline{\mathbb{G}}_{m,K})\subset\pi^{\lambda}U$ for some $\lambda\in\mathbb{N}$ with $\lambda\ge 1$, Eq. (7.3) implies the following congruence of Laurent series:

$$\begin{aligned}\overline{w}(\xi_1,\xi_2)&=\overline{u}\big(\xi_1^{-1}\xi_2^{-1}\big)\cdot\overline{u}(\xi_1)\cdot\overline{u}(\xi_2)\\ &\equiv\sum_{\nu\in\mathbb{Z}}\big(\overline{u}_\nu\xi_1^{-\nu}\xi_2^{-\nu}+\overline{u}_\nu\xi_1^{\nu}+\overline{u}_\nu\xi_2^{\nu}\big)\ \big(\mathrm{mod}\ \pi^{2\lambda}\big).\end{aligned}$$

Comparing the coefficients shows

$$w_{\nu,\nu}\equiv\overline{u}_{-\nu}\bmod\pi^{2\lambda}.$$

Thus, we obtain by (7.4) that

$$\overline{u}(\overline{\mathbb{G}}_{m,K})\subset\pi^{\lambda}U\Longrightarrow|\overline{u}_\nu|\le\varepsilon^{2\cdot\min\{\lambda,|\nu|\}}\quad\text{for all }\nu\in\mathbb{Z}. \tag{7.5}$$

Moreover, we need an elementary calculation: Consider Laurent series

$$\begin{aligned}a(\xi)&=\sum_{\nu\in\mathbb{Z}}a_\nu\xi^{\nu}\in R\big\langle\xi,\xi^{-1}\big\rangle\quad\text{with }|a_\nu|\le\varepsilon^{2\cdot|\nu|}\text{ for all }\nu\in\mathbb{Z},\\ b(\xi)&=\sum_{\nu\in\mathbb{Z}}b_\nu\xi^{\nu}\in R\big\langle\xi,\xi^{-1}\big\rangle\quad\text{with }|b_\nu|\le\varepsilon^{2\cdot|\nu|}\text{ for all }\nu\in\mathbb{Z}.\end{aligned}$$

Then, for every $i,j\in\mathbb{N}$ we have for the product

$$a(\xi)^i b(\xi)^j=\sum_{\nu\in\mathbb{Z}}c_\nu\xi^{\nu}\in R\big\langle\xi,\xi^{-1}\big\rangle\quad\text{with }|c_\nu|\le\varepsilon^{2\cdot|\nu|}\text{ for all }\nu\in\mathbb{Z}. \tag{7.6}$$

Now we start the proof of the assertion (7.1). After enlarging U_K, we may assume $\overline{u}(\overline{\mathbb{G}}_{m,K})\subset\pi U$ so that $|\overline{u}_\nu|\le\varepsilon$ for all $\nu\in\mathbb{Z}$. It suffices to show by induction on λ that

$$|\overline{u}_\nu|\le\varepsilon^{2\cdot\min\{\lambda,|\nu|\}}\quad\text{for all }\nu\in\mathbb{Z}\text{ and all }\lambda\in\mathbb{N}.$$

The case $\lambda=1$ follows from (7.5), because $\overline{u}$ factorizes through πU. Now let $\lambda\ge 1$ and assume that the induction hypothesis is satisfied for λ. Thus, we can consider the map

$$u:\mathbb{G}_{m,K}\big(\varepsilon^2\big)\longrightarrow U_K$$

given by the Laurent series

$$u(\xi) = \sum_{|\nu| \le \lambda} \overline{u}_\nu \xi^\nu = \sum_{\nu \in \mathbb{Z}} u_\nu \xi^\nu.$$

The latter is a definition for the coefficients u_ν; i.e., $u_\nu = 0$ for $|\nu| > \lambda$. Due to the induction hypothesis the coefficients satisfy

$$|u_\nu| \le \varepsilon^{2|\nu|} \quad \text{for all } \nu \in \mathbb{Z}. \tag{7.7}$$

In particular, because of $\lambda \ge 1$ we know

$$u|_{\overline{\mathbb{G}}_{m,K}} \equiv \overline{u}|_{\overline{\mathbb{G}}_{m,K}} \left(\bmod \pi^{\lambda+1}\right). \tag{7.8}$$

Then we can consider the map

$$\beta := \alpha \cdot u^{-1} : \mathbb{G}_{m,K}\left(\varepsilon^2\right) \longrightarrow G_K.$$

Set $\overline{\beta} := \beta|_{\overline{\mathbb{G}}_{m,K}}$. Then we have that

$$\overline{v} := \overline{\beta} \cdot \overline{\varphi}^{-1} = \overline{u} \cdot u^{-1} : \overline{\mathbb{G}}_{m,K} \longrightarrow U_K.$$

By (7.8) the map $\overline{v}$ sends $\overline{\mathbb{G}}_{m,K}$ to $\pi^{\lambda+1}U$. By (7.5) the coefficients of the power series

$$\overline{v}(\xi) = \sum_{\nu \in \mathbb{Z}} \overline{v}_\nu \xi^\nu \in \left(R\langle \xi, \xi^{-1} \rangle\right)^d$$

satisfy the estimate

$$|\overline{v}_\nu| \le \varepsilon^{2 \cdot \min\{\lambda+1, |\nu|\}} \quad \text{for all } \nu \in \mathbb{Z}. \tag{7.9}$$

Then consider

$$\overline{u} = \overline{v} \cdot u = \overline{v} + u + q(\overline{v}, u).$$

Modulo $\pi^{2(\lambda+1)}$ we have to estimate the coefficients q_ν of

$$q(\overline{v}, u) = \sum_{i,j \in \mathbb{N}^d} q_{i,j} \overline{v}^i u^j = \sum_{\nu \in \mathbb{Z}} q_\nu \xi^\nu \in \left(R\langle \xi, \xi^{-1} \rangle\right)^d.$$

We see by (7.6) that

$$|q_\nu| \le \varepsilon^{2 \cdot \min\{\lambda+1, |\nu|\}} \quad \text{for all } \nu \in \mathbb{Z}$$

because of (7.7) and (7.9). Thus, we obtain

$$|\overline{u}_\nu| \le \varepsilon^{2 \cdot \min\{\lambda+1, |\nu|\}} \quad \text{for all } \nu \in \mathbb{Z}$$

by using (7.7) and (7.9) again, and hence the assertion (7.1) is verified.

Thus, we have shown that the homomorphism $\overline{\varphi} : \overline{\mathbb{G}}_{m,K} \to G_K$ extends to a rigid analytic map $\psi := \alpha \cdot \overline{u}^{-1} : \mathbb{G}_{m,K}(\varepsilon^2) \to G_K$. Obviously, ψ satisfies the functional equation

$$\psi(\xi_1) \cdot \psi(\xi_2) = \psi(\xi_1 \cdot \xi_2) \quad \text{for all } \xi_1, \xi_2 \in \mathbb{G}_{m,K}(\varepsilon).$$

Finally we obtain a homomorphism $\varphi : \mathbb{G}_{m,K} \to G_K$ by setting

$$\varphi\big(\pi^n \xi\big) := \varphi(\pi)^n \cdot \psi(\xi) \quad \text{for } \xi \in \mathbb{G}_{m,K}(\varepsilon) \text{ and } n \in \mathbb{Z}.$$

This finishes the proof. □

7.4 Morphisms from Curves to Groups

In this section we will study morphisms $u_K : C_K \to G_K$ from a curve C_K to a commutative smooth rigid analytic group G_K, where C_K has a semi-stable R-model C over its valuation ring R. The statement requires a result on the group generated by the smooth part $\overline{C}$ of C_K in the style of Theorem 7.2.3. Since we have shown the group generation only for morphisms u_K which are *flat*; cf. Theorem 7.2.3, we are limited to the case of relative curve fibrations $C_K \to S_K$ and flat morphisms $u_K : C_K \to G_K$.

Let us fix the notation for this section. Let G_K be a commutative quasi-compact smooth rigid analytic group of dimension d and $\overline{G}_K$ a connected open subgroup of G_K, which admits a smooth formal R-model $\overline{G}$. Let C_K be a connected smooth rigid analytic curve, which admits a semi-stable formal R-model C with precisely one singular point $\tilde{x}_0 \in \widetilde{C}$ on the reduction $\widetilde{C}$. Let $\xi : C_K \to \mathbb{G}_{m,K}$ be a rigid analytic morphism which restricts to a coordinate function on the formal fiber $C_+(x_0)$, where $x_0 \in C_K$ is a lifting of $\tilde{x}_0$. Denote by $\overline{C}$ the smooth part of C.

Consider a rigid analytic morphism $u_K : C_K \to G_K$ such that $u_K(x_0)$ is equal to the unit element e of G_K. Assume that $u_K : C_K \to G_K$ is generically unramified. Here a morphism of smooth rigid analytic spaces $\psi : X_K \to Y_K$ is called *generically unramified* if the canonical morphism $\psi^* \Omega^1_{Y_K/K} \to \Omega^1_{X_K/K}$ is surjective outside a thin closed analytic subvariety of X_K.

Theorem 7.4.1. *In the above situation assume, in addition, that*

$$u_K(x_1) \cdot u_K(x_2)^{-1} \in \overline{G}_K$$

for all pairs of points x_1, x_2 belonging to the same connectedness component of $\overline{C}_K$. Then there exists a rigid analytic group homomorphism $\varphi : \mathbb{G}_{m,K} \to G_K$ with $\varphi(\overline{\mathbb{G}}_{m,K}) \subset \overline{G}_K$ such that u factorizes into

$$u = (\varphi \circ \xi) \cdot \overline{u},$$

where $\overline{u} : C_K \to \overline{G}_K$ is a rigid analytic morphism.

Before we start the proof, we provide some preparations. We may assume that K is algebraically closed and that ξ gives rise to an isomorphism

$$\xi : C_+(x_0) \xrightarrow{\sim} \left\{ z \in \mathbb{G}_{m,K} ; \alpha < |z| < 1/\alpha \right\}$$

for some $\alpha \in |K^\times|$ with $\xi(x_0) = 1$. For a $\rho \in |K^\times|$ with $\alpha \le \rho \le 1$ put

$$\begin{aligned} C_+(x_0) \supset A(\rho) &:= \xi^{-1}\left(\left\{ z \in \mathbb{G}_{m,K} ; \rho \le |z| \le 1/\rho \right\}\right), \\ C_+(x_0) \supset A(\rho)^+ &:= \xi^{-1}\left(\left\{ z \in \mathbb{G}_{m,K} ; |z| = 1/\rho \right\}\right), \\ C_+(x_0) \supset A(\rho)^- &:= \xi^{-1}\left(\left\{ z \in \mathbb{G}_{m,K} ; |z| = \rho \right\}\right). \end{aligned}$$

Since G_K is smooth, there exists a closed subvariety D_K of an open neighborhood of the unit element of G_K with $e \in D_K$, where D_K is isomorphic to a $(d-1)$-dimensional ball $\mathbb{B}_K^{d-1}$, such that the morphism

$$\psi_K : C_K \times D_K \longrightarrow G_K, \ (x, y) \longmapsto u_K(x) \cdot y,$$

is generically unramified. In particular, the map ψ is étale outside a thin closed analytic subvariety of $C_K \times D_K$ due to the Jacobian criterion, and hence flat. Since $\overline{G}_K$ is open, we may assume $D_K \subset \overline{G}_K$.

Lemma 7.4.2. *For every $\rho \in |K^\times|$ with $\alpha < \rho \le 1$ the maps*

$$\psi : A(\rho)^\pm \times D_K \longrightarrow G_K, \ (x, y) \longmapsto u_K(x) \cdot y,$$

generate an open subgroup $H(\rho)_K := \langle A(\rho)^\pm \times D_K, \psi \rangle \subset G_K$ which admits a smooth formal R-model $H(\rho)$. The reduction of $H(\rho)$ is linear.

Proof. $A(\rho)^+ \times D_K$ has a smooth formal R-models. Since ψ is generically unramified, there exists a dense open formal subscheme $Z(\rho)$ of $A(\rho)^+ \times D_K$ such that $\psi : Z(\rho) \to G_K$ is flat. Then $Z(\rho)$ generates an open subgroup $H(\rho)_K$, which admits a smooth formal R-model with linear reduction by Theorem 7.2.3. Up to a translation the map ψ maps $A(\rho)^+ \times D_K$ into $H(\rho)_K$ due to Theorem 7.2.1. Likewise one deals with $A(\rho)^- \times D_K$. The union of both generates a group $H(\rho)_K$ as required. □

We need a lemma on the domain of convergence of Laurent series.

Lemma 7.4.3. *Let $X = \operatorname{Sp} B$ be an irreducible and reduced affinoid space and U a non-empty open subvariety of X. For $\rho \in |K^\times|$, $\rho < 1$, set*

$$\begin{aligned} A &:= \left\{ z \in \mathbb{G}_{m,K} ; \rho \le |z| \le 1/\rho \right\}, \\ A^- &:= \left\{ z \in \mathbb{G}_{m,K} ; |z| = \rho \right\}, \\ A^+ &:= \left\{ z \in \mathbb{G}_{m,K} ; |z| = 1/\rho \right\}. \end{aligned}$$

Let f be a holomorphic function on $(X \times A^-) \cup (U \times A) \cup (X \times A^+)$. Then f extends to a holomorphic function on $X \times A$.

Proof. Let ξ be the coordinate function of A and consider the Laurent series expansions

$$f|_{X\times A^-} = \sum_{\nu\in\mathbb{Z}} b_\nu^- \xi^\nu \in B\langle \xi/\rho, \rho/\xi\rangle,$$
$$f|_{X\times A^+} = \sum_{\nu\in\mathbb{Z}} b_\nu^+ \xi^\nu \in B\langle \rho\xi, 1/\rho\xi\rangle.$$

The condition of the convergence is given by

$$\lim_{\nu\to\pm\infty} \rho^\nu |b_\nu^-| = 0 \quad \text{and} \quad \lim_{\nu\to\pm\infty} \rho^{-\nu}|b_\nu^+| = 0.$$

Since f is defined on $U \times A$, it follows $b_\nu := b_\nu^+ = b_\nu^-$ for $\nu \in \mathbb{Z}$. Thus, we get $\lim_{|\nu|\to\infty} \rho^{|\nu|}|b_\nu| = 0$. This implies the extension of f to $X \times A$. □

Lemma 7.4.4. *In the above situation, for $\rho \in |K^\times|$ with $1 \geq \rho > \alpha$, let*

$$H(\rho) = \langle (A(\rho)^- \cup A(\rho)^+) \times D_K, \psi_K\rangle$$

be the smooth formal R-group scheme generated by the smooth formal part of $A(\rho) \times D_K$ via ψ_K and $H(\rho)_K$ the induced open subgroup of G_K. Then we have the following results:

(a) *$H(\rho_1)_K \subset H(\rho_2)_K$ for all $1 \geq \rho_1 \geq \rho_2 > \alpha$.*
(b) *$H(\rho)_K \subset \overline{G}_K$ for all $\rho \in |K^\times|$, $1 \geq \rho > \alpha$.*
(c) *If the reduction of $H(\rho)$ is unipotent, then $u_K(A(\rho)) \subset H(\rho)_K$.*
(d) *If the reduction of $H(1)$ is unipotent, then $H(\rho)$ has unipotent reduction for all $\rho \in |K^\times|$ with $1 \geq \rho > \alpha$.*
(e) *If the reduction of $H(1)$ is unipotent, then $u_K(A(\rho)) \subset \overline{G}_K$.*

Proof. (a) Fix some $\rho \in |K^\times|$ with $\alpha < \rho \leq 1$. The group $H(\rho)$ has linear reduction by Lemma 7.4.2, and hence the underlying rigid analytic variety is an open affinoid subvariety of G_K. Then consider the morphism

$$v : \overline{\mathbb{G}}_{m,K} \times (A(\rho)^+ \cup A(\rho)^-) \longrightarrow G_K,\ (z,x) \longmapsto u_K(zx) \cdot u_K(x)^{-1},$$

where $zx := \xi^{-1}(z\xi(x))$. In particular, one has that $v(1,x) = e$ for all $x \in A(\rho)$. Since $H(\rho)_K$ is an open subdomain of G_K, there exists an open neighborhood $U \subset \overline{\mathbb{G}}_{m,K}$ of 1 with $v(U \times A(\rho)) \subset H(\rho)_K$. Thus, $v(\overline{\mathbb{G}}_{m,K} \times A(\rho))$ is contained in $H(\rho)_K$ by Lemma 7.4.3 and hence (a) is clear.

(b) Similarly as in the proof of Proposition 3.6.6 one constructs a filter $(U_i; i \in \mathbb{N})$ of neighborhoods consisting of admissible open subsets U_i of G_K such that

$$\overline{G}_K \Subset_{G_K} U_i \quad \text{and} \quad \bigcap_{i\in\mathbb{N}} U_i = \overline{G}_K.$$

By reasons of continuity, for every $i \in \mathbb{N}$ there exists an integer $j \in \mathbb{N}$ such that the group law

$$U_j^{2d} \longrightarrow U_i, \quad (x, y) \longmapsto x_1 \cdot y_1^{-1} \cdot \ldots \cdot x_{2d} \cdot y_{2d}^{-1},$$

maps to U_i, where d is the dimension of G_K. For every $j \in \mathbb{N}$ there is a radius $\rho_j \in |K^\times|$, $\rho > \rho_j > \alpha$, with $u_K(z_1) \cdot u_K(z_2)^{-1} \in U_j$ for all pairs z_1, z_2 in $A(\rho_j)^+$ or in $A(\rho_j)^-$. Thus, we see that $H(\rho_j)_K \subset U_i$, because every element of $H(\rho_j)_K$ is a $2d$-fold product of pairs in $u(A(\rho_j)^+)$ and $u(A(\rho_j)^-)$, up to an element of $\overline{G}_K$ induced by D_K. Now it follows from (a) that $H(\rho)_K \subset H(\rho_j)_K \subset U_i$ for all $i \in \mathbb{N}$ and hence $H(\rho)_K \subset \overline{G}_K$.

(c) Since $H(\rho)$ has unipotent reduction, the underlying variety $H(\rho)$ is isomorphic to a d-dimensional ball B_K, which is relatively compact in G_K. Due to Corollary 3.6.18 there exists a ball B'_K in G_K such that $B_K \Subset B'_K$. Then B'_K generates an open subgroup H'_K which has a smooth formal R-model with unipotent reduction due to Theorem 7.2.3. For $\delta \in |K^\times|$ put

$$\Delta(\delta) := \left\{(x, y) \in A(\rho) \times A(\rho); \delta \le \left|\xi(x)\right| \big/ \left|\xi(y)\right| \le 1/\delta\right\}.$$

Then consider the morphism

$$v : \Delta(\delta) \longrightarrow G_K, \ (x, y) \longmapsto u_K(x) \cdot u_K(y)^{-1}.$$

Due to (a) one has $H(\varrho)_K \subset H(\rho)_K$ for all $\varrho \in |K^\times|$ with $1 \ge \varrho \ge \rho$. Thus, we have $v(\Delta(1)) \subset H(\rho)_K$. Since $H(\rho)_K \Subset H'_K$, there exists a element $\delta \in |K^\times|$ with $\delta < 1$ such that $v(\Delta(\delta)) \subset H'_K$.

Now consider two points x, y in $A(\rho)$. Then there exists a finite sequence $x_0 := x, x_1, \ldots, x_n = y$ such that $(x_{i-1}, x_i) \in \Delta(\delta)$ for $i = 1, \ldots, n$. Thus, we obtain

$$u_K(x) \cdot u_K(y)^{-1} = v(x_0, x_1) \cdot v(x_1, x_2) \cdot \ldots \cdot v(x_{n-1}, x_n) \in H'_K,$$

because H'_K is a subgroup of G_K. Since we can choose H'_K arbitrarily close to $H(\rho)_K$, we see that $u_K(A(\rho)) \subset H(\rho)_K$.

(d) Assume the contrary. The reduction of $H := H(\rho)$ is a linear group, because the reduction of $A(\rho)$ is rational. Since we assumed that the residue field k is algebraically closed, the canonical reduction $\widetilde{H}$ of H is a product of a torus $\widetilde{T}$ and a unipotent group $\widetilde{U}$ by the theorem of Chevalley.

Then denote by Ω the subset of H_K consisting of all the points which specialize in $\widetilde{H}$ outside $\{\tilde{e}_T\} \times \widetilde{U}$, where $\tilde{e}_T \in \widetilde{T}$ is the unit element of $\widetilde{T}$. The preimage $u_K^{-1}(\Omega) \subset A(\rho)$ is a quasi-compact subdomain of $A(\rho)_K$. Since $H(1)$ has unipotent reduction, $A(1)$ is disjoint from $u_K^{-1}(\Omega)$. Due to the maximum principle the assumption implies that there exists a maximal $\beta \in |K^\times|$ with $1 > \beta \ge \rho$ such that $u_K(A(\beta)) \cap \Omega \ne \emptyset$.

So the reduction $\widetilde{H}(\beta)$ contains a torus. Indeed, otherwise we would have $u_K(A(\beta)) \subset H(\beta)_K$ due to (c) and hence $u_K(A(\beta))$ would not meet Ω, because the reduction $\widetilde{H}(\beta)$ would be contained in $\{\tilde{e}_T\} \times \widetilde{U}$. Thus, the reduction map

$\widetilde{H}(\beta) \to \widetilde{H}(\rho) \to \widetilde{T}$ followed by the projection onto the torus is not constant. In particular, the reduction map $\widetilde{A}(\beta) \to \widetilde{T}$ is not constant.

