Around the GS locus.

We want to know that the formal family this process builds genuinely smooths the singularities, whenever it lifts to a geometric smoothing. The following is a criterion for this

Defn: $\mathfrak{X} \xrightarrow{f} \mathfrak{S}$ a map of formal schemes has smooth generic fibre if it is adically flat, pure relative dim d and for $\mathfrak{Z} \subset \mathfrak{X}$ the sing locus the map $\mathcal{O}_{\mathfrak{S}} \to f_* \mathcal{O}_{\mathfrak{Z}}$ is not injective.

Examples are formal completions at a closed fibre of generically smooth maps. It is stable under surjective pullback.

In particular we are interested in neighbourhoods not just of the origin in $\operatorname{Spec} k[P]$, but of whole strata, this motivates the definition of the GS locus, an analogue of the Mumford family as follows.

The Mumford degeneration has a choice of polarisation made at the start, but we could write it in a coordinate free manner and then pair the construction with a polarisation (which is why our φ is P valued not $\mathbb{Z}$ valued). We're going to choose such a polarisation "almost" to reduce to a toric model.

So recall toric models.

Toric blow up: Blow up of (Y, D) at nodes of D to get $(\tilde{Y}, \tilde{D})$

Toric model: $(\tilde{Y}, \tilde{D}) \to (\bar{Y}, \bar{D})$ with $(\bar{Y}, \bar{D})$ toric and $\tilde{D} \to \bar{D}$ an iso.

And of course theorem: up to performing toric blow ups toric models exist. Toric blow ups don't affect scattering diagrams (they add in cones but not walls)

So we can assume $(Y, D) \xrightarrow{\rho} (\bar{Y}, \bar{D})$ is a toric model and we want to choose H ample on $\bar{Y}$ (so it contracts curves contracted by the map). Up to changing P we can assume there is a face σ whose intersection with $NE(Y) \subset P$ is $(\rho^* H)^\perp \cap NE(Y)$. Write G for the complement, a prime monomial ideal of $k[P]$.

The quotient $\operatorname{Spec} k[P/G]$ is a closed subscheme of $\operatorname{Spec} k[P]$ and the GS locus (wrt H) is $T^{GS} \subset \operatorname{Spec} k[P/G]$ the open torus orbit, $\operatorname{Spec} k[\operatorname{Span} \sigma]$

[Aside: The consistency of the scattering diagram for $(\bar{Y}, \bar{D})$ is used to show consistency for the one coming from (Y, D).

There is a PL function $\nu : \underset{B}{\underset{\shortparallel}{\Sigma(Y, D)}} \to \underset{\bar{B}}{\underset{\shortparallel}{\Sigma(\bar{Y}, \bar{D})}}$ and they use this to work with a diagram of $\bar{B}$.]

So they now prove smoothness using explicit charts

For all I with $\sqrt{I} = G$ we have

$$\begin{tikzcd} T^{GS} \times \mathbb{V}^n \arrow[r, phantom, "\subset"] \arrow[d, shift right] & X_I \arrow[d] \\ T^{GS} \arrow[r, phantom, "\subset"] \arrow[u, bend right, "0"'] & \operatorname{Spec} k[P/I] \end{tikzcd} \qquad \begin{tikzcd} T^{GS} \times \mathbb{V}^n \arrow[r, phantom, "\subset"] \arrow[d, shift right] & \bar{X}_I \arrow[d] \\ T^{GS} \arrow[r, phantom, "\subset"] \arrow[u, bend right, "0"'] & \operatorname{Spec} k[P/I] \end{tikzcd}$$

Trivial over T^{GS}

Mumford family associated to PL for $K_i = \rho^*[\bar{D}_i]$

Goal: There is an iso $U_I \xrightarrow{\sim} \bar{U}_I$ with

$U_I \supset \mathbb{V}_n \times \{0\} \subset \bar{U}_I$ and there is a nonzero

$g \in k[P]$ whose pullback to X_I lies in the stalk

at any point $x \in \{0\} \times T^{\sigma_s} \subset \mathbb{V}_n \times T^{\sigma_s}$ of $\mathrm{Sing}_{\mathfrak{X}_I}$ for

all I

Unpacking this: We know that the Mumford family

is smooth and using [GS07] the deformation theory of

U_I should be formally unique, so we can argue.

