1. Tropical hypersurface via discrete Legendre transformation $\Rightarrow$ Balancing Condition

2. Marked tropical curve & their moduli

3. $N^{o,trop}_{\Delta,\Sigma} = N^{o,hol.}_{\Delta,\Sigma}$

4. Mumford degeneration induced by trop curve

$\boxed{\text{다쓰고 2,3,4는 자유다.}}$

$\mathbb{R}^{trop} = (\mathbb{R}, \oplus, \odot)$

$a \oplus b := \min(a,b)$ $\quad a \odot b := a+b$

$\mathbb{R}^{trop}[x_1, \cdots, x_k] := \{ f : \mathbb{R}^n \to \mathbb{R} \mid f$ is given by tropical polynomial $f(x_1,\cdots,x_k) = \sum_{(i_1,\cdots,i_k)\in S} a_{i_1\cdots i_k} x_1^{i_1} \cdots x_k^{i_k}$, $S \subseteq \mathbb{Z}^k$ is a finite subset $\}$

※ Index can be negative.

$\boxed{\text{Usual sense}}$ $f(x_1,\cdots,x_n) = \min \{ a_{i_1,\cdots,i_n} + \sum_{j=1}^{k} i_j x_j \mid (i_1,\cdots,i_k) \in S \}$

tropical hypersurface as a set.

$V(f) \subseteq \mathbb{R}^k \to$ the locus where f is not linear.

As weighted polyhedron cplex.

⊙ $M = \mathbb{Z}^k$, $M_{\mathbb{R}} \otimes_{\mathbb{Z}} \mathbb{R}$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

$\langle n, m \rangle := n(m)$

$m \in M \setminus \{0\}$.

⊙ index of m : largest positive integer r such that there exists $m' \in M$ $\quad m = r \cdot m'$

⊙ primitive : $\quad \mathrm{ind}(m) = 1$.

⊙ Polyhedron :

⊙ face

⊙ boundary

⊙ interior

⊙ lattice polyhedron.

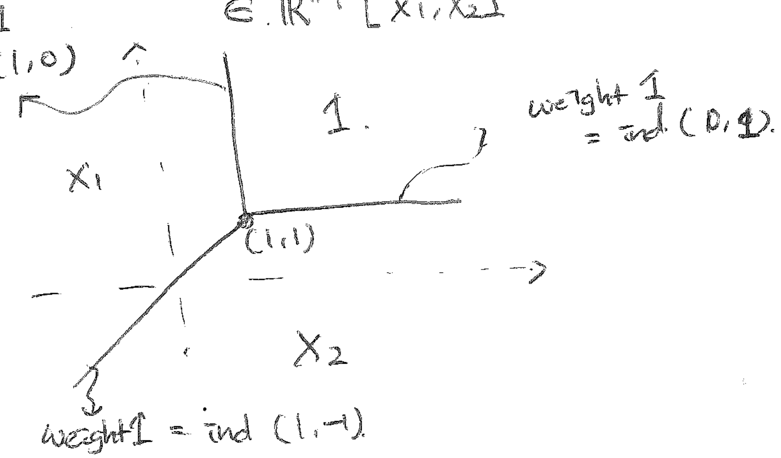
⊙ Polytope.

$\sigma \in V(f)$ $\quad$ codim 1. $\boxed{\text{(정의 말고 설명)}}$

(Ex) $f(x_1, x_2) = 1 \oplus (0 \odot x_1) \oplus (0 \odot x_2)$

$= (1 \odot x_1^0 x_2^0) \oplus (0 \odot x_1^1 x_2^0) \oplus (0 \odot x_1^0 x_2^1)$

$\in \mathbb{R}^{trop}[x_1, x_2]$



~~[illegible]~~ $\boxed{\text{simpler}}$

Characterizing trop hypersurface.

by discrete Legendre transformation.

$\boxed{\text{Sketch of general strategy}}$

$f = \sum_{n \in S} a_n z^n \overset{①}{\Rightarrow} (\Delta_S, \varphi, P)$ triple.

② discrete Legendre transformation

$\hookrightarrow (M_{\mathbb{R}}, \check{P}, \check{\varphi})$

※ $\check{P}^{[k-1]} \equiv V(f)$, $f \equiv \check{\varphi}$

$\boxed{\text{Step①}}$ $f \Rightarrow (\Delta_S, \varphi, P)$

a) $\Delta_S := \mathrm{conv}(S) \subseteq N_{\mathbb{R}}$

$\boxed{S = \text{set of exponents that appears in } f}$

유지할것

b) φ : $\tilde{S} = \{(n, a_n) \mid n \in S\} \subseteq N_{\mathbb{R}} \times \mathbb{R}$

define $\tilde{\Delta}_S = \{(n, a) \in N_{\mathbb{R}} \times \mathbb{R} \mid \exists (n, \alpha) \in \mathrm{conv}(\tilde{S})$ with $a \geq \alpha\}$