We may assume that its restriction to $A(\beta)^+$ is not constant; otherwise, consider $A(\beta)^-$. The reduction of $A(\beta)^+$ is isomorphic to $\mathbb{G}_{m,k}$ and hence the morphism of the reductions is given by a multiplicative group homomorphism $\widetilde{\varphi}$ up to a constant of $\widetilde{H}(\rho)$. Due to Proposition 5.6.7 the map $\widetilde{\varphi}$ lifts to a homomorphism $\overline{\varphi} : \overline{\mathbb{G}}_{m,K} \to H(\beta)_K$. Then it follows from Proposition 7.3.1 that $\overline{\varphi}$ extends to a group homomorphism $\varphi : \mathbb{G}_{m,K} \to G_K$. Thus, we can write

$$u_K = w_K \cdot (\varphi \circ \xi) : A(\rho) \longrightarrow G_K$$

such that $w_K|_{A(\beta)^+}$ has a reduction, which maps to the unipotent part $\widetilde{U}$. In particular, $w_K|_{A(\beta)^+}$ generates a subgroup H'_K with smooth formal R-model H', which has unipotent reduction. The underlying rigid analytic variety is a d-dimensional polydisc, so we can enlarge H'_K by a polydisc D'_K with $H'_K \Subset D'_K$ due to Corollary 3.6.18. Since D'_K generates an open subgroup H''_K, which has a smooth formal R-model with unipotent reduction due to Theorem 7.2.3, we see that there is a $\gamma > \beta$ such that $w_K|_{A(\gamma)^+}$ generates a similar subgroup with unipotent reduction. Thus, $u_K|_{A(\gamma)^+}$ generates a subgroup with a non-trivial torus part which is induced by $(\xi \circ \varphi)|_{A(\gamma)^+}$. This implies a contradiction to the maximality of β.

(e) Due to (b) it suffices to show $u(A(\rho)) \subset H(\rho)_K$ for all $\rho \in |K^\times|$ with $1 \geq \rho > \alpha$. Since the reduction of $H(\rho)$ is unipotent by (d), the assertion follows from (c). □

Proof of Theorem 7.4.1. The group $H(1) := \langle A(1), u \rangle$ has linear reduction. If the reduction $\widetilde{H}(1)$ of $H(1)$ is unipotent, the map u_K maps C_K into $\overline{G}_K$ by Lemma 7.4.4(e), and hence the assertion is true.

If the reduction of $H(1)$ is not unipotent, it contains a non-trivial maximal torus part $\widetilde{T}$. By the theorem of Chevalley, $\widetilde{H}(1)$ splits into a product of $\widetilde{T}$ and a unipotent group $\widetilde{U}$; cf. [15, 9.2/2]. Then the reduction map

$$\widetilde{\varphi} : \mathbb{G}_{m,k} \xrightarrow{\tilde{\xi}^{-1}} \widetilde{A}(1) \xrightarrow{\tilde{u}} \widetilde{H}(1) \xrightarrow{p} \widetilde{T}$$

followed by the projection is not constant and hence a non-trivial group homomorphism, because $u_K(x_0) = e$ and $\xi(x_0) = 1$. Due to Proposition 5.6.7 the map lifts to a group homomorphism $\overline{\varphi} : \overline{\mathbb{G}}_{m,K} \to H(1)_K$. Since $A(1) \Subset C_K$, the image $u_K(A(1)) \Subset G_K$ is relatively compact in G_K. Thus, it extends to a group homomorphism $\varphi : \mathbb{G}_{m,K} \to G_K$ by Proposition 7.3.1. Now consider the map

$$w_K := u_K \cdot (\varphi \circ \xi)^{-1} : C_K \longrightarrow G_K, \; x \longmapsto u_K(x) \cdot \varphi\big(\xi(x)\big)^{-1}.$$

The group $\langle A(1), w_K \rangle$ has unipotent reduction and $\langle A(1), w_K \rangle \subset H(1)_K$, because $\varphi(\overline{\mathbb{G}}_{m,K}) \subset H(1)_K$. Thus, it follows from Lemma 7.4.4(e) that w_K maps C_K to $\overline{G}_K$ as discussed before. □

7.5 Stable Reduction of Relative Curves

In this section we will show the stable reduction of rigid analytic curve fibrations. As before let K be a non-Archimedean field with valuation ring R and S_K be a normal quasi-compact rigid analytic space. By a *smooth rigid analytic curve fibration* we mean a quasi-compact, separated, smooth rigid analytic morphism $f_K : C_K \to S_K$ with purely 1-dimensional fibers. We say a model $f : C \to S$ over $\operatorname{Spf} R$ of such a fibration has *semi-stable* reduction if f is flat and if its geometric fibers are reduced and have at most ordinary double points as singularities. The geometric meaning of such singularities for the rigid analytic fibers is completely clarified in Proposition 4.1.12. For the existence of such models one has to allow certain extensions of the base S_K. Therefore, we introduce the following notions.

Definition 7.5.1. Let $f_K : X_K \to S_K$ be a morphism of quasi-compact, separated rigid analytic spaces. We say f_K admits *locally for the rigid analytic topology* or *for the rigid analytic étale topology*, respectively, on S_K a model with a certain property if there exists a quasi-compact surjective morphism $S'_K \to S_K$ which is locally with respect to the rigid analytic topology on S'_K an open immersion or étale, respectively, such that $f_K \times_{S_K} S'_K$ has a model with that property. Sometimes it is necessary, in addition, to localize X_K after such a base change, then we will call it locally (étale locally) on S_K and locally on X_K.

For proper smooth curve fibrations the existence of semi-stable models follows from the compactness of the moduli space of stable curves.

Theorem 7.5.2. *Let S_K be an affinoid space and $f_K : C_K \to S_K$ a smooth projective rigid analytic curve fibration with connected geometric fibers. Let $s_1, \ldots, s_n : S_K \to C_K$ be sections of f_K for some $n \geq 3$, which do not meet each other.*

Then, étale locally on S_K, there exists a semi-stable model $f : C \to S$ of f_K such that the sections $s_1, \ldots, s_n$ extend to sections $\sigma_1, \ldots, \sigma_n : S \to C$, which factorize through the smooth locus of C/S and do not meet each other.

Actually, one can choose $f : C \to S$ as an n-marked stable curve with respect to the sections $\sigma_1, \ldots, \sigma_n$; cf. Definition 4.4.2.

Proof. We deduce the assertion from the existence of the algebraic stack which classifies n-marked stable projective curves.

Deligne and Mumford have shown in [21, Theorem 5.2] that there exists the moduli space $\overline{\mathfrak{M}}_g$ classifying stable curves of genus g for $g \geq 2$ as an algebraic stack which is smooth and proper over $\operatorname{Spec} \mathbb{Z}$. Later Mumford and Knudsen proved in [54, 77] and [53] that there exists a moduli space $\overline{\mathfrak{M}}_{g,n}$ classifying n-marked stable curves if $2g + n \geq 3$ as an algebraic stack which is smooth and proper over $\operatorname{Spec} \mathbb{Z}$; cf. [53, II, Theorem 2.7]. The moduli space $\mathfrak{M}_g$ classifying smooth curves of genus g is a dense open substack of $\overline{\mathfrak{M}}_g$, likewise for $\mathfrak{M}_{g,n}$.

In our application we perform the base change $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$ and consider the algebraic stack $\overline{\mathfrak{M}}_{g,n}$ only over $\operatorname{Spec} R$. There exists an étale covering $\mathcal{U} \to \overline{\mathfrak{M}}_{g,n}$ and a "universal" n-marked stable curve $\mathcal{C} \to \mathcal{U}$ over $\mathcal{U}$, where

$\mathcal{U}$ is a smooth R-scheme of finite type equipped with an "equivalence relation" $\mathcal{R} = \mathrm{Isom}(p_1^*\mathcal{C}, p_2^*\mathcal{C}) = \mathcal{U} \times_{\overline{\mathfrak{M}}_{g,n}} \mathcal{U} \longrightarrow \mathcal{U} \times \mathcal{U}$, which is a finite and unramified morphism of schemes and whose projections $\mathcal{R} \to \mathcal{U}$ are étale surjective; cf. [21, 1.11].

With these tools our assertion can be verified in the following way. We choose an admissible R-algebra A with $A_K = A \otimes_R K$. Then we put $S = \mathrm{Spf}\, A$ and $\overline{S} = \mathrm{Spec}\, A$. We can view S_K as the set of closed points in the complement $\overline{S} - S_0$, where $S_0 := \mathrm{Spec}\, A/A\pi$. Due to the GAGA-principle in Theorem 1.6.11 the curve $C_K \to S_K$ is induced by a smooth projective n-marked curve $\overline{C}_K \to \overline{S}_K$, where $\overline{S}_K = \mathrm{Spec}(A \otimes_R K)$. So our morphism f_K and our sections give rise to a morphism $S_K \to \overline{\mathfrak{M}}_{g,n}$.

Then we obtain the étale cover $p_1 : S_K^1 := S_K \times_{\overline{\mathfrak{M}}_{g,n}} \mathcal{U} \to S_K$ and the morphism $p_2 : S_K^1 \to \mathcal{U}$. The pull-back $p_1^* C_K$ and the pull-back $p_2^*\mathcal{C}$ of the universal curve $\mathcal{C} \to \mathcal{U}$ become isomorphic over S_K^1.

Now we choose an affinoid subdomain S_K' in S_K^1 which is étale surjective over S_K. In fact, since S_K^1 is of finite type over S_K, we can consider this as a rigid analytic space over S_K. Since S_K is quasi-compact, there exists a finite collection of open affinoid subschemes which cover S_K, and hence the disjoint sum of these subdomains is an affinoid space S_K' which maps étale surjective to S_K. For the following we replace S_K by S_K'.

Next we choose some R-model S of S_K. Then we can consider the schematic closure $\Gamma \subset S \times_R \mathcal{U}$ of the graph of $p_2 : S_K \to \mathcal{U}$. The projection $p_1 : \Gamma \to S$ is an isomorphism over S_K. By the flattening technique there exists a blowing-up $S' \to S$ with center in the special fiber S_0 such that the strict transform $p_1' : \Gamma' \to S'$ is flat. Then p_1' is an open immersion, and hence an isomorphism, because the moduli space $\overline{\mathfrak{M}}_{g,n}$ is proper over $\mathrm{Spec}\, \mathbb{Z}$ and $\mathcal{R} \to \mathcal{U} \times \mathcal{U}$ is finite. Thus, we see that the pull-back $p_2^*\mathcal{C}$ of the universal curve $\mathcal{C}$ under the projection $p_2' : \Gamma' \to \mathcal{U}$ gives rise to an extension of C_K to an n-marked stable curve over S' which extends the given curve C_K and its n sections. □

Remark 7.5.3. (a) The essential ingredient in the proof of Theorem 7.5.2 is the result of Deligne and Mumford. One can avoid to use the stack $\overline{\mathfrak{M}}_{g,n}$ but one has to worry about combinatorial problems.

In fact, if $g = 0$ and $n = 3$ there is nothing to show, because $C_K \cong \mathbb{P}^1_{S_K}$ and the three points are identified with the points $0, 1, \infty$.

If $n \geq 4$, then one adds the points one by one. To add a point one refines the base space S_K with respect to rational coverings and introduces relative annuli in the manner of Lemma 2.4.5 if the distance of two sections comes below a certain value, which is identified by the sections already treated.

In the case $g \geq 1$ the result of Deligne and Mumford provides a stable model $C \to S$ of $C_K \to S_K$. If $g = 1$, then one has $n \geq 1$ and their result works as well. Now if $n \geq 1$, one faces similar combinatorial problems as in the case $g = 0$. On the one hand one has to introduce blowing-ups in the relative singular locus. On the other hand one has to blow-up the locus where two sections specialize to the same point.

(b) One can also avoid the complicated use of stacks by the following idea of Gabber; cf. [20, §3]. Let $g \geq 1$ and $2g-2+n \geq 1$. As in the proof of Theorem 7.5.2 let $\overline{\mathfrak{M}}_{g,n}$ be the algebraic stack over $\operatorname{Spec}\mathbb{Z}$ classifying n-marked stable curves of genus g. Let $\mathfrak{M}_{g,n} \subset \overline{\mathfrak{M}}_{g,n}$ be the open sub-stack which classifies smooth n-marked projective curves. Let ℓ be a prime number $\ell \geq 3$ and different from the characteristic of the residue field k. Let

$$\mathfrak{M}_{g,n,\ell} \longrightarrow \mathfrak{M}_{g,n}[1/\ell] = \mathfrak{M}_{g,n} \times_{\mathbb{Z}} \mathbb{Z}[1/\ell]$$

be the finite étale cover given by trivializing the ℓ-torsion of the Jacobian of the universal curve of genus g over $\mathfrak{M}_{g,n}[1/\ell]$. Note that $\mathfrak{M}_{g,n,\ell}$ is a scheme $M_{g,n,\ell}$; cf. [20, 3.7]. Finally, let

$$\overline{\mathfrak{M}}_{g,n,\ell} \longrightarrow \overline{\mathfrak{M}}_{g,n}[1/\ell] := \overline{\mathfrak{M}}_{g,n} \times_{\mathbb{Z}} \mathbb{Z}[1/\ell]$$

be the normalization of $\overline{\mathfrak{M}}_{g,n}[1/\ell]$ in the function field of $M_{g,n,\ell}$. Note that $\overline{\mathfrak{M}}_{g,n,\ell} = \overline{M}_{g,n,\ell}$ is a projective scheme over $\mathbb{Z}[1/\ell]$; cf. [20, 3.7]. By pulling back from $\overline{\mathfrak{M}}_{g,n}[1/\ell]$ one obtains a "universal" stable n-marked proper curve of genus g over $\overline{M}_{g,n,\ell}$.

Then we can proceed as in the proof of Theorem 7.5.2 by replacing $\mathcal{U}$ by $\overline{M}_{g,n,\ell}$. After a finite étale extension $S'_K \to S_K$ one can add a level ℓ-structure to the given curve $C_K \to S_K$. In this case it is evident that the pull-back of the universal curve coincides with the given curve over $S'_K \times_{\mathfrak{M}_g} \mathcal{U}$, because $M_{g,n,\ell}$ is a fine moduli space.

The case of not necessarily proper rigid analytic curve fibrations will be reduced to Theorem 7.5.2 by showing the following compactification theorem, which involves the approximation techniques provided in Sect. 3.6.

Theorem 7.5.4. *Let $f_K : X_K \to S_K$ be a smooth rigid analytic curve fibration. Then, étale locally on S_K and locally on X_K, there exists a smooth projective S_K-compactification; i.e., in the local sense as defined above there exists an S_K-embedding $X_K \hookrightarrow P_K$ of X_K into a flat S_K-curve which is smooth and projective over its image in S_K; cf. Definition 7.5.7.*

The proof, which is pretty hard, is done in three steps. First, one shows that, after a suitable base change, there exists a flat model $f : X \to S$ of f_K which has reduced geometric fibers; cf. Theorem 3.4.8.

Then we compactify such a formal morphism with respect to a topology, which is given by subschemes of the special fiber of S and which is finer than the given one. This is somehow an infinitesimal compactification. This part is formally algebraic and not very hard.

As a final step we use approximation arguments in order to provide a compactification with respect to the rigid analytic topology. This is done by approximating morphisms to Hilbert schemes and by applying the approximation technique in Theorem 3.6.7.

Let us start with the second step. We want to compactify formal relative curves with reduced geometric fibers. We will do this only on certain formal levels, where the ideals defining these formal structures may be different from the ideal of definition we started with.

Proposition 7.5.5 (Infinitesimal compactification). *Let $f : X \to S$ be a flat affine morphism of formal schemes with reduced geometric fibers which are equidimensional of dimension* 1*. Then there exists a stratification*

$$S_0 := S_0^1 \cup \cdots \cup S_0^r$$

of the special fiber S_0 by subschemes S_0^ϱ of S_0 and a rig-étale cover $S' \to S$ of S such that $X' = X \times_S S'$ admits a projective compactification P^ϱ over the formal completion of S' with respect to each stratum S_0^ϱ for $\varrho = 1, \ldots, r$. The compactification can be chosen to be smooth at infinity. Moreover, there exists a closed subscheme Δ^ϱ of P^ϱ which is an ample relative Cartier divisor on P^ϱ.

This statement needs some explanations. Consider an open subscheme U of S such that S_0^ϱ is closed in U_0. So there exists a sheaf $\mathcal{J}$ of open ideals of $\mathcal{O}_U$ such that $\mathcal{J}\mathcal{O}_{U_0}$ has the vanishing locus S_0^ϱ. Over the completion $\widehat{U}(\mathcal{J})$ of U with respect to $\mathcal{J}$, there exists a projective $\widehat{U}(\mathcal{J})$-compactification of X; i.e., of $X \times_S \widehat{U}(\mathcal{J})$, which is smooth at infinity.

In particular, if U' is an open subscheme of S', whose special fiber U_0' is mapped to S_0^ϱ, there exists a formal compactification of $X \times_S U'$ with respect to the ideal of definition $\pi\mathcal{O}_{U'}$. The latter is an S-compactification of X over U'. For the proof of Proposition 7.5.5 we first provide an infinitesimal compactification around a given point s of S_0. Then the assertion of Proposition 7.5.5 follows by Noetherian induction.

Lemma 7.5.6. *In the situation of Proposition 7.5.5 let s be a point of S_0, and denote by $\overline{s}$ its schematic closure in S_0. Let $\mathcal{J}$ be a finitely generated open ideal of $\mathcal{O}_S$ whose vanishing locus is $\overline{s}$. In the following we denote by the index "n" the reduction modulo $\mathcal{J}^n$ for $n \in \mathbb{N}$. Then there exists*

(i) *an open neighborhood U of s in S and a rig-étale cover $U' \to U$,*
(ii) *a projective flat $U'(n)$-scheme $P'(n) \to U'(n)$ together with an open immersion $X'(n) \hookrightarrow P'(n)$ such that*
   (a) *the geometric fibers of $P'(n)/U'(n)$ are reduced,*
   (b) *$P'(n)$ is smooth over $U'(n)$ at $P'(n) - X'(n)$,*
   (c) *$X'(n)$ is $U'(n)$-dense in $P'(n)$,*
   (d) *there exists a $U'(n)$-ample invertible sheaf $\mathcal{O}_{P'(n)}(\Delta(n))$, where $\Delta(n)$ is an effective relative Cartier divisor, which contains $P'(n) - X'(n)$ and which is contained in the smooth locus of $P'(n)$.*

In particular, if n varies, the schemes $P'(n)$ and the divisors $\Delta'(n)$ can be chosen in a coherent way; i.e., they constitute a formal scheme $\widehat{P}'(\mathcal{J})$ and a relative Cartier *divisor $\widehat{\Delta}'(\mathcal{J})$ on $\widehat{P}'(\mathcal{J})$ over the $\mathcal{J}$-adic completion $\widehat{U}'(\mathcal{J})$ of U'. In particular, $\widehat{P}'(\mathcal{J})$ is projective over $\widehat{U}'(\mathcal{J})$.*

Proof. Let ℓ be the residue field $k(s)$ of the given point s.

First we consider the case, where the characteristic of ℓ is $p > 0$. Then there exists a finite radical field extension ℓ'/ℓ such that the natural normal compactification $\overline{X \times_R \ell'}$ of $X \times_R \ell'$ is smooth at infinity and such that the points at infinity have residue fields which are étale over ℓ'. We can also suppose that there is a closed subscheme Δ' of $\overline{X \times_R \ell'}$, which is an ample relative Cartier divisor and which contains the complement of $X \times_R \ell'$. We choose an affine open neighborhood $V(\ell')$ in $\overline{X \times_R \ell'}$ of infinity.

We can realize ℓ' as an extension of ℓ by adjoining elements $\alpha_1, \ldots, \alpha_r$, which satisfy equations of the type

$$\alpha_\varrho^{p^{e(\varrho)}} = \overline{a}_\varrho, \quad \text{where } \overline{a}_\varrho \in \ell^\times \text{ for } \varrho = 1, \ldots, r.$$

After replacing S by an open neighborhood U of s, we may assume that $\overline{a}_1, \ldots, \overline{a}_r$ are induced by invertible sections $a_1, \ldots, a_r$ of $\mathcal{O}_S$. Let $S^* \to S$ be the admissible formal blowing-up of the ideal $(p, \mathcal{J}^2)$. Then consider the open subschemes

$$S^*(1) := \{s^* \in S^*; \mathcal{J}^2 \text{ is generated by } p \text{ at } s^*\},$$
$$S^*(2) := \{s^* \in S^*; p \text{ is contained in } \mathcal{J}^2 \text{ at } s^*\}.$$

Now look at the finite rig-étale map

$$S^*(1) \longleftarrow V\big(\alpha_\varrho^{p^{e(\varrho)}} - a_\varrho\big) \subset \mathbb{D}^r_{S^*(1)}(\alpha_1, \ldots, \alpha_r),$$

where $(\alpha_1, \ldots, \alpha_r)$ are the coordinate functions on $\mathbb{D}^r_{S^*}$, which is rig-étale, because p is invertible over $S^*(1)_{\mathrm{rig}}$. In the case of $S^*(2)$ we may assume that $\mathcal{J}^2$ is principal; say generated by an element $b \in \mathcal{O}_{S^*}(S^*(2))$. Then look at the finite map

$$S^*(2) \longleftarrow V\big(\alpha_\varrho^{p^{e(\varrho)}} - b\alpha_\varrho - a_\varrho\big) \subset \mathbb{D}^r_{S^*(2)}(\alpha_1, \ldots, \alpha_r),$$

which is rig-étale, because $p = b^2 c$ for some $c \in \mathcal{O}_{S^*}(S^*(2))$. So we see that there exists an open neighborhood U of s and a rig-étale cover $S' \to S$ such that every point in S' above s has residue field ℓ'.

If the characteristic is zero, it is evident that there exists an étale extension $S' \to S$ and an open neighborhood U of s such that every point in S' above s has residue field ℓ'.