What is the set? On $\bar{B}$ we have a scattering diagram

$\bar{\mathfrak{D}} = \nu(\mathfrak{D})$ and an associated scheme

$$X_I \cong \mathrm{Spec}\, \Gamma(\bar{X}^{\circ}_{I,\bar{\mathfrak{D}}}) =: \bar{X}_{I,\bar{\mathfrak{D}}}$$

and we have the trivial scattering diagram on $\bar{B}$.

set $I_0 = \{(m,p) \in P_{\bar{\varphi}} \mid p - \bar{\varphi}_\sigma(m) \in I$ for some $\sigma \in \Sigma_{max}\}$

They say $\sqrt{I_0} = \mathfrak{m}_{P_\varphi}$, since $\sqrt{I} = G$ and set

$$h = \prod_{\mathfrak{d} \in \bar{\mathfrak{D}}_I} f_{\mathfrak{d}}$$

where $\bar{\mathfrak{D}}_I$ is only the walls not $\equiv 1$ mod I_0, and we only

walls we throw away higher order terms.

U and $\bar{U}$ are $D(h)$, and they contain the

desired sets since each one is $\equiv 1$ mod $\mathfrak{m}_p$

The claim is that U and $\bar{U}$ are isomorphic, they

are glued from rings

$$S_{\tau, I, \mathfrak{D}} := (\bar{R}_{\tau, I, \mathfrak{D}})_h$$

use nilpotency to identify the topological spaces.

$$S_{p_i, I, \bar{\mathfrak{D}}} \xrightarrow{\chi_\tau} S_{p, I, \emptyset}$$

invertible

$$\bar{X}_{i-1} \mapsto \bar{X}_{i-1}\left(g_{p_i} \prod \left(1 + b_{\mathfrak{d}}^{-1} X_i\right)\right),$$

$$\bar{X}_i \mapsto \bar{X}_i \qquad \bar{X}_{i+1} \mapsto \bar{X}_{i+1}$$

Also note that $f_{\mathfrak{d}}$ is now invertible, so we can form a

$$\begin{tikzcd}
S_{p_i, I, \bar{\mathfrak{D}}} \arrow[r, "\Theta_{p_i,\tau,q} \circ \chi_i"] \arrow[d, "\Theta_{p_i, p_{i+1}} \circ \chi_{p_i}"'] & S_{p_i, I, \emptyset} \arrow[d, "\chi_{p_i}"] \\
S_{\sigma_{i,i+1}, I, \bar{\mathfrak{D}}} \arrow[r, "\Theta_{p_{i,i+1},\tau,q}"] & S_{\sigma_{i,i+1}, I, \emptyset} \\
S_{p_{i+1}, I, \bar{\mathfrak{D}}} \arrow[r, "\Theta_{p_{i+1},\tau,q} \circ \chi_{i+1}"'] \arrow[u, "\chi_{p_{i+1}}"] & S_{p_{i+1}, I, \emptyset} \arrow[u, "\chi_{p_{i+1}}"']
\end{tikzcd}$$

These give isos $U \cong \bar{U}$ away from $\{0\} \times T^{\sigma_s}$, but

this is codim 2, so extends. Finally:

Theorem: The formal family has smooth generic fibre.

Pf: We have a section $s : S_G \to X_G = S_G \times \mathbb{V}_n$

There is $g \neq 0$ with $\mathrm{Supp}(g \cdot \mathcal{O}_{\mathfrak{Z}_G}) \subset s(S_G)$

They then show that (after multiplying through)

$\mathrm{Supp}(gh \cdot \mathcal{O}_{\mathfrak{Z}_G}) \subset s(S_G \setminus (S_G \cap U))$

So $\mathcal{O}_{\mathfrak{Z}_G}$ is torsion and the map $k[P] \to \Gamma(\mathfrak{Z}_G, \mathcal{O}_{\mathfrak{Z}_G})$

is not injective, $k[P] \subset \Gamma(\mathfrak{Z}_m, \mathcal{O}_{\mathfrak{Z}_m}) \to f_{m*}\mathcal{O}_{\mathfrak{Z}_m}$

is not injective. ↯