↳ upper convex hull of $\tilde{S}$

$\varphi(n) = \min\{a \in \mathbb{R} \mid (n, a) \in \tilde{\Delta}_S\}$

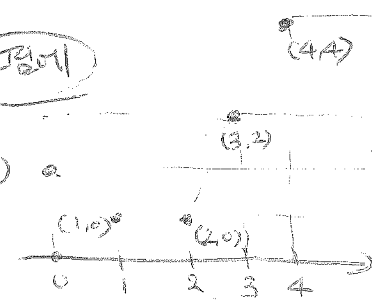
(c) P : the set of images under the projection $N_{\mathbb{R}} \times \mathbb{R} \to N_{\mathbb{R}}$ of proper faces of $\tilde{\Delta}_S$.

Ex $\underset{\mathbb{R}^{trop}[x]}{\overset{\cap}{f}} = 1 \oplus (0 \odot x) \oplus (0 \odot x^2) \oplus (2 \odot x^3) \oplus (4 \odot x^4)$

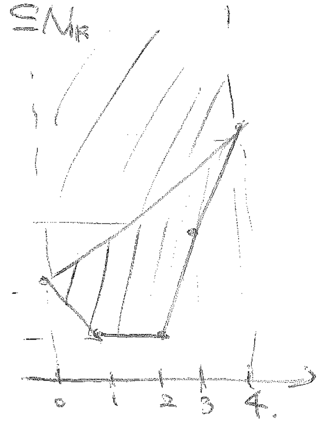
$1 \to (1 \odot x^0)$

a) $\Delta_3 = [0, 4] \subseteq N_{\mathbb{R}}$

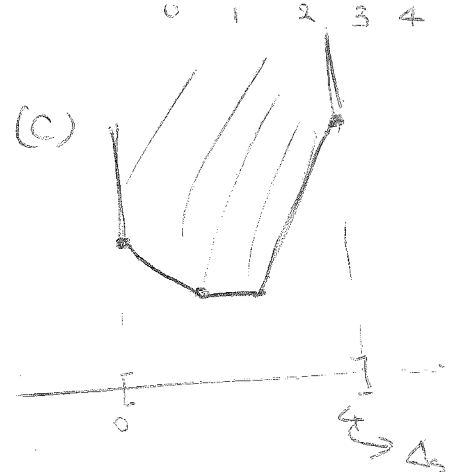
b) 한걸음에



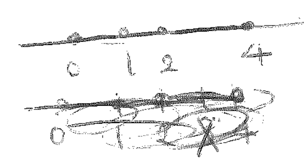
$\Rightarrow$



(c)



$\Rightarrow$



$P = \{\{0\}, \{1\}, \{2\}, \{4\},$
$[0, 1], [1, 2], [2, 4]\}$

Step ③ $(\Delta_S, \varphi, P) \xrightarrow{DLT} (\underbrace{M_{\mathbb{R}}}_{\hookrightarrow \text{clear}}, \check{P}, \check{\varphi})$.

① $\check{P} := \{ \check{\tau} \mid \tau \in P \}$.

$\boxed{\text{def}}$ $\check{\tau} = \{ m \in M_{\mathbb{R}} \mid \exists a \in \mathbb{R}$ such that $\langle -m, n \rangle + a \leq \varphi(n)$ for all $n \in \Delta_S$, with equality for $n \in \tau$. $\}$

$\boxed{\text{Meaning}}$ $\check{\tau}$ is the set of negative of → 나중에 써넣기. allowable slopes of the hyperplane.

① contacting the face of $\tilde{\Delta}_S$ above τ.

② lying below $\tilde{\Delta}_S$.

(2) $\check{\varphi} : M_{\mathbb{R}} \longrightarrow \mathbb{R}$.

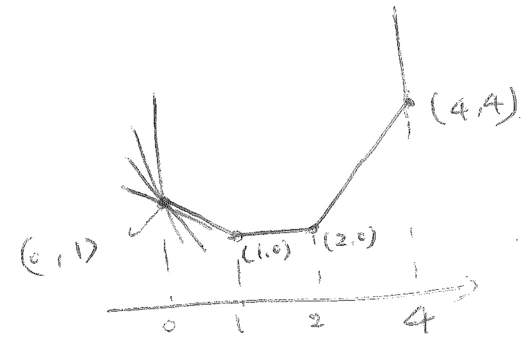
$\boxed{\text{Def}}$ $\check{\varphi}(m) = \max \{ a \mid \langle -m, n \rangle + a \leq \varphi(n) \ \ \forall n \in \Delta_S \}$

$\boxed{\text{Meaning}}$ $\check{\varphi}(m)$ is the highest y-intercept of the hyperplane

① with slope $-m$

② lying below $\tilde{\Delta}_S$.