Now we assert that after a suitable shrinking of U there exists a flat projective U'-scheme, which satisfies the assertion of the lemma.

Indeed, let s' be the reduced pull-back of s in S'. We have a projective compactification $P(s')$ of $X(s')$ over s' which is smooth at infinity and we have a Cartier divisor $\Delta(s')$ on $P(s')$, which is étale over s' containing infinity and which intersects every irreducible component of $P(s')$ in the required way. There exists a coherent open ideal $\mathcal{J}'$ of $\mathcal{O}_{S'}$ which is of finite type with $V(\mathcal{J}') = V(\mathcal{J}\mathcal{O}_{S'})$ such that, after shrinking U, the compactification $P(\mathcal{J}')$ and the Cartier divisor $\Delta(\mathcal{J}')$ exists over $U \cap V(\mathcal{J}')$. Moreover, we may assume that the affine open neighborhood $V(s')$ in $P(s')$ of infinity is defined over $U \cap V(\mathcal{J}')$, say $V^* \to U \cap V(\mathcal{J}')$, and that it is smooth over $U \cap V(\mathcal{J}')$.

Since $\mathcal{J}'/\mathcal{J}\mathcal{O}_{S'}$ is nilpotent, we can lift everything to a smooth affine scheme V' over $U \cap V(\mathcal{J}\mathcal{O}_{S'})$. Likewise we can lift the gluing map of $X \times_U V(\mathcal{J}')$ and V^*, which defines $P(\mathcal{J}')$, to a gluing map of $X \times_U V(\mathcal{J}\mathcal{O}_{S'})$ and V'. Thus, we obtain a proper flat relative curve P' over $U \cap V(\mathcal{J}\mathcal{O}_{S'})$. Likewise we can lift the relative Cartier divisor $\Delta(\mathcal{J}')$ to a relative Cartier divisor Δ' on P'. Since Δ' is ample, P' is projective over $U \cap V(\mathcal{J}\mathcal{O}_{S'})$. By a similar procedure one lifts from $U \cap V(\mathcal{J}\mathcal{O}_{S'})$ to the level $U \cap V(\mathcal{J}^n\mathcal{O}_{S'})$ for every $n \in \mathbb{N}$. □

Proof of Proposition 7.5.5. If the residue field of R has characteristic zero, it is easy to see that there exists a stratification of S_0 such that X_0 admits a compactification over the strata, which are smooth at infinity. Indeed, take a normal projective closure of X over (generic) points in S_0. The infinitesimal compactification at such a point can be constructed from such a compactification by a lifting procedure as described at the end of the proof of Lemma 7.5.6. The existence of the stratification follows by Noetherian induction on S_0.

In the case, where the residue field of R has positive characteristic, one has to perform a base change $S' \to S$ as described in the proof of Lemma 7.5.6 in order to get compactifications which are smooth at infinity. The method of Lemma 7.5.6 gives us a coherent system of compactifications on every level and hence a formal compactification as desired. As above the existence of the stratification follows by Noetherian induction on S_0. □

As a next problem we will discuss the third step; namely, to provide a compactification of X/S with respect to the rigid analytic topology, which is coarser than the one considered in Proposition 7.5.5. Since we have to work locally on X, we introduce the following definition.

Definition 7.5.7. Let $X \to S$ be a flat morphism of admissible formal schemes. Assume that the associated rigid analytic morphism $X_K \to S_K$ is a smooth rigid analytic curve fibration.

X is said to be *S-compactifiable* if there exists an open S-immersion of X into a flat projective formal relative curve $C \to U$, where U is the image of X under the morphism $X \to S$ and where the associated rigid analytic curve $C_K \to U_K$ is smooth.

X is said to be *locally S-compactifiable* if for every closed point x_0 of X_0 there exists an open neighborhood V which is *S-compactifiable*.

If U is an open subscheme of S and if $X \times_S U$ is (locally) compactifiable over U, we say X is (locally) S-compactifiable over U.

Proposition 7.5.8 (Extension of compactifications). *Let $X \to S$ be an affine flat morphism of admissible formal schemes such that the associated rigid analytic map*

$X_K \to S_K$ is a smooth rigid analytic curve fibration. Let $V = \mathrm{Spf}(B)$ be an open affine subscheme of X which is faithfully flat over an open affine subscheme $U = \mathrm{Spf}(A)$ of S. Assume that V is S-compactifiable.

Then, after base change by an admissible formal blowing-up $S' \to S$, there exist an admissible formal blowing-up $X' \to X$ and an open subscheme V' of X' such that the schematic closure of $(X' \times_X V)_0$ is contained in V'_0 and such that V' is S-compactifiable.

The last condition means that $V_K \Subset_{X_K} V'_K$ for the generic fibers; i.e., V_K is relatively compact with respect to X_K contained in V'_K.

Proof. Consider the following situation:

$$\begin{array}{ccccc} V & \hookrightarrow & C & \hookrightarrow & \mathbb{P}^N_U \\ & \searrow & \downarrow & \swarrow & \\ & & U & & \end{array}$$

where $V \hookrightarrow C$ is an open U-immersion into a flat projective formal U-curve C with smooth rigid analytic fibers and where C is immersed into $\mathbb{P}^N_U$ by a very ample line bundle. We may assume that U is connected; then the fibers of C_0 over U_0 have a constant Hilbert polynomial Φ with respect to the given very ample invertible sheaf. We may assume that the degree d with respect to the embedding into $\mathbb{P}^N$ of a fiber of C/U satisfies $d \geq (2g-1)$, where g is the genus of the rigid analytic fibers.

Due to Grothendieck's formal GAGA [39, III_1, 5.4], the morphism $C \to U$ is induced by an algebraic morphism $\overline{C} \longrightarrow \overline{U} = \mathrm{Spec}\, A$, where $\overline{C}$ is a closed subscheme of $\mathbb{P}^N_{\overline{U}}$, which is flat over $\overline{U}$ with Hilbert polynomial Φ. Let $\overline{H}$ be the Hilbert scheme parameterizing subschemes of $\mathbb{P}^N_{\mathbb{Z}}$ with Hilbert polynomial Φ. Note that $\overline{H}$ is proper over $\mathbb{Z}$; cf. [15, §8.2/9]. The morphism $\overline{C} \to \overline{U}$ corresponds to a morphism $\overline{\varphi} : \overline{U} \to \overline{H}$, and hence to a morphism $\varphi : U \to \overline{H}$.

Now consider the associated rigid analytic situation

$$\begin{array}{ccccc} V_K & \hookrightarrow & C_K & \hookrightarrow & \mathbb{P}^N_{U_K} \\ & \searrow & \downarrow & \swarrow & \\ & & U_K. & & \end{array}$$

Over the rigid analytic part, the family $C_K \to U_K$ is assumed to be a flat family of smooth curves of genus g. Let $H \subset \overline{H}$ be the open subscheme parameterizing smooth curves of genus g in $\mathbb{P}^N_{\mathbb{Z}}$. Note that $H \otimes_{\mathbb{Z}} K$ is a K-scheme of finite type and denote by H_K the associated rigid analytic space. Thus, due to Proposition 3.6.6 the rigid analytic space H_K has no boundary; i.e., every quasi-compact open rigid analytic subvariety of H_K is relatively compact in H_K. Moreover, H_K is smooth over K due to Lemma 7.5.9 below.

Thus, there exists an admissible formal blowing-up $H' \to \overline{H} \otimes_{\mathbb{Z}} R$ and a quasi-compact formally open subscheme Y of H' such that its associated rigid analytic space Y_K is contained in H_K and such that φ factorizes through Y with relatively compact image in Y. Thus, we can assume that we have morphisms

$$U \xrightarrow{\varphi} \widetilde{Y} \Subset Y \longrightarrow H'.$$

Let $\lambda_1 \in \mathbb{N}$ be an integer such that the A-algebra B, which is associated to V, satisfies the property of Proposition 3.6.9. Due to Theorem 3.6.7 there exist an admissible formal blowing-up $S' \to S$ and an open subscheme U' of S' such that the schematic closure of $(U \times_S S')_0$ in S'_0 is contained in U'_0, and there exists a morphism $\varphi' : U' \longrightarrow Y$ such that $\varphi'|_{U_K}$ extends to a morphism $\varphi' : U \to \widetilde{Y}$ and such that φ' coincides with φ up to the level $\lambda = \lambda_1$.

We may replace S' by S. Let $C' \to U'$ be the curve, which is obtained by the pull-back of the universal curve over H' or $\overline{H} \otimes_{\mathbb{Z}} R$ via the morphism φ'. Set

$$D := C' \times_{U'} U.$$

Due to the universal property we have a U_λ-isomorphism

$$\begin{array}{ccc} C_\lambda & \xrightarrow{\quad\quad} & D_\lambda \\ & \searrow \quad \swarrow & \\ & U_\lambda . & \end{array}$$

The open immersion $V_\lambda \hookrightarrow C_\lambda \xrightarrow{\sim} D_\lambda$ induces an isomorphism $V_\lambda \xrightarrow{\sim} W_\lambda$, where W_λ is an open affine subscheme of D_λ. By Proposition 3.6.9 we see that this isomorphism lifts to an isomorphism $V \xrightarrow{\sim} W$, where W is the open subscheme associated to W_λ. Thus, we have that the open immersion $V_\lambda \hookrightarrow D_\lambda$ lifts to an open U-immersion

$$\begin{array}{ccc} V & \xhookrightarrow{\quad\quad} & D \\ & \searrow \quad \swarrow & \\ & U . & \end{array}$$

Now we want to apply Theorem 3.6.7 to the situation

$$\varphi : V \to D \hookrightarrow C'.$$

The image of φ is relatively U'-compact in C', because C' is proper over U'. Furthermore, φ_K is an open immersion. Then it follows from Theorem 3.6.7 that there exist an admissible formal blowing-up $X' \to X$, an open subscheme V' of X' and a morphism $\varphi' : V' \to C'$ such that the schematic closure of $(X' \times_X V)_0$ is contained in V'_0 and such that φ'_K is an open immersion. By Corollary 3.3.8(c) there exists an admissible formal blowing-up $C'' \to C'$ such that, after taking strict transforms, $\varphi' \times_{C'} C''$ is an open immersion. Finally, choose an admissible formal blowing-up of S' to make the model C'' flat over S'; cf. Theorem 3.3.7. This completes the proof of the extension of compactifications in Proposition 7.5.8. □

Lemma 7.5.9. *Let $g, d \in \mathbb{N}$ be integers. Fix some projective space $\mathbb{P}^N_{\mathbb{Z}}$ and let $H(d, g, N)$ be the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^N$ which are smooth curves of genus g and of degree d. Then $H(d, g, N)$ is smooth over $\mathbb{Z}$ if $d \geq 2g - 1$.*

Proof. One can test the smoothness by the criterion using Artinian rings; cf. [40, Exp. III, Theorem 3.1]. Thus, it suffices to show the infinitesimal lifting property of smooth curves and the lifting property of the closed embedding into $\mathbb{P}^N$. The lifting of curves follows from [40, Exp. III, Theorem 6.3]. The lifting of the embedding follows from the lifting property of relative Cartier divisors; use [39, 0_{IV}, 15.1.16] and the vanishing of the $H^1(C, \mathcal{O}_C(\Delta))$ for a divisor Δ of degree $d \geq (2g - 1)$ on a curve of genus g. The latter follows from Serre's duality theorem. □

Proof of Theorem 7.5.4. A first major step is done by the reduced fiber theorem in Theorem 3.4.8. Thus, we may assume that the curve fibration $f_K : X_K \to S_K$ has a flat R-model $f : X \to S$ with reduced geometric fibers. Since the assertion is local on X and on S, we may assume that X_0 and S_0 are affine and that f is faithfully flat. Due to Proposition 7.5.5 we may assume that, in addition, there exists a finite partition of S_0

$$S_0 = S_0^1 \cup S_0^2 \cup \cdots \cup S_0^r$$

by affine (locally closed) subschemes such that there exists a compactification of X/S over the completion of S with respect to the strata S_0^ϱ for $\varrho = 1, \ldots, r$, which is smooth at infinity. Furthermore, we can choose the subschemes of the following special type:

$$S_0^1 := \left\{ s \in S_0;\ f_0^1(s) \neq 0 \right\},$$
$$S_0^\varrho := \left\{ s \in S_0;\ f_0^j(s) = 0 \text{ for } j = 1, \ldots, \varrho - 1 \text{ and } f_0^\varrho(s) \neq 0 \right\},$$

where $f^\varrho \in \mathcal{O}(S)$ and where f_0^ϱ denotes the restriction of f^ϱ to S_0. For $\varepsilon_1, \ldots, \varepsilon_r$ in $\sqrt{|K^*|}$ with $\varepsilon_\varrho < 1$, which will be specified later, we define open subvarieties

$$S_K(1) := \left\{ s \in S_K;\ \left|f^1(s)\right| \geq 1 \right\},$$
$$S_K(1)' := \left\{ s \in S_K;\ \left|f^1(s)\right| \geq \varepsilon_1 \right\},$$
$$S_K(\varrho) := \left\{ s \in S_K;\ \left|f^j(s)\right| \leq \varepsilon_j \text{ for } j = 1, \ldots, \varrho - 1 \text{ and } \left|f^\varrho(s)\right| \geq 1 \right\},$$
$$S_K(\varrho)' := \left\{ s \in S_K;\ \left|f^j(s)\right| \leq \varepsilon_j \text{ for } j = 1, \ldots, \varrho - 1 \text{ and } \left|f^\varrho(s)\right| \geq \varepsilon_\varrho \right\}.$$

Then $\{S_K(1)', \ldots, S_K(r)'\}$ is an open affinoid covering of S_K such that $S_K(\varrho) \subset S_K(\varrho)'$ for $\varrho = 1, \ldots, r$. Note that we have an S-compactification of X over $S(\varrho)$, where $S(\varrho) \to S$ is a model of $S_K(\varrho) \hookrightarrow S_K$. Now it follows from Proposition 7.5.8 that we can choose $\varepsilon_1, \ldots, \varepsilon_r$ such that X_K admits locally a smooth projective compactification over $S_K(\varrho)'$ for $i = 1, \ldots, r$.

Indeed, let $U = S(1)$ and consider $V = X \times_S U$ which is S-compactifiable over U. Due to Proposition 7.5.8 there exist admissible formal blowing-ups $S' \to S$ and $X \to (X \times_S S')$ and there exists an open subscheme V' of X', which contains the schematic closure $\overline{(X' \times_X V)_0}$ in X'_0 and which is S-compactifiable. Then let ε_1^2 be the maximum of f_1 on the complement of $\overline{(X' \times_X V)_0}$ in X'_0. After we have found a suitable $\varepsilon_1 < 1$, we turn to $S_K(2)$ and start the same procedure as before to find ε_2 with $\varepsilon_2 < 1$ and so on. It is clear that this implies the assertion of Theorem 7.5.4. □

The stable reduction theorem for smooth rigid analytic curve fibrations can easily be deduced from Theorems 7.5.2 and 7.5.4.

Theorem 7.5.10. *Let $f_K : X_K \to S_K$ be a quasi-compact separated smooth rigid analytic curve fibration over a normal rigid analytic space S_K.*

Then, étale locally on S_K and locally on X_K, there exists a semi-stable model of f_K; i.e., a formal model $f : X \to S$ of f_K such that f is flat with geometrically reduced fibers such that its singularities are at most ordinary double points.

Proof. Due to Theorem 7.5.4 we may assume that f_K is compactifiable by a smooth projective curve $\overline{f}_K : \overline{X}_K \to S_K$. Moreover, we may assume that f_K admits an R-model $f : X \to S$ which is flat with reduced geometric fibers. Thus, we may assume that f admits sections, and hence that the geometric fibers of f are connected by [39, IV_3, 15.6.4]. Thus, we may assume that every fiber of X/S is marked by sections $\sigma_1, \ldots, \sigma_n \in X(S)$, which factorize through the smooth locus of X/S and do not meet each other, such that every irreducible component of the smooth part of every fiber of X/S is marked by at least three of them.

Now by Theorem 7.5.2, after replacing S_K by an étale cover, there exists a n-marked stable R-model $\overline{f} : \overline{X} \to S$ of $\overline{f}_K$ with respect to the sections $\sigma_1, \ldots, \sigma_n$. Moreover, we may assume that S is normal. Then X_K is induced by an open formal subscheme of $\overline{X}$, after a suitable blowing-up of S. Indeed, the latter follows from Lemma 7.5.11 below. With the notation of Lemma 7.5.11 the open subspace X_K is induced by the subscheme V of Y. Thus, we see that $V \to S$ is a semi-stable model of $X_K \to S_K$. □

Lemma 7.5.11. *Let S be a normal admissible affine formal R-scheme. Let $X \to S$ and $Y \to S$ be flat relative formal S-curves with connected and reduced geometric fibers. Assume that their generic fibers $Y_K \to S_K$ and $X_K \to S_K$ are smooth and that $Y \to S$ is proper. Let $\varphi_K : X_K \to Y_K$ be an open S_K-immersion.*

Let $\sigma_1, \ldots, \sigma_n \in X(S)$ be sections which factorize through the smooth locus of X/S such that every irreducible component of the smooth part of every fiber of X/S

is marked by at least three of them. Let $\tau_1, \ldots, \tau_n$ *be* S*-valued points of* Y*, which extend* $\varphi_K \circ \sigma_1, \ldots, \varphi_K \circ \sigma_n$*.*

If $Y \to S$ *is an* n*-marked stable curve with respect to* $\tau_1, \ldots, \tau_n$*, then there exists an open subscheme* $V \subset Y$ *such that* $\psi_K := (\varphi_K|_{V_K})^{-1}$ *extends to a surjective* S*-morphism* $\psi : V \to X$*.*

Proof. The assertion is true if S is the spectrum of a valuation ring. Indeed, consider the schematic closure $\Gamma := \overline{\Gamma(\varphi_K)} \subset X \times_S Y$ of the graph of φ_K.

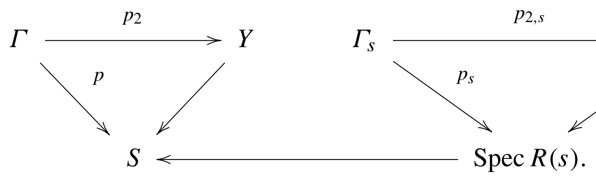
Then $\Gamma \to S$ is flat and $p_1 : \Gamma \to X$ is surjective, because $Y \to S$ and hence $p_1 : \Gamma \to X$ is proper. So Γ is an R-model of X_K. The second projection $p_2 : \Gamma \to Y$ is quasi-finite. In fact, assume that there exists an irreducible component $\widetilde{\Gamma_i}$ of $\Gamma_k := \Gamma \otimes_R k$, which is mapped to a point under p_2. There is a component $\widetilde{X}_i$ of $X \otimes_R k$ such that $\widetilde{\Gamma_i}$ is mapped surjectively to $\widetilde{X}_i$. However, this is impossible, because the irreducible component $\widetilde{X}_i$ of $X \otimes_R k$ is marked by at least three of the points $\sigma_1, \ldots, \sigma_n$ and $Y \otimes_R k$ is also n-marked with respect to the images of these points.

Let $V_k := p_2(\Gamma_k) \subset Y_k := Y \otimes_R k$ be the image of Γ_k. If $y \in V_k$, then there exists an étale neighborhood $Y'_k \to Y_k$ of y such that $\Gamma_k \times_Y Y'_k \to Y'_k$ is finite. Let $Y' \to Y$ be a lifting of Y'_k. Then we have that $\Gamma \times_Y Y' \to Y'$ is finite. Since φ_K is an open immersion, the projection $p_2|_{\Gamma_K} : \Gamma_K \to Y_K$ is an open immersion. Thus, we see by Zariski's main theorem [78, Chap. IV, Corollaire 2] that $\Gamma' := \Gamma \times_Y Y' \to Y'$ is an open immersion, because S is normal, and hence Y is normal. In particular, the subscheme V_k is open in Y_k. Let $V \subset Y$ be the lifting of V_k. By faithfully flat descent we obtain that $\Gamma \to V$ is an isomorphism. Thus, $\psi := p_1 \circ (p_2|_\Gamma)^{-1} : V \to X$ is an extension of ψ_K and ψ is surjective.

In the relative case we also consider the schematic closure

$$\Gamma := \overline{\Gamma(\varphi_K)} \subset X \times_S Y$$

of the graph of φ_K and the second projection $p_2 : \Gamma \to Y$. After performing a suitable admissible blowing-up of S, we may assume that the structure morphism $p : \Gamma \to S$ is flat. Then the morphism $p_2 : \Gamma \to Y$ is an open immersion. Indeed, if s is a closed point of S_0 let $R(s)/R$ be an extension of valuation rings such that $\operatorname{Spec} R(s) \to S$ is an R-morphism, which lifts the closed point s. Thus, we obtain the base change



Since p is flat, p_s is flat as well. As discussed before, $p_{2,s}$ is an open immersion. Thus, $p_2 : \Gamma \to Y$ is quasi-finite. Likewise as before one concludes by Zariski's main theorem that $p_2 : \Gamma \to Y$ is an open immersion, because Y is normal and $p_2 \otimes_R K$ is an open immersion. Then we see as above that

$$\psi := p_1 \circ (p_2|_\Gamma)^{-1} : V := p_2(\Gamma) \longrightarrow X$$

is an extension of ψ_K, which is proper and surjective. □

Remark 7.5.12. In [44] Hartl has generalized Theorem 7.5.10 to higher dimensions if the valuation ring R is a discrete valuation ring. He shows that every quasi-compact smooth rigid analytic space has a rig-étale cover which admits a strictly semi-stable formal R-model after a suitable finite étale extension of the base ring R. His proof uses our compactification result in Theorem 7.5.4 to deduce the proof of his result to the semi-stable reduction theorem for curve fibrations in Theorem 7.5.10. After applying the same procedure to the base of the curve fibration he shows how to resolve the singularities which occur in such a situation.

7.6 The Structure Theorem

Instead of abeloid varieties, we will more generally study bounded rigid analytic groups. A smooth rigid analytic group is called *bounded* if it is quasi-compact and commutative; for example a proper smooth rigid group is bounded. In this section we will only present the results; the proofs are postponed to Sect. 7.7.

Theorem 7.6.1. *Let G_K be a smooth rigid analytic group assumed to be bounded and connected. After a suitable finite separable field extension of K there exists an open connected analytic subgroup $\overline{G}_K$ of G_K, which admits a smooth formal model $\overline{G}$ over R, and there exists a rigid analytic group homomorphism $\varphi : T_K \to G_K$ of a split affine torus $T_K = \mathbb{G}^r_{m,K}$ for some $r \in \mathbb{N}$ to G_K with the following properties:*

(i) *The restriction $\overline{\varphi}$ of φ to the open analytic subtorus $\overline{T}_K = \overline{\mathbb{G}}^r_{m,K}$ of units of T_K induces a closed immersion $\overline{T} = \overline{\mathbb{G}}^r_{m,R} \hookrightarrow \overline{G}$, which splits in the maximal formal torus of $\overline{G}$.*
(ii) *There is a bounded part $T_K(\alpha)$ of T_K with $\varphi(T_K(\alpha)) \cdot \overline{G}_K = G_K$.*

The subgroup $\overline{G}_K$ of G_K is uniquely determined by these conditions. In particular, $\overline{G}_K$ is the largest open connected analytic subgroup of G_K, which admits a smooth formal R-model.

The subtorus of units $\overline{\mathbb{G}}_{m,K}$ of $\mathbb{G}_{m,K}$ is the open analytic subset of $\mathbb{G}_{m,K}$ consisting of the points $z \in \mathbb{G}_{m,K}$ with $|z| = 1$. Denote by E_K the push-forward of $\overline{G}_K = \overline{E}_K$ via the morphism $\overline{T}_K \hookrightarrow T_K$. By Proposition A.2.5 the extension $\overline{G} \to \overline{G}/\overline{T}$ admits a formal rational section. Therefore, the extension is given by formal line bundles $E_1, \ldots, E_r$, where r is the rank of the maximal subtorus $\overline{T}_K$ of

$\overline{G}_K$ which extends. Therefore, as in the case of Raynaud extensions in Sect. 6.1 we have an absolute value function

$$-\log|\_| : E_K \longrightarrow \mathbb{R}^r, \ (z_1,\dots,z_r) \longmapsto \big(-\log|z_1|,\dots,.\log|z_r|\big).$$

Thus, we obtain a surjective homomorphism $p : E_K \to G_K$ of rigid groups via φ. The kernel $M = \ker(p)$ is a discrete subgroup of E_K.

Corollary 7.6.2 (Structure theorem). *In the situation of Theorem 7.6.1, after a suitable finite separable field extension of K, there is a canonical isomorphism $G_K \simeq E_K/M$, where E_K is an extension of the generic fiber of the smooth formal group scheme $B := \overline{G}/\overline{T}$ by an affine torus T_K, and where M is a lattice of E_K of rank equal to the rank of* $\operatorname{Hom}(\mathbb{G}_{m,K}, G_K)$.

The group $\overline{G}_K$ has the following mapping properties; cf. Propositions 5.4.6 and 5.4.8.

Proposition 7.6.3 (Mapping property). *In the situation of Theorem 7.6.1, the subgroup $\overline{G}_K$ has the following property*:

(i) *Let $u_K : Z_K \to G_K$ be a rigid analytic morphism from a connected rigid analytic space Z_K with a K-rational point z_0 satisfying $u_K(z_0) = e$, where e is the unit element of G_K.*

If Z_K admits a smooth formal model Z, then u_K maps Z_K to $\overline{G}_K$ and $u_K : Z_K \to \overline{G}_K$ extends to a formal morphism $u : Z \to \overline{G}$.

(ii) *Let Z_K be a connected affine rig-smooth formal curve which has a semi-stable model Z with precisely one singular point $\tilde{z}_0$. Let $\xi : Z_K \to \mathbb{G}_{m,K}$ be a rigid analytic morphism which restricts to a coordinate function on the formal fiber of $Z_+(\tilde{z}_0)$ of the double point $\tilde{z}_0$. Let $z_0 \in Z_+(\tilde{z}_0)$ be a K-rational point with $\zeta(z_0) = 1$.*

If $u_K : Z_K \to G_K$ is a rigid analytic morphism, then there exists a unique group homomorphism $\varphi : \mathbb{G}_{m,K} \to G_K$ with $\varphi(\overline{\mathbb{G}}_{m,K}) \subset \overline{G}_K$ such that u_K factorizes into $u_K = (\varphi \circ \xi) \cdot \overline{u} \cdot u_K(z_0)$, where $\overline{u} : Z_K \to \overline{G}_K$ is a rigid analytic morphism.

In the case of proper groups one can say more:

Theorem 7.6.4. *Let G_K be a smooth rigid analytic group assumed to be proper and connected. After a suitable finite separable field extension G_K admits a Raynaud representation*:

(a) *There exists a largest open connected analytic subgroup $\overline{G}_K$ of G_K which admits a smooth formal R-model $\overline{G}$. The formal group $\overline{G}$ is an extension of a formal abelian R-scheme B by a formal torus $\overline{T}$.*

(b) *The canonical maps* (*restriction and reduction*)

$$\operatorname{Hom}(\mathbb{G}_{m,K}, G_K) \xrightarrow{\sim} \operatorname{Hom}(\overline{\mathbb{G}}_{m,K}, \overline{G}_K) \xrightarrow{\sim} \operatorname{Hom}(\mathbb{G}_{m,k}, \overline{G}_k)$$

are bijective. In particular, there exists a rigid analytic group homomorphism from a split affine torus $\varphi : T_K \to G_K$ *with the following properties*:

(i) $\varphi|_{\overline{T}_K}$ *induces an isomorphism from* $\overline{T}$ *to the maximal torus of* $\overline{G}$.
(ii) *There is a bounded part* $T_K(\alpha)$ *of* T_K *with* $\varphi(T_K(\alpha)) \cdot \overline{G}_K = G_K$.

(c) *Let* E_K *be the push-forward of* $\overline{G}_K$ *via the morphism* $\overline{T}_K \hookrightarrow T_K$. *The map* φ *induces a rigid analytic group homomorphism* $p : E_K \to G_K$. *The kernel of* p *is a lattice* M *of rank* $\dim(\overline{T}_K)$ *and* G_K *is the quotient of* E_K *by* M.

This representation of G_K is called Raynaud representation in Definition 6.1.5:

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow h & \searrow \phi & \\ T_K & \longrightarrow & \widehat{G}_K & \xrightarrow{q} & B_K \\ & & \downarrow p & & \\ & & G_K = \widehat{G}_K/M & & \end{array} \qquad \widehat{=} \qquad \phi' : M' \to B'.$$

In this case every multiplicative group homomorphism $\mathbb{G}_{m,k} \to \overline{G}_k$ of the reductions lifts to a group homomorphism $\overline{\mathbb{G}}_{m,R} \to \overline{G}$ of the formal groups and extends to a rigid homomorphism $\mathbb{G}_{m,K} \to G_K$. In general, the latter is not valid for bounded groups G_K; for example take $G_K = \overline{\mathbb{G}}_{m,K}$.

If G_K is proper, then there is no additive group $\overline{\mathbb{G}}_{a,k}$ in the reduction of $\overline{G}$. This corresponds to Grothendieck's Stable Reduction Theorem. That follows from the fact that the quotient $\overline{G}/\overline{T}$ is proper if G_K is proper.

From Theorems 7.6.4 and 6.3.3 it follows by using Galois descent that every smooth proper rigid group has a dual.

Corollary 7.6.5. *Let* G_K *be a smooth rigid analytic group which is proper and connected. Then the rigid analytic Picard functor* $\mathrm{Pic}^{\tau}_{G_K/K}$ *of translation invariant line bundles is representable by a smooth rigid analytic group* G'_K, *which is proper and connected.*

Furthermore, the dual of G'_K *is canonically isomorphic to* G_K.

The representability of $\mathrm{Pic}_{X_K/K}$ of a smooth proper rigid analytic space X_K with potentially semi-stable reduction is shown in [45].

For the remaining part of this section we assume that the non-Archimedean field has a discrete valuation. In this case there is a theory of formal Néron models; cf. [16]. A *formal Néron model* of G_K is the largest open subgroup N_K, which admits a smooth formal R-model N; it depends on the base field K. More precisely, a formal Néron model of G_K is defined by a universal property:

If $f_K : Z_K \to G_K$ is a morphism of rigid analytic spaces, where Z_K admits a smooth formal R-scheme Z, then f_K maps Z_K to N_K and extends uniquely to a morphism of formal R-schemes $f : Z \to N$.

One can show that every bounded group has a formal Néron model. A formal Néron model is called *stable* if its 1-component is compatible with finite separable field extensions.

Corollary 7.6.6. *If G_K is a smooth proper rigid analytic group, then there exists a finite separable field extension K'/K such that the formal Néron model of $G_K \otimes_K K'$ has semi-abelian reduction.*

In particular, a formal Néron model with semi-abelian reduction is stable.

Corollary 7.6.7. *If G_K is a smooth bounded rigid analytic group, then there exists a finite separable field extension K'/K such that the formal Néron model of $G_K \otimes_K K'$ is stable.*

Remark 7.6.8. Let K be an arbitrary non-Archimedean field with valuation ring R. Let G_K be a connected bounded rigid analytic group with a unit element e_K. One can think of the existence of a *marked formal Néron model* of G_K; i.e., a *connected* admissible open subgroup $\overline{G}_K$ of G_K, which has smooth formal R-model $\overline{G}$, and satisfies the universal property:

Let Z_K be any rigid analytic spaces, which has a K-rational point z and admits a connected smooth formal R-scheme Z. If $f_K : Z_K \to G_K$ is a morphism of rigid analytic spaces with $f_K(z) = e$ equal to the unit element of G_K, then f_K maps Z_K to $\overline{G}_K$ and factorizes uniquely to a morphism of the formal R-schemes $f : Z \to \overline{G}$.

Due to the existence of formal Néron models it exists if R is a discrete valuation ring. Moreover, our Theorem 7.6.1 implies that such a marked Néron model exists after a finite separable field extension. Raynaud gave an example of a group G_K such that G_K does not admit a marked formal Néron model over the given valuation ring R with a non-discrete valuation.

As an example, let R_0 be a discrete valuation ring with uniformizer π_0. Consider an infinite increasing sequence of discrete valuation rings R_n with uniformizers π_n satisfying $\pi_n = \pi_{n+1}^3$. Let R be the completion of the inductive limit of the R_n. Then consider the formal torus $\overline{\mathbb{G}}_{m,R}$ over R and let G_K be the rigid analytic K-group obtained from $\overline{\mathbb{G}}_{m,K}$ by twisting the inversion on $\mathbb{G}_{m,K}$ by π_0. Then G_K has a marked Néron model $G(n)$ over each R_n. The generic fiber $G(n)_K$ of $G(n)$ is strictly bigger than the one of $G(n-1)_K$; namely, $G(n)_K$ consists of all points $x \in \overline{K}$ with $|1 - x| \leq |\pi_n/\sqrt{\pi_0}|$. Thus, when we pass to the limit, there cannot exist a marked Néron model of finite type over R, because it would already exists over a some R_n and hence the sequence of the $G(n)_K$ would not increase strictly.

Over the quadratic extension $K' := K[\varrho]$ with $\varrho^2 = \pi_0$ there exists a Néron model with good reduction. For sake of simplicity assume that the characteristic of the residue field is unequal 2. So the extension R'/R is even tame.

7.7 Proof of the Structure Theorem

Using Lemma 3.1.13 and Corollary 3.4.3 one easily reduces the proof to the case, where the non-Archimedean field K is algebraically closed. Therefore, we assume in the following that K is algebraically closed.

We start with the *proof of Theorem 7.6.1 and Corollary 7.6.2.*

Let d be the dimension of G_K. Since G_K is smooth over K, there exists an étale surjective map

$$u_K : X_K \longrightarrow G_K$$

from a smooth affinoid space X_K which admits a flat fibration by smooth affinoid curves over a smooth affinoid space S_K of dimension $(d-1)$; say

$$f_K : X_K \longrightarrow S_K.$$

Indeed, since G_K is bounded and smooth, there exists a finite admissible open covering by smooth affinoid subvarieties $\{U_K^i; i = 1, \ldots, m\}$ of G_K such that the module of differential forms $\Omega^1_{G_K/K}$ is generated over U_K^i by differentials $dg_1^i, \ldots, dg_d^i$, where $g_1^i, \ldots, g_d^i$ are holomorphic functions on U_K^i. Let

$$f_K^i : U_K^i \longrightarrow \mathbb{A}_K^{d-1}$$

be the map given by $g_1^i, \ldots, g_{d-1}^i$. Due to Corollary 3.3.8(d) the image of f_K^i is an admissible open subvariety of $\mathbb{A}_K^{d-1}$; so we can replace $\mathbb{A}_K^{d-1}$ by the image of f_K^i. Then let $f_K : X_K \to S_K$ be the disjoint sum of the maps $f_K^1, \ldots, f_K^m$. This map is smooth and faithfully flat due to the construction. Due to Theorem 7.5.10 we may assume that f_K admits a model

$$f : X \longrightarrow S$$

which is faithfully flat with reduced geometric fibers such that their singularities are at most ordinary double points.

Let Y be the smooth part of X/S. Since $X \to S$ is faithfully flat with geometrically reduced fibers, Y is dense in every fiber of X/S and Y is faithfully flat over S. After a further étale surjective base change of S, we may assume that Y can be marked by a finite family of S-sections

$$\sigma_j : S \longrightarrow Y \hookrightarrow X \quad \text{for } j = 1, \ldots, n$$

which meet every connected component of a fiber of Y/S. Let X_j be the open subscheme of X which is the union of all connected components of fibers of X/S which meet σ_j; cf. [39, IV_3, 15.6.4]. After replacing $f : X \to S$ by the sum

$$\coprod (f|_{X_j}) : \coprod X_j \longrightarrow \coprod S_j$$

of the induced maps $f|_{X_j} : X_j \to S_j$ we may assume that $f : X \to S$ has geometrically connected fibers. Furthermore, we have the section

$$\sigma := \coprod \sigma_j : S \longrightarrow Y \hookrightarrow X.$$

Then set

$$\overline{X} := \text{smooth part of } (X/\operatorname{Spf} R),$$
$$Y := \text{smooth part of } (X/S).$$

Next we apply the same procedure to S_K which we did for G_K above; i.e., we choose a smooth fibration of S_K by relative curves. In order to do it in a coherent way, set

$$f_K^d := f_K : X_K^d := X_K \longrightarrow X_K^{d-1} := S_K$$
$$f^d := f : X^d := X \longrightarrow X^{d-1} := S$$
$$u_K^d := u_K : X_K^d \longrightarrow G_K.$$

Moreover, fix the section

$$\sigma^{d-1} = \sigma : X^{d-1} \longrightarrow Y \hookrightarrow X^d$$

of f^d. Thus, we obtain a morphism

$$u_K^{d-1} := u_K^d \circ \sigma_{\mathrm{rig}}^{d-1} : X_K^{d-1} \longrightarrow G_K.$$

By the same procedure, after étale surjective base change, we obtain a fibration

$$f_K^{d-1} : X_K^{d-1} \longrightarrow X_K^{d-2}$$

by smooth curves, which admits a flat model

$$f^{d-1} : X^{d-1} \longrightarrow X^{d-2}$$

such that its geometric fibers are reduced and connected and such that the singularities of the geometric fibers are at most ordinary double points. Set

$$Y^{d-1} := \text{smooth part of } \left(X^{d-1}/X^{d-2}\right).$$

Also, we fix a section

$$\sigma^{d-2} : X^{d-2} \longrightarrow Y^{d-1} \hookrightarrow X^{d-1}$$

of f^{d-1}. Thus, we obtain a morphism

$$u_K^{d-2} := u_K^{d-1} \circ \sigma_K^{d-2} : X_K^{d-2} \longrightarrow G_K.$$

Continuing this way, we obtain a system of maps with sections

$$\begin{array}{ccccccccc}
X^d := X & \overset{u_K}{\dashrightarrow} & G_K & & & & & & \\
{\scriptstyle f^d}\big\downarrow\big\uparrow{\scriptstyle \sigma^{d-1}} & & & & & & & & \\
X^{d-1} := S & \underset{f^{d-1}}{\overset{\sigma^{d-2}}{\rightleftarrows}} & X^{d-2} & \underset{f^{d-2}}{\overset{\sigma^{d-3}}{\rightleftarrows}} & X^{d-3} \ldots & \longrightarrow & X^1 & \underset{f^1}{\overset{\sigma^0}{\rightleftarrows}} & X^0,
\end{array}$$

where

$$f^\delta : X^\delta \longrightarrow X^{\delta-1} \quad \text{for } \delta \in \{d, \ldots, 1\}$$

is faithfully flat with reduced and connected geometric fibers such that the singularities of the fibers are at most ordinary double points. We have sections

$$\sigma^{\delta-1} : X^{\delta-1} \to Y^\delta \hookrightarrow X^\delta$$

of f^δ for $\delta = 1, \ldots, d$, where

$$Y^\delta := \text{smooth part of } \left(X^\delta / X^{\delta-1}\right).$$

Moreover, we obtain morphisms

$$u_K^{\delta-1} := u_K^\delta \circ \sigma_K^{\delta-1} : X_K^{\delta-1} \longrightarrow G_K.$$

After a suitable finite separable field extension, we may assume that every connected component of X^0 is a rational point. For $\delta = 1, \ldots, d$ put

$$\overline{X}^\delta := \text{smooth part of } \left(X^\delta / \operatorname{Spf} R\right)$$

Then we let

$$\overline{G}_K := \left\langle \overline{X}^d, u_K^d \right\rangle$$

be the open admissible subgroup of G_K, which is generated by X_K. This has a smooth formal model $\overline{G}$ over R by Theorem 7.2.3; see Definition 7.2.8 for the notation.

Note that X^0 is a disjoint union of rational points after the extension of the base ring R. Therefore, X^0 is smooth over $\operatorname{Spf} R$ and hence $Y^1 = \overline{X}^1$. Since each σ^δ maps $X^{\delta-1}$ into Y^δ, the image

$$\sigma^{d-1} \circ \cdots \circ \sigma^{\delta-1}\left(\overline{X}^{\delta-1}\right) \subset \overline{X}$$

is contained in the smooth part of X over R. Now consider the map

$$u_K^\delta := u_K \circ \sigma^{d-1} \circ \cdots \circ \sigma^\delta : X^\delta \longrightarrow G_K.$$

Then we have that

$$u_K^\delta(x_1) \cdot u_K^\delta(x_2)^{-1} \in \overline{G}_K$$

for all points x_1, x_2, which belong to the same connectedness component of $\overline{X}^\delta$ for $\delta = 1, \ldots, d$. Indeed, $\sigma^\delta(x) \in \overline{X}^{\delta+1}$ for $x \in \overline{X}^\delta$, because $\sigma^\delta(x)$ belongs to $Y^{\delta+1}$ and $Y^{\delta+1} \to X^\delta$ is smooth. Thus, we see that $\sigma^\delta(x)$ is a smooth point of $X^{\delta+1}$. So the assertion follows from the definition of $\overline{G}_K$.

Moreover, from Corollary 7.2.9 we can deduce the stronger statement:

Lemma 7.7.1. *In the above situation we have that*

$$u_K^\delta(x_1) \cdot u_K^\delta(x_2)^{-1} \in \overline{G}_K$$

for all points x_1, x_2, which belong to the same connectedness component of the smooth part of a fiber of $X^\delta/X^{\delta-1}$ for $\delta = 1, \ldots, d$; i.e., of a connectedness component of $Y^\delta \times_{X^{\delta-1}} s$, where s is a closed point of $X^{\delta-1}$.

Proof. The group generated by $\overline{X}_\delta$ via u_K^δ is contained in $\overline{G}_K$, because the composition $\sigma^\delta \circ \cdots \circ \sigma^{d-1}$ maps $\overline{X}^\delta$ into $\overline{X}^d$. Then the assertion follows by Corollary 7.2.9. □

Next consider the maximal formal torus

$$\overline{T} = \overline{\mathbb{G}}_{m,R}^r \subset \overline{G}.$$

This torus is split, because the base field K is algebraically closed. $\overline{T}$ is a lifting of the maximal torus $\widetilde{T} \subset \widetilde{G}$ of the reduction $\widetilde{G}$ of $\overline{G}$ due to Proposition 5.6.7. The torus $\overline{T}$ canonically splits into a product

$$\overline{T} = \overline{T}_1 \times \overline{T}_2.$$

Here $\overline{T}_1 \cong \overline{\mathbb{G}}_{m,R}^{r_1}$ is the maximal subtorus such that the inclusion $\overline{\mathbb{G}}_{m,R}^{r_1} \hookrightarrow \overline{G}$ extends to a group homomorphism $\mathbb{G}_{m,K}^{r_1} \longrightarrow \overline{G}_K$. In fact, this follows from Proposition 7.3.1. Indeed, if $\overline{\varphi} : \overline{\mathbb{G}}_{m,K} \to \overline{G}_K$ is a group homomorphism such that $\overline{\varphi}^N$ extends for some positive integer N, then $\overline{\varphi}^N(\mathbb{G}_{m,K}) \Subset G_K$ and, $\overline{\varphi}(\mathbb{G}_{m,K}) \Subset G_K$, as well. Thus, it follows from Proposition 5.6.7 that $\overline{\varphi}$ extends also.