(1) Ex • $\check{\{0\}}$



allowed slope m : $(-\infty, -1]$

$\overset{\Rightarrow}{\text{negate}}$ $[1, \infty) = \check{\{0\}} \subseteq M_{\mathbb{R}}$

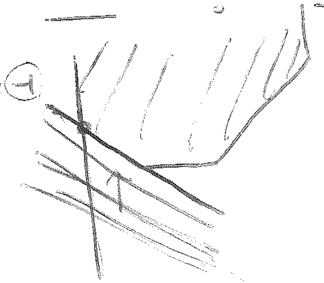
• $\check{[2,4]}$ ⇒ allowed slope m : $\{2\}$

⇒ , $\check{\{0\}} = [1, \infty)$, $\check{\{1\}} = [0, 1]$

$\check{\{2\}} = [-2, 0]$, $\check{\{4\}} = (-\infty, -2]$.

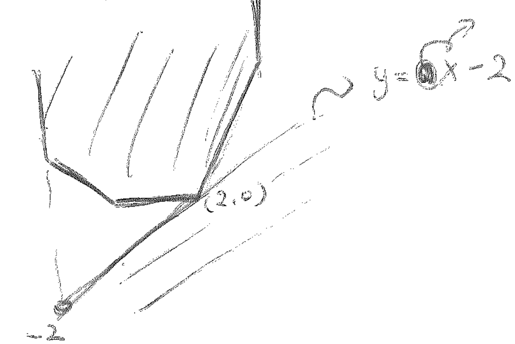
$\check{[0,1]} = \{1\}$, $\check{[1,2]} = \{0\}$, $\check{[2,4]} = \{-2\}$

(2). $\check{\varphi}(1)$ ⇒ slope -1



$\check{\varphi}(1) = 1$.

$\check{\varphi}(-1)$ ⇒ slope 1



$\check{\varphi}(-1) = -2$

Details | Facts

① $\tau \in P \Rightarrow \check{\tau} \in \check{P}$

$\dim \tau = \operatorname{codim} \check{\tau}$.

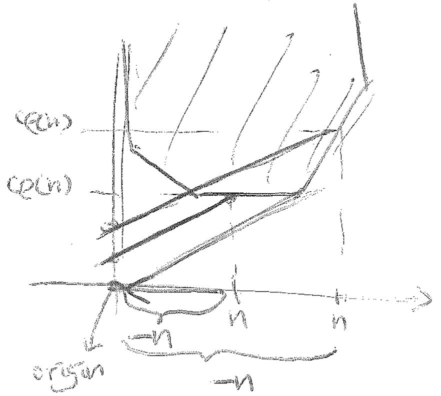
[illegible] believable!: less the dim of τ.
less the contact condition. ⇒ more degree of freedom on allowable slopes
restriction on

if $v \in P^{[0]} \Rightarrow \check{v} \in \check{P}^{[\dim M_{\mathbb{R}}]}$

② On $\check{v}$, $\check{\varphi}$ is ~~affine~~ linear with slope v

$\check{\varphi}(m) = \max\{a \mid \langle -m, n\rangle + a \leq \varphi(n) \ \forall n \in \Delta_S\}$
$= \max\{a \mid a \leq \langle m, n\rangle + \varphi(n) \ \forall n \in \Delta_S\}$
$= \min\{\varphi(n) + \langle m, n\rangle \mid n \in \Delta_S\}$.

$\langle -m, -n\rangle + \varphi(n)$



⇒ minimum is obtained when n is some vertex of P or equivalently, when $(n, \varphi(n))$ is a vertex of $\tilde{\Delta}_S$.

Moreover, if $m \in \check{N}$

$\check{\varphi}(m) = \min\{\varphi(n) + \langle m, n\rangle \mid n \in \Delta_S\}$

minimum is obtained when $n = v$.

So on $\check{N}$, $\check{\varphi}(m) = \langle v, m\rangle + \varphi(v)$

③ $\check{\varphi} = \sum_{v \in P^{[0]}} \varphi(v) z^v$

and $\check{\varphi} \equiv f$ ⟶ defines some $\mathbb{R}^k \to \mathbb{R}$

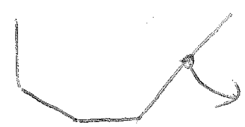
Ex $f = 1 \oplus (0 \odot x) \oplus (0 \odot x^2) \oplus (2 \odot x^3) \oplus (4 \odot x^4)$

$\check{\varphi} = \sum_{v \in P^{[0]}} \varphi(v) z^v = 1 \oplus (0 \odot x) \oplus (0 \odot x^2) \oplus (4 \odot x^4)$

※ $2 \odot x^3$ term in f ~~is~~ redundant.

※ $m \in M_{\mathbb{R}}$, hyperplane with slope $-m$ below $\tilde{\Delta}_S$ touches $(n, \varphi(n))$ where $n \in S$
means the value of $f(m)$ is given by the monomial corresponding to n.

Ex



⟶ redundant because whenever line touches this point the line also touches other two points at the end.

$\therefore f \equiv \check{\varphi}$.