This settles the assertion (i) of Theorem 7.6.1.

Lemma 7.7.2. *In the above situation let $\varphi : T_1 := \mathbb{G}_{m,K}^{r_1} \to G_K$ be the extension of the inclusion $\overline{T}_1 = \overline{\mathbb{G}}_{m,K}^{r_1} \hookrightarrow G_K$. Then there exists a bounded part $T_1(\alpha)$ of T_1 such that $G_K = \varphi(T_1(\alpha)) \cdot \overline{G}_K$.*

Proof. Firstly, consider two points $x_1, x_2 \in C_K$ of a *fiber* C_K of $X_K^\delta/X_K^{\delta-1}$, for some $\delta \in \{1, \ldots, d\}$. Then C_K is a rigid analytic curve with semi-stable model $C := X^\delta \times_{X^{\delta-1}} \operatorname{Spf} R(s)$, where $R(s)$ is the valuation ring of the non-Archimedean field $K(s)$ at the base point of the fiber. By Lemma 7.7.1 we know that the group $\langle \overline{C}, u_K^\delta \rangle \subset \overline{G}_K$ generated by the smooth part $\overline{C}$ is contained in $\overline{G}_K$. Then, Theorem 7.4.1 implies

$$u_K^\delta(x_1) \cdot u_K^\delta(x_2)^{-1} \in \varphi(T_1) \cdot \overline{G}_K.$$

Secondly, consider two points $x_1, x_2 \in X_K$ which belong to the same *connected component* of X^d. Then there exists a maximal $\delta \in \{0, \dots, d-1\}$ such that

$$f^{\delta+1} \circ \cdots \circ f^d(x_1) = f^{\delta+1} \circ \cdots \circ f^d(x_2) \in X_K^\delta,$$

because X_K^0 is a disjoint union of isolated points. For $i = 1, 2$ consider

$$\begin{aligned} x_{i,d} &:= x_i, \\ x_{i,\delta} &:= f^{\delta+1} \circ \cdots \circ f^d(x_i) \in X^\delta \quad \text{for } \delta \in \{1, \dots, d-1\}. \end{aligned}$$

Thus, we have $x_{1,\delta} = x_{2,\delta} \in X^\delta$ for some $\delta \in \{0, \dots, d\}$. Therefore

$$u_K^\delta(x_{1,\delta}) \cdot u_K^\delta(x_{2,\delta})^{-1} = e \in \varphi(T_1) \cdot \overline{G}_K.$$

Furthermore $x_{i,\delta+1}$ and $\sigma^\delta(x_{i,\delta})$ belong to the same fiber of $X^{\delta+1}/X^\delta$. Thus, we see by induction

$$u_K^{\delta+1}(x_{i,\delta+1}) \cdot u_K^\delta(x_{i,\delta})^{-1} \in \varphi(T_1) \cdot \overline{G}_K$$

by the above reasoning. Since $\varphi(T_1) \cdot \overline{G}_K$ is a subgroup, we obtain

$$\begin{aligned} u_K^d(x_1) \cdot u_K^d(x_2)^{-1} &= u_K^d(x_{1,d}) u_K^{d-1}(x_{1,d-1})^{-1} \dots u_K^{\delta+1}(x_{1,\delta+1}) u_K^\delta(x_{1,\delta})^{-1} \\ &\quad \cdot u_K^\delta(x_{2,\delta}) u_K^{\delta+1}(x_{2,\delta+1})^{-1} \dots u_K^{d-1}(x_{2,d-1}) u_K^d(x_{2,d})^{-1} \\ &\in \varphi(T_1) \cdot \overline{G}_K. \end{aligned}$$

Thirdly, we choose a point z_i in every connectedness component of the generic fiber of the smooth part $\overline{X} \subset X$; note that the connectedness components are related to the points in X^0. Thus we see by the second step that $u_K(x_i)$ is contained in $U_i := u_K(z_i) \cdot \varphi(T_1) \cdot \overline{G}_K$ if x_i belongs to the same component as z_i. Since $\varphi(T_1) \cdot \overline{G}_K$ is a group, we have that $U_i = U_j$ or $U_i \cap U_j = \emptyset$. For each connectedness component we need only a bounded part $T_1(\alpha)$ with $\varphi(T_1(\alpha)) \cdot \overline{G}_K = \varphi(T_1) \cdot \overline{G}_K = G_K$. Indeed, the heights of the annuli, which appear in the fibers of our fibrations is bounded from below, because we deal with finitely many quasi-compact formal schemes. Thus, the connectedness of G_K implies that $\varphi(T_1) \cdot \overline{G}_K = G_K$, as we will see below. □

We define the push forward

$$\widehat{G}_K := \overline{G}_K \times_{\overline{T}_1} T_1$$

by $\overline{T}_1 \to T_1$. Then consider the induced group homomorphism

$$\Phi : \widehat{G}_K \longrightarrow G_K, \; (g, t) \longmapsto g \cdot \varphi(t),$$

for $g \in \overline{G}_K$ and $t \in T_1$ and put

$$M := \operatorname{Ker} \Phi \subset \widehat{G}_K.$$

Then M is a discrete subgroup of $\widehat{G}_K$, because $\overline{\Phi} := \Phi|_{\overline{G}_K} : \overline{G}_K \to \overline{G}_K$ is an isomorphism and hence $\overline{G}_K$ is a connectedness component of $\Phi^{-1}(\overline{G}_K)$. Thus, M is a lattice in the sense of Sect. 6.1. Since there is a bounded part $T_1(\alpha)$ with $\varphi(T_1(\alpha)) \cdot \overline{G}_K = \varphi(T_1) \cdot \overline{G}_K$, the lattice M is of full rank. Denote by H_K the group $\widehat{G}_K/M$, which is an open admissible subgroup of G_K. Moreover, our considerations have shown that G_K is disjoint union of finitely many translates of H_K, the connectedness of G_K implies $H_K = G_K$.

The latter implies the assertion (ii) and Corollary 7.6.2; for the existence of the quotient $\overline{G}/\overline{T}$ see Lemma 5.5.2. The assertion on the uniqueness follows by the mapping property in Proposition 7.6.3. Hereby we complete the proof of Theorem 7.6.1 and of Corollary 7.6.2. □

We continue with the proof of the mapping property of Proposition 7.6.3.

(i) If $\overline{G}$ has semi-abelian reduction, the assertion follows from Proposition 5.4.6. Now consider the general case. The group G_K is a quotient E_K/M due to Corollary 7.6.2. Since the reduction of Z is irreducible, there exists an element $t \in T_K$ and a formal dense open part $Z'_K \subset Z_K$ such that $u_K(Z'_K) \subset t \cdot \overline{G}_K$. Indeed, one can cover G_K by finitely many charts $U_K^1, \ldots, U_K^n$ of the following type. Take a cell decomposition of the absolute value set

$$\Delta := \mathbb{R}^r / \log(M)$$

and denote by $U_K^1, \ldots, U_K^n$ the pre-image of such a decomposition.

Due to Proposition 3.1.12 there exists an index $j \in \{1, \ldots, m\}$ and a formal dense open part Z'_K of Z_K such that $u_K(Z'_K) \subset U_K^j$, because the reduction of Z is irreducible. On the chart U_K^j there are the character functions, which do not have zeros. Thus, we see by Remark 1.4.6 that the absolute value functions of their pull-backs to Z'_K are constant, because the reduction of Z' is irreducible. If we choose the point $t \in T_K$ such that the character functions take the same absolute value at t, then we see by Theorem 7.2.1 that Z_K is mapped to $t \cdot \overline{G}_K$. Since $u_K(z_0) = 1$, we obtain that $u_K(Z_K) \subset \overline{G}_K$.

(ii) The proof is close to the proof of Theorem 7.4.1 which is a similar statement. In Theorem 7.4.1 we assumed that the map u_K is unramified; this assumption was only made to have the existence of the group $H(\rho)_K$ in Lemma 7.4.2. For the present statement this can be avoided due to the assertion (i) already proved above.

We may assume that $\xi(z_0) = 1$ and that there is an $\alpha \in |K^\times|$ such that $|\xi(x)| = \alpha$ or $|\xi(x)| = 1/\alpha$ for all $x \in \overline{Z}_K$ in the smooth formal part $\overline{Z}$ of Z. Then, for any $\rho \in |K^\times|$ with $\alpha < \rho \leq 1$ set

$$Z_K(x_0) \supset A(\rho) := \xi^{-1}\big(\{z \in \mathbb{G}_{m,K}; \rho \leq |z| \leq 1/\rho\}\big),$$
$$Z_K(x_0) \supset A(\rho)^+ := \xi^{-1}\big(\{z \in \mathbb{G}_{m,K}; |z| = 1/\rho\}\big),$$
$$Z_K(x_0) \supset A(\rho)^- := \xi^{-1}\big(\{z \in \mathbb{G}_{m,K}; |z| = \rho\}\big).$$

Since $u_K(A(1))$ is relatively compact in G_K, one shows as in Theorem 7.4.1 that there exists a group homomorphism $\varphi : \mathbb{G}_{m,K} \to G_K$ such that u_K decomposes into

$$u_K = (\varphi \circ \xi) \cdot \overline{u}_K,$$

where $\overline{u}_K$ maps $A(1)$ to a unipotent subgroup of $\overline{G}_K$. After replacing u_K by $\overline{u}_K$ we may assume that u_K maps $A(1)$ into a subgroup which admits a smooth formal model with unipotent reduction. Thus, it remains to see that u_K maps Z_K to $\overline{G}_K$. Now we proceed in the same manner as in the proof of Lemma 7.4.4 to show that $u_K(A(\rho)) \subset \overline{G}_K$ for all $\rho \in |K^\times|$ with $\alpha < \rho \leq 1$. Indeed, the group $H(\rho) := \langle A(\rho)^+ \cup A(\rho)^-, u_K \rangle$ is contained in $\overline{G}_K$ for all $\rho \in |K^\times|$ with $\alpha < \rho \leq 1$ due to (i). Thus, we see as in the proof of Lemma 7.4.4(e) that $u_K(A(\rho)) \subset \overline{G}_K$ for all ρ with $\alpha < \rho \leq 1$. So $u_K^{-1}(\overline{G}_K)$ contains a formal open part of Z_K, which is a neighborhood of the formal fiber $Z_+(z_0)$. Due to (i) we know that the smooth part $\overline{Z}_K$ is mapped to $\overline{G}_K$ and hence the whole Z_K is mapped to $\overline{G}_K$. □

Now we turn to the proof of Theorem 7.6.4.

(a) Now we have, in addition, that G_K is proper. Denote by $B = \overline{G}/\overline{T}$ the quotient of $\overline{G}$ by the maximal formal torus $\overline{T}$ of $\overline{G}$; cf. Lemma 5.5.2. We will show that B is proper. Then $\overline{G}$ has semi-abelian reduction and $\overline{T}$ is the maximal torus, which is a lifting of the maximal torus of the reduction $\overline{G}_0$. In particular, $\overline{G}$ is a formal torus extension of B, which is a formal abelian R-scheme.

For this it suffices to show that there is no subgroup of type $\mathbb{G}_{a,k}$ contained in the reduction of $\overline{G}$. Assume the contrary and consider a closed embedding $\mathbb{G}_{a,0} \hookrightarrow \overline{G}_0$. Since $\overline{G}$ is smooth, the closed embedding lifts to a formal morphism $\mathbb{D}_R \to \overline{G}$, where $\mathbb{D}_R$ is the formal affine line over R.

Let d be the dimension of G_K. There exists a closed subvariety B_K of an open neighborhood of the unit element such that B_K is isomorphic to a $(d-1)$-dimensional ball $\mathbb{D}_K^{d-1}$ and

$$\mathbb{D}_K^d \cong (\mathbb{D}_K \times B_K) \longrightarrow \overline{G}_K, \quad (x, y) \longmapsto x \cdot y,$$

is an open immersion. Since G_K is proper, we see by Corollary 3.6.18 that there exists an open immersion $\mathbb{D}_K^d(c) \hookrightarrow G_K$ of a ball $\mathbb{D}_K^d(c)$ into G_K, where $\mathbb{D}_K^d$ is relatively compact in $\mathbb{D}_K^d(c)$. By Theorem 7.2.3 the d-dimensional ball $\mathbb{D}_K^d(c)$ generates a subgroup with unipotent reduction. Due to Proposition 7.6.3 this subgroup is contained in $\overline{G}_K$. Thus, we see that the reduction of the map $\mathbb{D}_K \to \overline{G}_K$ maps to the unit element, and hence we arrive at a contradiction.

(b) The restriction map is well-defined by Proposition 7.6.3. Moreover, the isomorphism $\overline{\varphi} : \overline{\mathbb{G}}^r_{m,K} \xrightarrow{\sim} \overline{T}_K$ extends to a homomorphism $\varphi : \mathbb{G}^r_{m,K} \to G_K$ by Proposition 7.3.1, because G_K is proper. Thus, we see that the restriction map is bijective. The reduction is bijective due to the lifting property of tori; cf. Proposition 5.6.7. The remaining assertion follows from Theorem 7.6.1.

(c) This follows from Corollary 7.6.2. □

The Corollaries 7.6.6 and 7.6.7 follow immediately from the theorems and the mapping property. Moreover, we know from Proposition 5.4.6 that a formal Néron model with semi-abelian reduction is stable. Finally let us give an interpretation of the induced map from the affine torus $T_K = \mathbb{G}_{m,K}^r$ to G_K.

Remark 7.7.3. If G'_K is the dual of an abeloid variety G_K and if G'_K admits an open subgroup $\overline{G}'_K$ which has a smooth formal model $\overline{G}'$ with semi-abelian reduction, whose formal subtorus is split, the map

$$\varphi : T_K = \mathbb{G}_{m,K}^r \longrightarrow G_K$$

can be presented in the following way: If $\underline{a} = (a_1, \ldots, a_r)$ is a basis of $H^1(G'_K, \mathbb{Z})$, then $\underline{a}$ gives rise to a map

$$\underline{a} : \mathbb{G}_{m,K}^r \longrightarrow G_K = G''_K, \ (t_1, \ldots, t_r) \longmapsto \big(a_1(t_1) \otimes \cdots \otimes a_r(t_r)\big),$$

where $a(t)$ is the line bundle given by the cocycle $(\zeta(t)^{n_{ij}}) \in H^1(G'_K, \mathcal{O}_{G'_K}^{\times})$ for a cycle $a = (n_{ij}) \in H^1(G'_K, \mathbb{Z})$ and ζ is a coordinate function of $\mathbb{G}_{m,K}$. We view $a(t)$ as a point of the dual of G'_K, which is isomorphic to G_K.

Thus, all the statements of Sect. 7.6 are completely proved. □

Appendix
Miscellaneous

A.1 Some Notions about Graphs

Here we fix some notions about graphs which are used in this book. The definitions are taken form the book of Serre [90].

Definition A.1.1. A *graph* G consists of sets $V = \mathrm{vert}(G)$, $E = \mathrm{edge}(G)$, and two maps

$$E \longrightarrow V \times V; \quad e \longmapsto \big(o(e), t(e)\big),$$

and

$$E \longrightarrow E; \quad e \longmapsto \overline{e},$$

which satisfy the following conditions: for each $e \in E$ we have $\overline{\overline{e}} = e$, $\overline{e} \neq e$ and $o(e) = t(\overline{e})$.

In this definition an edge is oriented. Sometimes, when it is not necessary to have an orientation on the graph, we identify e with $\overline{e}$. Then we will talk about *geometric edges* and *geometric graphs*, respectively.

An element $v \in V$ is called a *vertex of* G; an element $e \in E$ is called an *(oriented) edge*, $\overline{e}$ is called the *inverse* edge of e. The vertex $o(e) = t(\overline{e})$ is called the *origin* of e, and the vertex $t(e) = o(\overline{e})$ is called the *terminus* of e. These two vertices are called the *extremities* of e. We say two vertices are *adjacent* if they are the extremities of some edge.

There is an evident notion of *morphism* for graphs. We say a morphism is *injective* if the corresponding maps on the vertices and edges are injective. Moreover, there is an evident notion of a *subgraph* of G.

An *orientation* of a graph G is a subset E_+ of the set of edges E such that E is the disjoint union of E_+ and $\overline{E}_+$. It always exist. An *oriented graph* is defined, up to isomorphism, by giving the two subsets V and E_+ and a map $E_+ \to V \times V$. The corresponding sets of edges is $E = E_+ \amalg \overline{E}_+$ where $\overline{E}_+$ denotes a copy of E_+.

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Definition A.1.2. Let G be a graph and $E = \text{vert}(G)$, $E = \text{edge}(G)$. We form a topological space T which is a disjoint union of V and $E \times [0, 1]$, where V and E are provided with the discrete topology. Let R be the finest equivalence relation on T for which $(e, t) \equiv (\overline{e}, (1 - t))$, $(e, 0) \equiv o(e)$ and $(e, 1) \equiv t(e)$ for $e \in E$ and $t \in [0, 1]$.

The quotient space $\text{real}(G) = T/R$ is called the *realization* of G. The realization $\text{real}(G)$ is a *CW-complex of dimension* ≤ 1 in the sense of H.C. Whitehead.

Definition A.1.3. A *path (of length n)* in a graph G is a finite sequence of edges $c = (e_1, \ldots, e_n)$ with $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, n - 1$. We say that c leads from $o(e_1)$ to $t(e_n)$. The vertices $o(e_1)$ and $t(e_n)$ are called the *extremities of the path*. It is called *closed* if $t(e_n) = o(e_1)$.

A pair of the form $(e_i, e_{i+1}) = (e_i, \overline{e}_i)$ in a path is called a *backtracking*. If there is a path from v_0 to v_1 in G, then there is one without backtracking. An infinite path is a direct limit of paths of finite length; i.e., it is an infinite sequence $(e_1, e_2, \ldots)$ of edges such that $t(e_i) = o(e_{i+1})$ for all $i \in \mathbb{N}$.

A graph is *connected* if every two vertices are the extremities of at least one path.

Definition A.1.4. A *circuit (of length n)* in a graph G is a finite path c of length n as in Definition A.1.3 without backtracking such that $t(e_n) = o(e_1)$.

A *loop* in G is a circuit of length 1.

Definition A.1.5. A *tree* is a connected non-empty graph without circuits.

Next we list some facts about graphs; for proofs we refer to [90, I.2].

Proposition A.1.6. *Let v_0 and v_1 be two vertices in a tree G. Then there is exactly one path leading from v_0 to v_1 without backtracking.*

Definition A.1.7. Let G be a graph and let $V = \text{vert}(G)$, $E = \text{edge}(G)$. Let v be a vertex and let E_v be the set of edges such that $v = t(e)$. The cardinal n of E_v is called the *index* of v.

If $n = 0$ one says that v is *isolated*; if G is connected this is not possible unless $V = \{v\}$, $E = \emptyset$. If $n \leq 1$ one says v is a *terminal vertex*.

Let v be a vertex of G. We denote by $G - v$ the subgraph of G with vertex set $V - \{v\}$ and edge set $E - (E_v \cup \overline{E}_v)$.

Proposition A.1.8. *Let v be a non-isolated terminal vertex of a graph G.*

(a) *G is connected if and only if $G - v$ is connected.*
(b) *Every circuit of G is contained in $G - v$.*
(c) *G is a tree if and only if $G - v$ is a tree.*

Let G be a graph. The set of subgraphs of G which are trees, ordered by inclusion, is evidently directed. By Zorn's lemma it has a maximal element; such an element is called a *maximal tree* in G.

Proposition A.1.9. *Let T be a maximal tree of a connected non-empty graph G. The T contains all the vertices of G.*

Definition A.1.10. Let G be a connected graph with finitely many edges. Let

$$v(G) := \#(\operatorname{vert} G) \quad \text{and} \quad e(G) := \frac{1}{2}\#(\operatorname{edge} G)$$

be the number of vertices and of geometric edges, respectively. Then we have that $e(g) \geq v(G) - 1$. Equality holds if and only if G is a tree. The integer

$$z(G) := e(G) - v(G) + 1$$

is called the *cyclomatic number* of G.

Note that $e(G)$ is the number of "geometric edges" of G; i.e., number of lines in the realization $\operatorname{real}(G)$ of G. The cyclomatic number equals the first Betti number of $\operatorname{real}(G)$.

Proposition A.1.11. *Let G be a connected graph with finitely many edges and let v_0 be a vertex of G. Then there exists a tree $\widehat{G}$ with a vertex $\widehat{v}_0$ and a morphism $p : \widehat{G} \to G$ with $p(\widehat{v}_0) = v_0$ such that the induced map $\operatorname{real}(\widehat{G}) \to \operatorname{real}(G)$ of their realizations is the universal covering in the sense of topological spaces. The pair $(\widehat{G}, \widehat{v}_0)$ is uniquely determined up to canonical isomorphism.*

We call $(\widehat{G}, \widehat{v}_0)$ the universal covering *of G.*

The fundamental group $\pi_1(\operatorname{real}(G), v_0)$ is free and acts as Deck transformation group on $\widehat{G}$.

Proof. The proof is done by resolving all the circuits of G.

At first we choose a maximal tree T in G with $v_0 \in \operatorname{vert}(T)$ and an orientation A of T. Put $T_0 = T$ and let $p_0 : T_0 \to G$ be the inclusion. Then, for each $n \in \mathbb{N}$, we will construct trees T_n with injective maps $\iota_n : T_n \to T_{n+1}$ and surjective maps $p_n : T_n \to G$, where $p_{n+1} : T_{n+1} \to G$ restricts to p_n. The inductive limit $\widehat{G}$ of the system (T_n, ι_n) with the induced map $p : \widehat{G} \to G$ is universal in the sense that each map from a tree (F, f_0) to G, which sends the vertex f_0 to v_0, factorizes uniquely through $p : (\widehat{G}, \widehat{v}_0) \longrightarrow (G, v_0)$, where $\widehat{v}_0$ is the limit of the vertices $v_{n+1} := \iota_n(v_n)$.

We start with $T_0 := T$ and $p_0 := \iota$, where ι is the inclusion of T into G. Let F be a copy of T, where we add open edges e_+, e_- for each $e \in T - (A \cap T)$; by an open edge we mean an edge with one extremity. We put $o(e_+) := o(e)$ and $t(e_-) := t(e)$. Note that there is neither $t(e_+)$ nor $o(e_-)$ defined. Let F_e^+ and F_e^- be copies of F for $e \in A - (T \cap A)$. Then we glue F with the new copies by identifying the open edges e_+ of F with e_- of F_e^+ and e_- of F with e_+ of F_e^-; i.e., $t(e_+) := t(e)$ in F_e^+ and $o(e_-) := t(e)$ in F_e^-. It is evident how to introduce the reversed edges of the edges which connect F with the new copies. Thus we end up with a tree F_1 with open edges. By removing the remained open edges we obtain a tree T_1, an evident inclusion $\iota_0 : T_0 \hookrightarrow T_1$, and a surjective morphism $p_1 : T_1 \to G$. Now we can repeat the process by gluing new copies of F along the open edges of F_1. By iterating such constructions we obtain the desired directed system $(T_n, \iota_n)_{n\in\mathbb{N}}$, where $\iota_n : T_n \to T_{n+1}$ is the inclusion. Then the inductive limit $\varinjlim(T_n, \iota_n)$ is a tree $\widehat{G}$ equipped with a map $p : \widehat{G} \to G$. We leave it to the reader to show the universal property of $(\widehat{G}, \widehat{v}_0)$.

Let Γ be the free group generated over the edges $e \in A - (T \cap A)$. Then Γ acts on $\widehat{G}$ by mapping a generator $e \in A - (T \cap A)$ to the shift which sends e_- to e_+.

It is clear that the induced map $\mathrm{real}(p) : \mathrm{real}(\widehat{G}) \to \mathrm{real}(G)$ is the universal covering of $\mathrm{real}(G)$. Moreover, the group Γ can be identified with the deck transformation group of $\mathrm{real}(p)$, and hence with the fundamental group $\pi_1(\mathrm{real}(G), x_0)$, which is free; cf. [90, I, §3, Theorem 4]. □

A.2 Torus Extensions of Formal Abelian Schemes

In this section we want to give an overview of the theory of torus extensions of formal abelian varieties. In principle, one can cite the book of Serre [88, Chap. VII], where the case of algebraic groups is widely explained. We recall the main results and point out, where some modification have to be made in the formal algebraic case. As always we mean by a formal scheme an admissible formal scheme over $\mathrm{Spf}\, R$. The residue field k of R is assumed to be algebraically closed.

In the following we consider only smooth formal group schemes G, E, B which are *commutative and connected*. We write the group laws on G, E multiplicatively and on B additively. We assume that G is affine and B will be a formal abelian scheme at the end. A sequence of formal algebraic homomorphisms

$$0 \to G \to E \to B \to 0$$

is called *strictly exact* if it is exact in the usual sense, and if the sequence of their tangent bundles

$$0 \to \mathfrak{t}_G \to \mathfrak{t}_E \to \mathfrak{t}_B \to 0 \tag{A.1}$$

is exact. The latter means that $E \to B$ is smooth and $G \to E$ is a closed immersion. A strictly exact sequence (A.1) is called an *extension* of B by G.

As in the algebraic case, two extensions E_1 and E_2 are *isomorphic* if there is a morphism $f : E_1 \to E_2$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & E_1 & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_G} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_B} & & \\ 0 & \longrightarrow & G & \longrightarrow & E_2 & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

commutative. Then f is automatically an isomorphism.

The set of isomorphism classes of commutative extensions of B by G is denoted by $\mathrm{Ext}(B, G)$. It is a contravariant functor in B and a covariant functor in G. Indeed, for every $\gamma : G \to G'$ one obtains a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\iota} & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \mathrm{id}_B} & & \\ 0 & \longrightarrow & G' & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

where

$$E' := \gamma_* E := G' \amalg_G E$$

is the fibered coproduct in the category of formal group schemes; it is also called the *push-out*. This is the quotient of $G' \times E$ by the image of the closed immersion

$$(\gamma, -\iota) : G \hookrightarrow G' \times E$$

which exists in the category of formal group schemes. In fact, it exists on every level of the formal schemes; that is the algebraic case.

If $\varphi : B' \to B$ is a homomorphism, one obtains a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\iota} & E' & \longrightarrow & B' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_G} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & G & \xrightarrow{\iota} & E & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

where

$$E' := \varphi^* E := E \times_B B'$$

is the fibered product; it is also called *pull-back*. This exists in the category of formal group schemes.

On $\mathrm{Ext}(B, G)$ one has a law of composition: For $E_1, E_2 \in \mathrm{Ext}(B, G)$ set

$$E_1 + E_2 := \delta^* \mu_* (E_1 \times E_2),$$

where $\delta : B \to B \times B$ is the diagonal map of B and $\mu : G \times G \to G$ is the group law of G. As in [88, VII, §1, Props. 1, 2, 3] we have the results:

Proposition A.2.1. *The law of composition defined above turns* $\mathrm{Ext}(B, G)$ *into an abelian group. If* $\mathcal{C}$ *denotes the category of commutative formal group schemes, the functor* $\mathrm{Ext}(B, G)$ *is an additive bi-functor on* $\mathcal{C} \times \mathcal{C}$.

We have two exact sequences:

Proposition A.2.2. *Consider the strictly exact sequences of formal algebraic groups*

$$0 \to B' \to B \to B'' \to 0,$$
$$0 \to G' \to G \to G'' \to 0.$$

Then the sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}\big(B'', G\big) \to \operatorname{Hom}(B, G) \to \operatorname{Hom}\big(B', G\big) \xrightarrow{d} \\ &\to \operatorname{Ext}\big(B'', G\big) \to \operatorname{Ext}(B, G) \to \operatorname{Ext}\big(B', G\big) \end{aligned}$$

is exact, where $d(\varphi) := \varphi_*(B)$. Similarly, the sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}\big(B, G'\big) \to \operatorname{Hom}(B, G) \to \operatorname{Hom}\big(B, G''\big) \xrightarrow{d} \\ &\to \operatorname{Ext}\big(B, G'\big) \to \operatorname{Ext}(B, G) \to \operatorname{Ext}\big(B, G''\big) \end{aligned}$$

is exact, where $d(\varphi) := \varphi^*(G)$.

An important tool in the theory of extensions of B by G is the notion of a *factor system*. This is a morphism of the underlying formal schemes $f : B \times B \to G$ satisfying the relations

$$\frac{f(y, z) \cdot f(x, y+z)}{f(x+y, z) \cdot f(x, y)} = 1 \quad \text{for all } x, y, z \in B.$$

If $g : B \to G$ is a morphism, then the function $\delta(g) : B \times B \to G$ defined by the formula

$$\delta(g)(x, y) := \frac{g(x+y)}{g(x) \cdot g(y)}$$

is a factor system; such a system is called *trivial*. The classes of factor systems modulo the trivial ones is denoted by $H^2(B, G)$. A factor system is called *symmetric* if

$$f(x, y) = f(y, x) \quad \text{for all } x, y \in B.$$

The classes of symmetric systems is a subgroup $H^2(B, G)_{sym} \subset H^2(B, G)$.

One can also define a rational factor system; i.e., a rational morphism $f : B \times B \dashrightarrow G$ which is defined only on a nonempty open subset of the formal scheme $B \times B$. It is called *trivial* if $f = \delta(g)$ for a rational morphism $g : B \dashrightarrow G$. Thus, we obtain the groups $H^2_{rat}(B, G)$ and $H^2_{rat}(B, G)_{sym}$, respectively. If the systems are required to be defined on the whole of B, one writes $H^2_{reg}(B, G)$ and $H^2_{reg}(B, G)_{sym}$, respectively.

There are important extensions which admit a rational section

$$0 \longrightarrow G \longrightarrow E \underset{\xleftarrow{\;s\;}}{\longrightarrow} B \longrightarrow 0.$$

If s is defined on B, then it is called *regular*. If it is defined on a dense formally open subset, then it is called *rational*. There is the result:

Proposition A.2.3. *In the above situation we have*:

(a) *The group* $H^2_{reg}(B,G)_{sym}$ *is isomorphic to the subgroup of* $\mathrm{Ext}(B,G)$ *given by the formal extensions which admit a regular section.*
(b) *The group* $H^2_{rat}(B,G)_{sym}$ *is isomorphic to the subgroup of* $\mathrm{Ext}(B,G)$ *given by the formal extensions which admit a rational section.*
(c) *The canonical homomorphism* $H^2_{reg}(B,G)_{sym} \to H^2_{rat}(B,G)_{sym}$ *is injective.*

Proof. One can proceed as in [88, VII, §1, Prop. 4] except for the statement that one can associate a formal group to a formal birational group law. If an extension $G \to E \to B$ admits a rational section $s : B \dashrightarrow E$, then

$$f : B \times B \dashrightarrow G, (b_1, b_2) \dashrightarrow s(b_1 + b_2) \cdot s(b_1)^{-1} \cdot s(b_2)^{-1},$$

defines a symmetric rational factor system, as the extension is commutative.

Conversely, to a symmetric rational factor system $f : B \times B \dashrightarrow G$ one associates a birational group law on $G \times B$ by setting

$$(g_1, b_1) \star (g_2, b_2) := \big(g_1 g_2 f(b_1, b_2), b_1 + b_2\big),$$
$$(g, b)^{-1} := \big(\big(g \cdot f(b, -b)\big)^{-1}, -b\big)$$

where we may assume that $f(0_B, 0_B) = 1_G$. Then, similarly as in the proof of [15, Theorem 5.1/5], this birational law gives rise to a formal group scheme. In fact, the formal group can be constructed by gluing certain copies (as translates) of the chart of the open part, where the law of composition is defined. This extension obviously admits a formal rational section. □

We remind the reader that the push-out of a morphism $\gamma : G \to G'$ of a rational factor system $f : B \times B \dashrightarrow G$ is given by the group associated to $\gamma_*(f) := \gamma \circ f$ and the pull-back of a morphism $\varphi : B' \to B$ is given by the group associated to $\varphi^*(f) := f \circ (\varphi \times \varphi)$.

As before consider an extension with a rational section; say with a regular section $s : U \to E$, where $U \subset B$ is a nonempty open subset of B. Using translates of U, one obtains an open covering $\{U_i; i \in I\}$ of B and regular sections $s_i : U_i \to E$. Let $q : E \to B$ be the morphism of the extension, then the section s_i gives rise to an isomorphism

$$G \times U_i \xrightarrow{\sim} q^{-1}(U_i), \quad (x, y) \longmapsto x \cdot s_i(y).$$

Let $\mathcal{G}_B$ be the sheaf which associates to an open subset V of B the set $\mathrm{Mor}(V, G)$ of regular morphisms from V to G. There is a map

$$\psi : H^2_{rat}(B, G)_{sym} \longrightarrow H^1(B, \mathcal{G}_B),\ s \longmapsto (\lambda_{i,j}) := (s_j/s_i)_{i,j},$$

which associates to a rational section s the class of the cocycle $((s_j/s_i)_{i,j})$.

Proposition A.2.4. *The kernel of ψ is $H^2_{reg}(B, G)_{sym}$.*

A main result for our application is the following statement.

Proposition A.2.5. *If G is linear, then $H^2_{rat}(B, G)_{sym} = \mathrm{Ext}(B, G)$.*

Proof. In view of Proposition A.2.3, one has to show that every extension E of B by G has a rational section. It is shown in [88, VII, §1, Prop. 6] that over the special fiber of the given formal extension there exists a rational section. This follows from a result of Lang-Tate [58, Chap. V, no. 21]. Indeed, one can view $E \otimes_B \mathrm{Spec}(\ell)$ as a principal homogeneous space under the group $G \otimes_R \ell$ over the generic point $\mathrm{Spec}(\ell)$ of $B \otimes_R k$. The set of classes of principal homogeneous spaces over $\mathrm{Spec}(\ell)$ is isomorphic to $H^1(\mathrm{Gal}(\ell_{sep}/\ell), G(\ell_{sep}))$, where ℓ_{sep} is a separable algebraic closure of the generic point $\mathrm{Spec}(\ell)$ of the special fiber of B. Since G is commutative and smooth, G is a successive extension of groups of type $G = \mathbb{G}_m$ or $\mathbb{G}_a$. The vanishing of $H^1(\mathrm{Gal}(\ell_{sep}/\ell), \mathbb{G}_m(\ell_{sep}))$ is a consequence of "Theorem 90" of Hilbert. The vanishing of $H^1(\mathrm{Gal}(\ell_{sep}/\ell), \mathbb{G}_a(\ell_{sep}))$ follows from the normal basis theorem. Thus, let $U_k \subset B_k$ be a nonempty open affine subset of the special fiber which admits a section $s_k : U_k \to E_k$. Due to the smoothness of $E \to B$ the section s_k lifts to a formal section $s : U \to B$. □

Finally, we focus on the case, where B is a formal abelian scheme. Recall form of Propositions A.2.5 and A.2.4 that we have a canonical morphism

$$\psi : \mathrm{Ext}(B, G) = H^2_{rat}(B, G)_{sym} \longrightarrow H^1(B, \mathcal{G}_B),$$

where $\mathcal{G}_B$ is the sheaf of germs of regular maps from B to G. One can determine the kernel and the image of ψ; cf. [88, Chap. VII, §3, Theorem 5]. For explaining the result, we need some notations. For $i = 1, 2$ let

$p_i : B \times B \longrightarrow B$ be the i-th projection,

$\iota_i : B \to B \times B$ be the i-th injection defined by the unit element e of B.

Then, via pull-backs, one obtains morphisms

$$\begin{aligned} (p_1^*, p_2^*) &: H^1(B, \mathcal{G}_B) \times H^1(B, \mathcal{G}_B) \longrightarrow H^1(B \times B, \mathcal{G}_{B\times B}), \\ (\iota_1^* \times \iota_2^*) &: H^1(B \times B, \mathcal{G}_{B\times B}) \longrightarrow H^1(B, \mathcal{G}_B) \times H^1(B, \mathcal{G}_B). \end{aligned}$$

Then, $(\iota_1^* \times \iota_2^*)$ is a left inverse of (p_1^*, p_2^*). Furthermore, let

$$m_B : B \times B \longrightarrow B$$

be the group law. Thus, for every $x \in H^1(B, \mathcal{G}_B)$ we have the pull-back $m_B^*(x) \in H^1(B \times B, \mathcal{G}_{B\times B})$. One says that an $x \in H^1(B, \mathcal{G}_B)$ is *primitive* if $m_B^*(x) = (p_1^*, p_2^*)(x, x)$. In the case $G = \mathbb{G}_m$, the notion "primitive" for a line bundle L means $m_B^* L \cong p_1^* L \otimes p_2^* L$ which is equivalent to L being translation invariant; cf. [74, II.8 & II.13].

Theorem A.2.6. *Let B be a smooth formal group scheme and G be a connected commutative formal linear group. If every morphism $B \to G$ is constant, then the canonical homomorphism*

$$\psi : \mathrm{Ext}(B, G) \longrightarrow H^1(B, \mathcal{G}_B)$$

is injective and its image is the set of primitive elements in $H^1(B, \mathcal{G}_B)$.

Proof. Let $G \to E \to B$ be an extension. The extension E corresponds to a formal rational factor system $f \in H^2_{\mathrm{rat}}(B, G)$ due to Proposition A.2.5. Assume that E belongs to the kernel of ψ. Then $E \to B$ is a trivial G-torsor. Let $s : B \to E$ be a regular section of $E \to B$. After a transformation by a constant we may assume that s maps the unit of B to the unit of E. Now E corresponds to

$$f : B \times B \longrightarrow G, (b_1, b_2) \longmapsto f(b_1, b_2) = s(b_1 + b_2) \cdot s(b_1)^{-1} \cdot s(b_2)^{-1}.$$

The map f is regular on all of $B \times B$ and maps $B \times B$ to G. Since any map $B \to G$ is constant, f is constant and equal to the unit element. So $s : B \to E$ is a group homomorphism, and hence E is the trivial extension. Thus, we see that ψ is injective. The identification of the image of ψ follows as in [88, VII, §3, Theorem 5]. □

Remark A.2.7. On can even construct the group law on a primitive G-torsor L over B explicitly; cf. [88, p. 182]. In fact, the isomorphism $m_B^* L \cong p_1^* L \otimes p_2^* L$ gives rise to a regular function $m_L : L \times L \to L$ which is compatible with the group law of B. Indeed, one checks the identity

$$m_L(\ell_1 g_1, \ell_2 g_2) = m_L(\ell_1, \ell_2) g_1 g_2 \quad \text{for } \ell_1, \ell_2 \in L,\ g_1, g_2 \in G.$$

There exists a point $e \in L$ projecting to the unit element of B. After applying a translation by an element of G, one can suppose $m_L(e, e) = e$. Then it follows that (L, m_L) is a group, and hence a G-extension of B.

Theorem A.2.8. *Let B be a formal abelian R-scheme. Then the group* $\mathrm{Ext}(B, \mathbb{G}_m^r)$ *is canonically isomorphic to the set of R-valued points $B'(R)^r$ of $(B')^r$, where B' is the dual abelian scheme of B which represents the functor* $\mathrm{Pic}^\tau_{B/R}$ *of translation invariant line bundles on B; cf. Theorem 6.1.1.*

Without introducing coordinates on the torus $T \cong \mathbb{G}_m^r$, an element of $\mathrm{Ext}(B,T)$ *is equivalent to a group homomorphism $\varphi : \mathbb{X}(T) \to B'$ from the character group $\mathbb{X}(T)$ to the dual B'.*

In particular, for every $\sigma \in \mathrm{Hom}(T, \mathbb{G}_m)$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \sigma_* E & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

and the class of the $\mathbb{G}_m$-torsor $\sigma_* E$ is given by $\varphi(\sigma)$.

A.3 Cubical Structures

In this section we give a survey on cubical structures. This notion was invented by L. Breen [19]. Here we follow the exposition of Moret-Bailly [70, Chap. I] and add some proofs.

In the following let A and G be *commutative* group objects in the category of schemes or of rigid analytic spaces. The group law on A is written additively "+" and on G multiplicatively "·". The unit element of A and of G is denoted by "0" and by "1", respectively. Then we will consider G-torsors on A. In our applications A will be an abelian scheme, an abeloid variety or a Raynaud extension and $G = \mathbb{G}_m$. Recall that $\mathbb{G}_m$-torsors L on A correspond to line bundles on A.

Notation A.3.1. Let S be scheme or a rigid analytic space and let $A \to S$ and $G \to S$ be smooth S-group objects with connected fibers. If L is a G-torsor over A, for integers $n \geq 1$ we obtain G-torsors on A^n

$$\mathcal{D}_n(L) := \bigotimes_{\emptyset \neq I \subset \{1,\ldots,n\}} \mu_I^* L^{\otimes(-1)^{n+\mathrm{card}(I)}},$$

where the tensor product runs through all non-empty subsets I of $\{1,\ldots,n\}$ and where $\mu_I : A^n \to A$ is the morphism

$$\mu_I : A^n \longrightarrow A,\ (x_1,\ldots,x_n) \longmapsto \sum_{i\in I} x_i .$$

For the empty set I the map μ_I is the zero map. Sometimes one also adds the tensor product with $\mu_\emptyset^* L^{\otimes(-1)^n}$, then the modified $\mathcal{D}_n(L)$ is canonically rigidified along the zero section. Since we mostly work with rigidified line bundles, we will omit the empty set. On the other hand, any rigidificator of $\mathcal{D}_n(L)$ in the above given definition gives rise to a rigidificator of L.

Note that $\mathcal{D}_n(L)$ is functorial in L and compatible with pull-backs by group homomorphisms and with tensor products of G-torsors. We are especially interested in the cases $n = 2, 3$. For S-valued points x, y, z of A we have

$$\mathcal{D}_2(L)_{x,y} = L_{x+y} \otimes L_x^{-1} \otimes L_y^{-1},$$
$$\mathcal{D}_3(L)_{x,y,z} = L_{x+y+z} \otimes L_{x+y}^{-1} \otimes L_{x+z}^{-1} \otimes L_{y+z}^{-1} \otimes L_x \otimes L_y \otimes L_z.$$

Here we denote the fiber of a G-torsor at a point x by L_x. For any permutation $\sigma \in \mathfrak{S}_n$ and points $x_1, \ldots, x_n \in A(S)$ there is a canonical isomorphism

$$(\chi_\sigma)_{x_1,\ldots,x_n} : \mathcal{D}_n(L)_{x_1,\ldots,x_n} \xrightarrow{\sim} \mathcal{D}_n(L)_{x_{\sigma(1)},\ldots,x_{\sigma(n)}}.$$

The action of $\mathfrak{S}_2$ gives rise to isomorphisms

$$\xi_{x,y} : \mathcal{D}_2(L)_{x,y} \xrightarrow{\sim} \mathcal{D}_2(L)_{y,x}$$

for points $x, y \in A(S)$ satisfying $\xi_{x,x} = \mathrm{id}$ and $\xi_{x,y} \circ \xi_{y,x} = \mathrm{id}$. There is a canonical isomorphism

$$\varphi_{x,y,z} : \mathcal{D}_2(L)_{x+y,z} \otimes \mathcal{D}_2(L)_{x,y} \xrightarrow{\sim} \mathcal{D}_2(L)_{x,y+z} \otimes \mathcal{D}_2(L)_{y,z}, \tag{A.2}$$

which can be viewed as a cocycle condition. In particular, one has an isomorphism of rigidificators

$$\mathcal{D}_2(L)_{0,y} \simeq L_0^{-1} \simeq \mathcal{D}_2(L)_{x,0}, \tag{A.3}$$

where 0 denotes the unit element of A. The isomorphisms are compatible with the symmetry; i.e., the following diagram is commutative

$$\begin{array}{ccc} \mathcal{D}_2(L)_{y,z} \otimes \mathcal{D}_2(L)_{0,y} & \xrightarrow{\varphi_{0,y,z}} & \mathcal{D}_2(L)_{0,y+z} \otimes \mathcal{D}_2(L)_{y,z} \\ & \searrow \quad \swarrow & \\ & L_0^{-1} \otimes \mathcal{D}_2(L)_{y,z}. & \end{array}$$

Assume now that there is a section τ of $\mathcal{D}_2(L)$ over A^2. Then τ gives rise to a trivialization $\tau(x, y)$ of $L_{x,y} \otimes L_x^{-1} \otimes L_y^{-1}$; i.e., to an isomorphism

$$L_x \otimes L_y \xrightarrow{\sim} L_{x+y}, \quad (\ell_x, \ell_y) \longmapsto \ell_x \star \ell_y,$$

and hence to a law of composition on L, which is compatible with the group law of A under the projection $p : L \to A$. In particular, we have that

$$(g\ell_1) \star \ell_2 = \ell_1 \star (g\ell_2) = g(\ell_1 \star \ell_2) \quad \text{for all } g \in G, \ell_1, \ell_2 \in L. \tag{A.4}$$

Obviously, the sections τ of $\mathcal{D}_2(L)$ correspond to the laws of composition on L. Indeed, a law of composition "$\star$" on L defines a section

$$\tau\big(p(\ell_1), p(\ell_2)\big) := (\ell_1 \star \ell_2) \otimes \ell_1^{-1} \otimes \ell_2^{-1} \quad \text{for } \ell_1, \ell_2 \in L.$$

Such a section τ gives rise to two sections $\varepsilon_1, \varepsilon_2 : A \to L_0 = 0^*L$ via the isomorphisms of rigidificators (A.3); they satisfy $\varepsilon_1(x) \star \ell = \ell$ and $\ell \star \varepsilon_2(y) = \ell$ for $x, y \in A$ and $\ell \in L$. If $\varepsilon : 0 \to L_0$ is the original rigidificator, then $\varepsilon_1(0) = \varepsilon$ and $\varepsilon_2(0) = \varepsilon$. Due to (A.4) one has $(g_1\varepsilon) \star (g_2 \star \varepsilon) = (g_1 \cdot g_2)\varepsilon$ for $g_1, g_2 \in G$. Thus, the restriction of "$\star$" onto L_0 can be viewed as a group law on L_0 with unit element ε and is isomorphic to G.

The law of composition "$\star$" is commutative if and only if τ is invariant under $\mathfrak{S}_2$ and "$\star$" is associative if and only if τ is compatible with the *cocycle condition*

$$\varphi_{x,y,z}\big(\tau(x+y,z) \otimes \tau(x,y)\big) = \tau(x, y+z) \otimes \tau(y,z).$$

Regarding ε as a unit element of L, the law of composition "$\star$" is a group law on L which can be viewed as a G-extension of A. There is a bijective correspondence; cf. [70, I.2.3.10].

Proposition A.3.2. *Let L be a G-torsor over A. Then there is a one-to-one correspondence between*

(i) *Commutative G-extensions of A on L, which are compatible with the structure of the G-torsor on L.*
(ii) *Sections τ of $\mathcal{D}_2(L)$ over $A \times A$ which are symmetric and satisfy the cocycle condition.*

Now we turn to the discussion of $\mathcal{D}_3(L)$. Starting with $\mathcal{D}_2(L)$, one has two canonical isomorphisms

$$\mathcal{D}_2(L)_{x+y,z} \otimes \mathcal{D}_2(L)_{x,z}^{-1} \otimes \mathcal{D}_2(L)_{y,z}^{-1} \xrightarrow{(\alpha_1)_{x,y,z}} \mathcal{D}_3(L)_{x,y,z}$$

$$\mathcal{D}_2(L)_{x,y+z} \otimes \mathcal{D}_2(L)_{x,y}^{-1} \otimes \mathcal{D}_2(L)_{x,z}^{-1} \xrightarrow{(\alpha_2)_{x,y,z}} \mathcal{D}_3(L)_{x,y,z}.$$

One verifies that $(\alpha_2)_{x,y,z}^{-1} \circ (\alpha_1)_{x,y,z} = \varphi_{x,y,z}$ from (A.2). There is a canonical isomorphism

$$\psi_{x,y,z,t} : \mathcal{D}_3(L)_{x+y,z,t} \otimes \mathcal{D}_3(L)_{x,y,t} \xrightarrow{\sim} \mathcal{D}_3(L)_{x,y+z,t} \otimes \mathcal{D}_3(L)_{y,z,t},$$

which will serve as a cocycle isomorphism for $\mathcal{D}_3(L)$.

Definition A.3.3. A *cubical structure* on a G-torsor L on A is a section τ of the G-torsor $\mathcal{D}_3(L)$ over A^3 with the following properties:

(i) τ is invariant under $\mathfrak{S}_3$; i.e. $\chi_\sigma(\tau) = \sigma^*(\tau)$ for all $\sigma \in \mathfrak{S}_3$,
(ii) τ is a 2-cocycle; i.e., for all S-valued points x, y, z, t of A holds

$$\psi_{x,y,z,t}\big(\tau(x+y, z, t) \otimes \tau(x, y, t)\big) = \tau(x, y+z, t) \otimes \tau(y, z, t).$$

A *cubical G-torsor* is a couple (L, τ), where L is a G-torsor on A and τ is a cubical structure on the G-torsor associated to L.

A *morphism* $(L, \tau) \to (L', \tau')$ *of cubical G-torsors* is a morphism $f : L \to L'$ of G-torsors such that $\mathcal{D}_3(f)(\tau) = \tau'$.

A *trivialization of a cubical G-torsor* (L, τ) is a section $\sigma : A \to L$ such that $\mathcal{D}_3(\sigma) = \tau$.

In the case $G = \mathbb{G}_m$ we denote by $\mathrm{Cub}_{A/S}$ the functor, which associates to an S-scheme S' the set of isomorphy classes (L, τ) of line bundles L on $A \times_S S'$ with a cubical structure τ.

Note that a cubical G-torsor is automatically rigidified. For abeloid varieties A and $\mathbb{G}_m$-torsors the converse is also true by the theorem of the cube Theorem 7.1.6(a).

Proposition A.3.4. *Let A be an abeloid variety over a non-Archimedean field K and $G = \mathbb{G}_m$. Then the canonical morphism of functors*

$$\mathrm{Cub}_{A/K} \xrightarrow{\sim} (\mathrm{Pic}_{A/K}, 1),$$

which associates to a line bundle with cubical structure the line bundle with the induced rigidificator at the unit section, is an equivalence.

Proof. It follows from the rigidity of Lemma 7.1.2 that $\mathcal{D}_3(L)$ is trivial for any line bundle L on A; cf. Theorem 7.1.6. The cocycle condition in Definition A.3.3(ii) is automatically fulfilled. Indeed, since A is a connected proper rigid analytic space, any global holomorphic function is constant due to Corollary 1.6.8. Moreover, for any rigid analytic space S the canonical map $\mathcal{O}_S \to p_*\mathcal{O}_{A\times S}$ is bijective. So one has to check the cocycle condition only at the point $x = y = z = t = 0$. There it is true because of the compatibility of the rigidificators. □

Mumford introduced the concept of $\mathbb{G}_m$-biextension in [73], which was extensively analyzed and amplified in [42, VII & VIII] by Grothendieck. Roughly speaking, a G-biextension of two commutative group spaces A_1, A_2 is a G-torsor over $A_1 \times A_2$ with two partial group laws which are compatible in the obvious sense. There is the following relationship between cubical structures and biextensions; cf. [70, I.2.5.4].

Proposition A.3.5. *Let L be a G-torsor. Then a cubical structure τ on L is equivalent to the structure of a symmetric biextension on $\mathcal{D}_2(L)$ over $A \times A$ by G.*

The laws of composition "$\star_1$" and "$\star_2$" on $\mathcal{D}_2(L)$ are associated to the sections σ_1 and σ_2 satisfying the relation $\alpha_1(\sigma_1) = \tau = \alpha_2(\sigma_2)$.

Proof. By the isomorphisms α_1 and α_2, the section τ induces

$$\sigma_1 := \alpha_1^{-1}(\tau) : A \times A \longrightarrow \mathcal{D}_2(L)_{x+y,z} \otimes \mathcal{D}_2(L)_{x,z}^{-1} \otimes \mathcal{D}_2(L)_{y,z}^{-1},$$
$$\sigma_2 := \alpha_2^{-1}(\tau) : A \times A \longrightarrow \mathcal{D}_2(L)_{x,y+z} \otimes \mathcal{D}_2(L)_{x,y}^{-1} \otimes \mathcal{D}_2(L)_{x,z}^{-1},$$

and hence two laws of composition

$$\star_1 : \mathcal{D}_2(L)_{x,z} \otimes \mathcal{D}_2(L)_{y,z} \longrightarrow \mathcal{D}_2(L)_{x+y,z},$$
$$\star_2 : \mathcal{D}_2(L)_{x,y} \otimes \mathcal{D}_2(L)_{x,z} \longrightarrow \mathcal{D}_2(L)_{x,y+z}.$$

The conditions of the cubical structure in Definition A.3.3 imply that the group laws "$\star_1$" and "$\star_2$" define symmetric biextensions. Here the symmetry is given by the morphism $\xi_{x,y} : \mathcal{D}_2(L)_{x,y} \longrightarrow \mathcal{D}_2(L)_{y,x}$. □

Now we turn to the special situation we are mainly interested in.

Notation A.3.6. In the following we will consider an extension of an abelian R_n-scheme B by a split affine torus T of rank r

$$0 \to T \xrightarrow{\iota} E \xrightarrow{q} B \to 0 \quad \widehat{=} \quad \phi' : M' \to B' \qquad (*)$$

which are smooth over $R_n := R/R\pi^{n+1}$ for some $n \in \mathbb{N}$, where $(R, \mathfrak{m}_R)$ is a valuation ring with $0 \neq \pi \in \mathfrak{m}_R$. In the following let S be an R_n-scheme and consider the extension $(*)$ after base change $S \to \operatorname{Spec} R_n$.

In our applications the above extension will be an extension of a formal torus by a formal abelian scheme. The results we will show below for the R_n-situation can be immediately carried over to the formal situation.

Proposition A.3.7. *Let $T \to S$ be a torus. Then the category* $\operatorname{Cub}(T, \mathbb{G}_m)$ *of cubical $\mathbb{G}_m$-torsors on T is equivalent to the category* $\operatorname{Ext}(T, \mathbb{G}_m)$ *of commutative $\mathbb{G}_m$-extensions of T and both categories consist of exactly one isomorphy class.*

Proof. The assertion mainly follows from [42, VIII, 3.4]; cf. [70, I.7.2.1]. For the convenience of the reader we will give the proof in the case $S = \operatorname{Spec} R_n$.

Let $L \to T$ be a $\mathbb{G}_m$-torsor. Over the residue field k of R there exists a section $s_0 : T \otimes k \to L \otimes k$, since $\mathcal{O}_T(T \otimes k)$ is factorial. Due to the smoothness of $L \to T$ the section lifts to a section $s : T \to L$. Thus, we know that $L = \mathbb{G}_{m,T}$ is the trivial torsor. Knowing this, one can proceed with a general affine base scheme $S = \operatorname{Spec} A$. From Proposition A.3.5 it follows that a cubical structure on $\mathbb{G}_{m,T}$ is equivalent to the structure of a symmetric biextension on $\mathcal{D}_2(\mathbb{G}_{m,T})$ over $T \times T$. This is a section

$$\sigma : T \times T \longrightarrow \mathcal{D}_2(\mathbb{G}_{m,T}) = \mathbb{G}_{m,T \times T}$$

with certain properties. The section σ is given by a Laurent polynomial

$$\sigma(\xi_1,\xi_2)=\xi_1^{m_1}\xi_2^{m_2}\cdot\bigl(1+h(\xi_1,\xi_2)\bigr)$$

with $m_1,m_2\in\mathbb{Z}^r$, where ξ_1,ξ_2 are the coordinate functions on $T\times T$, and h is a polynomial in $A[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ with coefficients contained in the Jacobson radical of A. Let $\mathfrak{a}\subset A$ be the ideal generated by the coefficients of h; in particular, $\mathfrak{a}$ is finitely generated and contained in the Jacobson radical of A, since the units in $A[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ are of type $\alpha\cdot\xi_1^{m_1}\xi_2^{m_2}$ with a unit $\alpha\in A^\times$ if A is reduced. The symmetry implies $m_1=m_2$ and $h(\xi_1,\xi_2)=h(\xi_2,\xi_1)$. Moreover,

$$\sigma(\xi_1,1)=1=\sigma(1,\xi_2)$$

implies $m_1=m_2=0$. It remains to show $h=0$. The cocycle condition

$$\sigma(\xi_1\xi_2,\zeta)\cdot\sigma(\xi_1,\xi_2)=\sigma(\xi_1,\xi_2\zeta)\cdot\sigma(\xi_2,\zeta)$$

implies

$$h(\xi_1\xi_2,\zeta)+h(\xi_1,\xi_2)=h(\xi_1,\xi_2\zeta)+h(\xi_2,\zeta)\ \mathrm{mod}\ \mathfrak{a}^2.$$

Now look at the Laurent expansion

$$h=\sum_{\mu,\nu\in\mathbb{Z}^r}h_{\mu,\nu}\xi_1^{\mu}\xi_2^{\nu}\in A_n\bigl[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}\bigr].$$

Then the cocycle condition yields

$$h_{\mu,\nu}\cdot\xi_1^{\mu}\xi_2^{\mu}\zeta^{\nu}+h_{\mu,\nu}\cdot\xi_1^{\mu}\xi_2^{\nu}=h_{\mu,\nu}\cdot\xi_1^{\mu}\xi_2^{\nu}\zeta^{\nu}+h_{\mu,\nu}\cdot\xi_2^{\mu}\zeta^{\nu}\ \mathrm{mod}\ \mathfrak{a}^2.$$

Thus, modulo $\mathfrak{a}^2$, we see $h_{\mu,\nu}=0$ if $\nu\neq 0$, and hence $h_{\mu,0}=0$ if $\mu\neq 0$. Thus, it follows $\mathfrak{a}=\mathfrak{a}^2$ and hence $\mathfrak{a}=0$. □

Next, we turn to the main result for our applications. This result was invented by Breen [19, 3.10]; we specialize the statement to the case we are mainly concerned with; cf. [70, I.7.2.2].

Theorem A.3.8. *In the situation of Notation A.3.6 let S be an R_n-scheme or an admissible formal R-scheme and let E be a (formal) torus extension of an abelian R_n-scheme or of a formal abelian R-scheme, respectively.*

Let (L,s) be a pair consisting of a cubical line bundle L on the torus extension $E\times_R S$ and a trivialization $s:T\times S\to L|_{T\times S}$ of the cubical line bundle $L|_{T\times S}$.

(a) *Then (L,s) descends to a cubical line bundle N on $B\times_R S$.*
(b) *The set of all possible descent data for L is a principal homogeneous space under the character group $M'=\mathrm{Hom}(T,\mathbb{G}_m)$ in a natural way.*

In particular, if L is the trivial line bundle, then s is a character m' of T and $N=P_{B\times\phi'(m')}$.

We will sketch a proof in the situation $S = \operatorname{Spec} R_n$.

Lemma A.3.9. *Let $T \to S$ be a split affine torus. Every trivialization $\varphi : T \to \mathbb{G}_{m,T}$ of the trivial cubical $\mathbb{G}_m$-torsor is a character $m' : T \to \mathbb{G}_m$.*

Proof. As in the proof of Proposition A.3.7 let $S = \operatorname{Spec} A$. Then φ can be written as

$$\varphi(\xi) = c \cdot \xi^{m'} \cdot \big(1 + h(\xi)\big),$$

where ξ denotes the coordinate functions on T, and $c \in A^\times$ is a unit, and $h \in A[\xi, \xi^{-1}]$ is a Laurent polynomial, whose ideal of coefficients $\mathfrak{a} \subset A$ is contained in the Jacobson radical. Since φ has to respect rigidificators, we have $c = 1$ and $h(1) = 0$. Moreover, φ induces a group homomorphism $\mathcal{D}_2(\varphi)$. Therefore, φ has to fulfill the relation

$$\varphi(\xi_1\xi_2) = \varphi(\xi_1) \cdot \varphi(\xi_2).$$

The latter implies

$$h(\xi_1\xi_2) = h(\xi_1) + h(\xi_2) \bmod \mathfrak{a}^2.$$

As in the proof of Proposition A.3.7, it follows from the last equation that $h = 0$. □

Sketch of the Proof of Theorem A.3.8. (a) Let us first consider the situation over the residue field $R = k$. Since E is a torus extension of B, by Proposition A.2.5 there exists an affine open covering $\mathfrak{B} := (B_1, \ldots, B_N)$ of B and sections

$$\sigma_i : B_i \longrightarrow E \quad \text{for } i = 1, \ldots, N.$$

The sections give rise to isomorphisms

$$T \times B_i \xrightarrow{\sim} E_i := q^{-1}(B_i), \quad (t, b) \longmapsto t \cdot \sigma_i(b).$$

Then the canonical map

$$\operatorname{Pic}(B_i) \xrightarrow{\sim} \operatorname{Pic}(T \times_R B_i), \quad N \longmapsto q^* N,$$

is bijective. Indeed, the pull-back by the section σ_i is a left inverse of the map, and so the mapping is injective. Since B_i is smooth over k, its local rings are factorial. Thus, for any given line bundle L on $E_i \cong T \times B_i$ there exists an affine open covering $\mathfrak{U} = \{U_1, \ldots, U_n\}$ of B_i such that $L|_{T \times U_\nu}$ is trivial. After refining the covering $\mathfrak{B}$ we may assume that $L|_{E_i}$ is trivial. Due to Theorem A.3.9 the trivialization $s : T \to L|_T$ is a character m', which extends to a trivialization $e_{m'} : E \to q^* P_{B \times \phi'(m')}$. Then the trivializations of $L|_{E_i}$ can be represented in the form

$$\ell_i = \varepsilon_i \otimes e_{m'}$$

where $\varepsilon_i : E_i \to L \otimes q^* P_{B\times\phi'(-m)}$ is a section. Since the invertible elements of the ring $A[\xi, \xi^{-1}]$ of Laurent polynomials over a connected reduced ring A are of the form $\alpha \cdot \xi^{m'}$ for an invertible element α of A and an integer $m' \in \mathbb{Z}$, the transition functions $\lambda_{i,j} \in \mathcal{O}_E(E_i \cap E_j)^\times$ defined via

$$\lambda_{i,j} \cdot \ell_j = \ell_i \quad \text{over } E_i \cap E_j$$

are of type

$$\lambda_{i,j} = \beta_{i,j} \cdot \xi_i^{m'_{i,j}} \tag{$*$}$$

with invertible functions $\beta_{i,j} \in \mathcal{O}_B^\times(B_i \cap B_j)$ and integers $m'_{i,j} \in \mathbb{Z}$. The integers $(m'_{i,j})$ constitute a cocycle. Since B is irreducible, the cocycle vanishes. Then, we can present the transition functions $\lambda_{i,j}$ in the form $\lambda_{i,j} = \beta_{i,j} \otimes e_{m'}$. Thus, the functions $(\beta_{i,j})$ constitutes a cocycle in $Z^1(\mathfrak{B}, \mathcal{O}_B^\times)$ and hence $(\beta_{i,j})$ induces a line bundle N on B. Therefore, the couple (L, s) is the pull-back of $N \otimes P_{B\times\phi'(m')}$ equipped with the canonical trivialization $e_{m'}|_T : T \to P_{B\times\phi'(m')}$. So the statement in Theorem A.3.8 is true in the case $R = k$.

Now consider the case $R_n := R/R\pi^{n+1}$. Generally speaking, this follows by lifting the results from the situation over the residue field due to the lifting property of smooth morphisms, but this can be done only locally on B. At first, they are defined only over the reduction $B_i \otimes_R k$, where $k = R/\mathfrak{m}_R$ is the residue field of R, then they lift to sections over $R/R\pi^n$ due to the smoothness of $E \to B$. The main property, which one looses in the non-reduced case, is the structure of the units as used in ($*$). That is precisely the point why the cubical structures are introduced. In fact, if one postulates that the sections have to respect the cubical structures, then the transition functions are of the type ($*$) as well. For example look at the proof of Proposition A.3.7.

Concerning the uniqueness, we may assume that $q^*N \cong \mathbb{G}_{m,E}$ and that the section $s : E \to \mathbb{G}_{m,E}$ is equal to the constant section $e_0 = 1$. Then the sections ℓ_i are equal to $\beta_i \cdot e_0$ and hence the cocycle $\lambda_{i,j} = \beta_i \cdot \beta_j^{-1}$ is solvable. Thus, the sections (β_i) fit together to build a trivialization of the $\mathbb{G}_m$-torsor N.

(b) It remains to analyze the possible descent data of the trivial cubical line bundle $\mathcal{O}_E$ on E. It follows from Theorem A.3.9 that trivializations of the trivial cubical line bundle on a torus can be given only by characters $m' \in M'$. Thus, the character group acts faithfully and transitively on the trivializations of s_T. □

Corollary A.3.10. *Let N_1 and N_2 be cubical line bundles on B. Then, for every morphism $\alpha : q^*N_1 \to q^*N_2$ respecting the cubical structures, there exists a unique character $m' \in M'$ such that*

$$\alpha \otimes e_{-m'} : q^*N_1 \longrightarrow q^*(N_2 \otimes P_{B\times\phi'(-m')})$$

descends to a morphism $\beta : N_1 \to N_2 \otimes P_{B\times\phi'(-m')}$.

Proof. We may assume that N_1 is trivial and the pull-back $L := q^* N_2$ is trivial. Then it follows from Theorem A.3.8 that L descends to $N_2 := P_{B \times \phi'(m')}$. Now $\alpha \otimes e_{-m'}$ is a pull-back of a global section $\beta : B \to N_2 \otimes P_{B \times \phi'(-m')}$ if and only if $\alpha \otimes e_{-m'}$ is constant on the fibers of q. The latter is equivalent to the fact that the restriction $(\alpha \otimes e_{-m'})|_T$ is constant to 1. Thus we see that the assertion is true. □

Glossary of Notations

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Q}_p$	natural, integer, rational, real, p-adic numbers
K	non-Archimedean field, Definition 1.1.1
R	valuation ring of K
k	residue field of R
$\mathfrak{m}_R$	maximal ideal of R
π	element of $\mathfrak{m}_R$ with $\pi \neq 0$ or uniformizer of R if R is a discrete valuation ring
K'	finite separable field extension of K
R'	valuation ring of K'
$\overline{K}$	complete algebraic closure of K
$\overline{R}$	valuation ring of $\overline{K}$
$K(x)$	the residue field of a point x
$\lvert K^\times\rvert$	value group of K
$\sqrt{\lvert K^\times\rvert}$	divisible hull of $\lvert K^\times\rvert$, equal to $\lvert\overline{K}^\times\rvert$
$\widehat{A}$	universal covering of an abelian variety A, Theorem 5.6.3
$\overline{A}$	largest connected subgroup of A with smooth formal R-model, Theorem 5.6.3
A'	dual of an abelian variety A, Definition 5.1.5
$\mathbb{A}^n_S$	affine n-space over a base S
$A_K = T_n/\mathfrak{a}$	affinoid K-algebra, Sect. 1.2
$\mathring{A}_K$	R-subalgebra of A_K of power bounded functions, Definition 1.4.4
$\check{A}_K$	ideal in $\mathring{A}_K$ of topologically nilpotent elements, Definition 1.4.4
$\widetilde{A}_K$	reduction of A_K, Definition 1.4.4
$A_K\langle\eta_1, \ldots, \eta_n\rangle$	relative Tate algebra over an affinoid K-algebra A_K, Definition 1.2.8
$\mathbb{B}^n := \operatorname{Sp} T_n$	n-dimensional unit ball, Sect. 1.3
$\operatorname{Cub}_{A/S}$	functor of cubical line bundles on A, Theorem 6.3.2
$\operatorname{Cub}^M_{E/S}$	functor of cubical line bundles on E with M-linearization, Theorem 6.3.2

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$\mathrm{Cub}^M_{E/B/S}$	functor of cubical line bundles of B with M-action, Theorem 6.3.2
$\mathbb{D}_K := \mathrm{Sp}\, K\langle\xi\rangle$	closed unit disc over K
$\mathbb{D}^-_K$	open unit disc, Definition 2.4.1
$\mathcal{D}_n(L)$	line bundle on A^n for a group space A, Notation A.3.1
$\mathbb{G}_{m,S}$	1-dimensional torus over a space S
$\overline{\mathbb{G}}_{m,K}$	1-dimensional torus of units over K, Proposition 5.4.6
$\mathrm{vert}(G)$	set of the vertices of a graph G, Definition A.1.1
$\mathrm{edge}(G)$	set of the edges of a graph G, Definition A.1.1
$\mathrm{real}(G)$	realization of a graph G, Definition A.1.2
$H_1(X_K, \mathbb{Z})$	homology group of a curve X_K, Proposition 5.5.3, Remark 2.4.18
$H_1(\widetilde{X}, \mathbb{Z})$	homology group of the reduction $\widetilde{X}$, Proposition 5.2.3 and Corollary 5.2.4
$H^1(X_K, \mathbb{Z})$	cohomology group of a curve X_K, Lemma 5.5.1 and Corollary 5.5.6
$H^1(\widetilde{X}, \mathbb{Z})$	cohomology group of the reduction $\widetilde{X}$, Remark 5.2.7 and Proposition 5.2.8
$J_K := \mathrm{Jac}(X)$	Jacobi variety, representing $\mathrm{Pic}^0_{X/K}$, Definition 5.1.2
$\widehat{J}_K$	universal covering of the Jacobian J_K, Theorem 5.5.11
$\overline{J}_K$	largest connected subgroup of J_K with smooth formal R-model, Proposition 5.3.3
$\overline{J}$	formal R-model of $\overline{J}_K$, Proposition 5.3.3
L_Γ	set of limit points of a Schottky group Γ
$\ell(\gamma)$	length of the presentation of $\gamma \in \Gamma$ with respect to a basis of a Schottky group Γ
$\mathcal{O}_X(D)$	invertible sheaf associated to a divisor D meromorphic functions f with $\mathrm{div}\, f + D \geq 0$
$\mathbb{P}^n_S$	projective n-space over a base S
$P_{A\times A'}$	Poincaré bundle on $A \times A'$, Theorem 6.1.1
$\mathcal{P}_{A\times A'}$	Poincaré invertible sheaf on $A \times A'$, Theorem 5.1.4
$\mathrm{Pic}^0_{X/K}$	Picard functor of a line bundles of degree zero on K-curve X, Theorem 5.1.1
$\mathrm{Pic}^\tau_{A/K}$	Picard functor of translation invariant line bundles on an abeloid variety A, Theorem 5.1.4
$\mathrm{Pol}(f)$	the set of poles of a meromorphic function f
$R\langle\xi_1, \ldots, \xi_n\rangle$	restricted power series ring over R
rig	generic fiber of an object over R, Definition 3.3.1, Theorem 3.3.3, Definition 3.3.6
$\mathrm{Sing}(X)$	singular locus of a curve X
Sp	maximal spectrum of an affinoid algebra, Sect. 1.3
Spf	formal spectrum of a formal R-scheme, Sect. 3.2
T_K	affine torus over K
$\overline{T}$	formal torus over R
$\overline{T}_K$	torus of units of T_K, generic fiber of $\overline{T}$

$T_n := K\langle\xi_1, \ldots, \xi_n\rangle$	Tate algebra, Sect. 1.2
$\mathring{T}_n$	subring of T_n of functions with Gauss norm ≤ 1, Example 1.4.5
$X(S)$	S-valued points of a space X
$\mathbb{X}(T)$	character group of a torus T
Ω_Γ	domain of ordinary points of a Schottky group Γ
$\lvert\cdot\rvert_X$	sup-norm on a rigid space X, Sect. 1.4

References

1. Abbes, A.: Éléments de Géométrie Rigide. Birkhäuser. PM 286 (2010)
2. Artin, M.: Algebraization of formal moduli I. In: Global Analysis (Papers in Honor of K. Kodaira), pp. 21–71. Princeton Univ. Press, Princeton (1969)
3. Artin, M.: Algebraic approximation of structures over complete local rings. Publ. Math. IHES **36**, 23–58 (1969)
4. Artin, M.: The implicit function theorem in algebraic geometry. In: Proceedings of the Bombay Colloquium on Algebraic Geometry 1968. Tata Institute of Fundamental Research Studies in Mathematics, vol. 4, pp. 13–34. Oxford University Press, London (1968)
5. Artin, M., Winters, G.: Degenerate fibres and stable reduction of curves. Topology **10**, 373–383 (1971)
6. Berkovich, V.G.: Spectral Theory and Analytic Geometry over Non-Archimedean Fields. American Mathematical Society, Washington (1990)
7. Bosch, S.: Eine bemerkenswerte Eigenschaft der formellen Fasern affinoider Räume. Math. Ann. **229**, 25–45 (1977)
8. Bosch, S.: Zur Kohomologietheorie rigid analytischer Räume. Manuscr. Math. **20**, 1–27 (1977)
9. Bosch, S.: Lectures on Formal and Rigid Geometry. Lecture Notes in Mathematics, vol. 2105. Springer, Berlin/Heidelberg/New York (2014)
10. Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean Analysis. Grundlehren, vol. 261. Springer, Berlin/Heidelberg/New York (1984)
11. Bosch, S., Lütkebohmert, W.: Stable reduction and uniformization of abelian varieties. I. Math. Ann. **270**, 349–379 (1985)
12. Bosch, S., Lütkebohmert, W.: Stable reduction and uniformization of abelian varieties. II. Invent. Math. **78**, 257–297 (1984)
13. Bosch, S., Lütkebohmert, W.: Degenerating abelian varieties. Topology **30**, 653–698 (1991)
14. Bosch, S., Lütkebohmert, W., Raynaud, M.: Formal and rigid geometry.
    I. Rigid spaces. Math. Ann. **295**, 291–317 (1993);
    II. Flattening techniques. Math. Ann. **296**, 403–429 (1993);
    III. The relative maximum principle. Math. Ann. **302**, 1–29 (1995);
    IV. The reduced fiber theorem. Invent. Math. **119**, 361–398 (1995)
15. Bosch, S., Lütkebohmert, W., Raynaud, M.: Néron Models. Springer, Ergeb. 3. Folge, vol. 21. Springer, Berlin/Heidelberg/New York (1990)
16. Bosch, S., Schlöter, K.: Néron models in the setting of formal and rigid geometry. Math. Ann. **301**, 339–362 (1995)
17. Bourbaki, N.: Commutative Algebra. Springer, Berlin/Heidelberg/New York (1980). Chapsters 1–7
18. Bourbaki, N.: Algèbre. Masson, Paris (1981). Chapters 4–7

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19. Breen, L.: Fonctions theta et théorème du cube. Lecture Notes in Mathematics, vol. 980. Springer, Berlin/Heidelberg/New York (1983)
20. Deligne, P.: Le Lemme de Gabber. Astérisque **127**(5), 131–150 (1985)
21. Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus. Publ. Math. IHES **36**, 75–109 (1969)
22. Deligne, P., Rapoport, M.: Les schémas de modules de courbes elliptiques. In: Modular Functions of One Variable, II. Lecture Notes in Mathematics, vol. 349. Springer, Berlin/Heidelberg/New York (1973)
23. Demazure, M., Gabriel, P.: Groupes algébriques. Tome I. Masson, Paris (1970). North-Holland, Amsterdam
24. Elkik, R.: Solutions d'équations à coefficients dans un anneau hensélien. Ann. Sci. Éc. Norm. Super. **6**(4), 553–604 (1973)
25. Epp, H.: Eliminating wild ramification. Invent. Math. **19**, 235–249 (1973)
26. Faltings, G.: Semi-stable vector bundles on Mumford curves. Invent. Math. **74**, 199–212 (1983)
27. Faltings, G., Chai, C.-L.: Degeneration of Abelian Varieties. Springer, Ergeb. 3. Folge, vol. 22. Springer, Berlin/Heidelberg/New York (1990)
28. Ford, L.R.: Fundamental regions for discontinuous groups of linear transformations. In: Proc. Internat. Congress Math., p. 392 (1950)
29. Forster, O.: Compact Riemann Surfaces. GTM, vol. 81. Springer, Berlin (1981)
30. Fresnel, J., van der Put, M.: Rigid Analytic Geometry and Its Applications. Progress in Mathematics, vol. 218. Birkhäuser Boston, Inc., Boston (2004)
31. Gerritzen, L.: Zur nichtarchimedischen Uniformisierung von Kurven. Math. Ann. **210**, 321–337 (1974)
32. Gerritzen, L.: Unbeschränkte Steinsche Gebiete. J. Reine Angew. Math. **297**, 21–34 (1978)
33. Gerritzen, L.: On non-Archimedean representations of abelian varieties. Math. Ann. **196**, 323–346 (1972)
34. Gerritzen, L., Grauert, H.: Die Azyklizität der affinoiden Überdeckungen. In: Global Analysis (Papers in Honor of K. Kodaira), pp. 159–184. Princeton University Press, Princeton (1969). University of Tokyo Press, Tokyo
35. Gerritzen, L., van der Put, M.: Mumford Curves and Schottky Groups. Lecture Notes in Mathematics, vol. 817. Springer, Berlin/Heidelberg/New York (1980)
36. Grauert, H., Remmert, R.: Über die Methode der diskret bewerteten Ringe in der nicht-archimedischen Analysis. Invent. Math. **2**, 87–133 (1966)
37. Grauert, H., Remmert, R.: Coherent Analytic Sheaves. Grundlehren, vol. 265. Springer, Berlin/Heidelberg/New York (1984)
38. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Pure & Applied Mathematics. Wiley, New York (1978)
39. Grothendieck, A., Dieudonné, J.: Eléments de Géométrie Algébrique. Publ. Math. IHES **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32** (1960–1967)
40. Grothendieck, A., et al.: Séminaire de Géométrie Algébrique 1. In: Revêtements Etales et Groupe Fondamental. Lecture Notes in Mathematics, vol. 224. Springer, Berlin/Heidelberg/New York (1971). New edition: Société Mathématique de France
41. Grothendieck, A., et al.: Séminaire de Géométrie Algébrique 3. In: Schémas en Groupes I, II, III. Lecture Notes in Mathematics, vols. 151, 152, 153. Springer, Berlin/Heidelberg/New York (1970). New edition: Société Mathématique de France
42. Grothendieck, A., et al.: Séminaire de Géométrie Algébrique 7. In: Groupes de Monodromie en Géométrie Algébrique. Lecture Notes in Mathematics, vols. 288, 340. Springer, Berlin/Heidelberg/New York (1972)
43. Gruson, L.: Fibrés vectoriels sur un polydisque ultramétrique. Ann. Sci. Éc. Norm. Super. **1**(4), 45–89 (1968)
44. Hartl, U.: Semi-stable models for rigid-analytic spaces. Manuscr. Math. **110**, 365–380 (2003)
45. Hartl, U., Lütkebohmert, W.: On rigid-analytic Picard varieties for rigid-analytic spaces. J. Reine Angew. Math. **528**, 101–148 (2000)

46. Herrlich, F.: Die Ordnung der Automorphismengruppe einer p-adischen Schottkykurve. Math. Ann. **246**, 125–130 (1979/1980)
47. Huber, R.: A generalization of formal schemes and rigid analytic varieties. Math. Z. **217**, 513–551 (1994)
48. Ihara, Y.: On discrete subgroups of the two by two projective linear group over p-adic fields. J. Math. Soc. Jpn. **18**, 219–235 (1966)
49. Keel, S., Mori, S.: Quotients by groupoids. Ann. Math. **145**, 193–213 (1997)
50. Kiehl, R.: Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie. Invent. Math. **2**, 191–214 (1967)
51. Kiehl, R.: Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. Invent. Math. **2**, 256–273 (1967)
52. Kiehl, R.: Analytische Familien affinoider Algebren. In: Heidelberger Sitzungsberichte math. nat. Kl, pp. 25–49 (1968)
53. Knudsen, F.F.: The projectivity of the moduli space of stable curves. II & III. Math. Scand. **52**, 161–212 (1983)
54. Knudsen, F.F., Mumford, D.: The projectivity of the moduli space of stable curves. I. Math. Scand. **39**, 19–55 (1976)
55. Knutson, D.: Algebraic Spaces. Lecture Notes in Mathematics, vol. 203. Springer, Berlin/Heidelberg/New York (1971)
56. Köpf, U.: Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. Schr. Math. Inst. Univ. Münster (2), vol. 7 (1974)
57. Kotissek, N.: Nichtarchimedische Schottky-Gruppen. Diplomarbeit. Münster (1983)
58. Lang, S., Tate, J.: Galois cohomology and principal homogeneous spaces. Am. J. Math. **80**, 659–684 (1958)
59. Lipman, J.: Desingularization of two-dimensional schemes. Ann. Math. **107**, 151–207 (1978)
60. Liu, Q.: Algebraic Geometry and Arithmetic Curves. Oxford Science Publications. Oxford University Press, Oxford (2002)
61. Lütkebohmert, W.: Formal algebraic and rigid analytic geometry. Math. Ann. **286**, 341–371 (1990)
62. Lütkebohmert, W.: The structure of proper rigid groups. J. Reine Angew. Math. **468**, 167–219 (1995)
63. Lütkebohmert, W.: From Tate's elliptic curve to abeloid varieties. Pure Appl. Math. Q. **5**, 1385–1427 (2009)
64. Manin, J., Drinfeld, V.G.: Periods of p-adic Schottky groups. J. Reine Angew. Math. **262/263**, 239–247 (1973)
65. Manin, J.: p-Adic automorphic forms. Itogi Nauki Teh., Ser. Sovrem. Probl. Mat. **3**, 5–92 (1974)
66. Martin, A.: Eine rigid analytische Version des Artinschen Glättungssatzes. Schr. Math. Inst. Univ. Münster (3), vol. 33 (2008)
67. Mehlmann, F.: Ein Beweis für einen Satz von Raynaud über flache Homomorphismen affinoider Algebren. Schr. Math. Inst. Univ. Münster (2), vol. 19 (1981)
68. Milne, J.S.: Jacobian varieties. In: Cornell, G., Silverman, J. (eds.) Arithmetic Geometry. Springer, Berlin (1986)
69. Milne, J.S.: Étale Cohomology. Princeton University Press, Princeton (1980)
70. Moret-Bailly, L.: Pinceaux de variétés abéliennes. In: Astérisque, vol. 129 (1985)
71. Morikawa, H.: On theta functions and abelian varieties over valuation fields of rank one. Nagoya Math. J. **20**, 1–27 (1962); **21**, 231–250 (1962)
72. Mumford, D., Fogarty, J.: Geometric Invariant Theory, Third Enlarged edn. Springer, Ergeb. 1. Folge, vol. 34. Springer, Berlin/Heidelberg/New York (1992)
73. Mumford, D.: Bi-extensions of formal groups. In: Algebraic Geometry, pp. 307–322. Oxford University Press, Oxford (1969)
74. Mumford, D.: Abelian Varieties. Tata Institute of Fundamental Research Studies in Mathematics, vol. 5. Oxford University Press, London (1970). Published for the Tata Institute of Fundamental Research, Bombay

75. Mumford, D.: An analytic construction of degenerating curves over complete local rings. Compos. Math. **24**, 129–174 (1972)
76. Mumford, D.: An analytic construction of degenerating abelian varieties over complete rings. Compos. Math. **24**, 239–272 (1972)
77. Mumford, D.: Stability of projective varieties. Enseign. Math. **23**, 39–110 (1977)
78. Raynaud, M.: Anneaux Locaux Henséliens. Lecture Notes in Mathematics, vol. 169. Springer, Berlin/Heidelberg/New York (1970)
79. Raynaud, M.: Variétés abéliennes et géométrie rigide. In: Actes du congrès international de Nice, vol. 1, pp. 473–477 (1970)
80. Raynaud, M.: Géométrie analytique rigide d'après Tate, Kiehl,.... Mém. Soc. Math. Fr. **39–40**, 319–327 (1974)
81. Raynaud, M.: Faiseaux amples sur les schémas en groupes et les espace homogènes. Lecture Notes in Mathematics, vol. 119. Springer, Berlin/Heidelberg/New York (1970)
82. Raynaud, M., Gruson, L.: Critères de platitude et de projectivité. Invent. Math. **13**, 1–89 (1971)
83. Raynaud, M.: Courbes semi-stables et groupe fondamental en géométrie algébrique. In: Bost, J.-B., Loeser, F., Raynaud, M. (eds.): Proceedings of the Conference on Fundamental Groups of Curves in Algebraic Geometry Held in Luminy, November 30–December 4, 1998. Progress in Mathematics, vol. 187. Birkhäuser Verlag, Basel (2000)
84. Reversat, M., van der Put, M.: Construction analytique rigide de variétés abéliennes. Bull. Soc. Math. Fr. **117**(4), 415–444 (1989)
85. Roquette, P.: Analytic Theory of Elliptic Functions over Local Fields. Hamburger Mathematische Einzelschriften (N.F.), vol. 1. Vandenhoeck & Ruprecht, Göttingen (1970)
86. Schottky, F.: Über eine spezielle Funktion, welche bei einer bestimmten linearen Änderung ihres Arguments unverändert bleibt. J. Reine Angew. Math. **101**, 227–272 (1887)
87. Serre, J.-P.: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier **6**, 1–42 (1956)
88. Serre, J.-P.: Algebraic Groups and Class Fields. Springer GTM, vol. 117. Springer, Berlin/Heidelberg/New York (1975)
89. Serre, J.-P.: Algèbre locale et Multiplicités. Lecture Notes in Mathematics, vol. 11. Springer, Berlin/Heidelberg/New York (1967)
90. Serre, J.-P.: Trees. Springer, Berlin/Heidelberg/New York (1980)
91. Soublin, J.-P.: Anneaux et Modules Cohérents. J. Algebra **15**, 455–472 (1970)
92. Tate, J.: Rigid analytic spaces. Invent. Math. **12**, 257–289 (1971)
93. Tate, J.: A review of non-Archimedean elliptic functions. In: Elliptic Curves, Modular Forms & Fermat's Last Theorem. Series in Number Theory I, pp. 162–184. International Press, Somerville (1995)
94. Temkin, M.: On local properties of non-Archimedean analytic spaces. Math. Ann. **318**, 585–607 (2000)
95. Ullrich, P.: Analytic lifting of algebraic tori. Math. Ann. **273**, 287–296 (1986)
96. van der Put, M.: Stable reductions of algebraic curves. Proc. K. Ned. Akad. Wet., Ser. A, Indag. Math. **46**(4), 461–478 (1984)
97. Zhang, B.: Sur les jacobiennes des courbes à singularités ordinaires. Manuscr. Math. **92**, 1–12 (1997)

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