Looijenga pairs

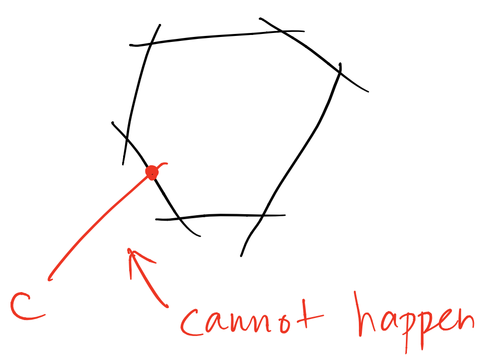
Def: A Looijenga pair (Y, D) is a smooth rat'l projective surface Y together with a reduced nodal curve $D \subset |-K_Y|$ w/ at least one singular point.

- $p_a(D) = 1$
- $h^0(\mathcal{O}_D) = h^1(\mathcal{O}_D)$

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$$

$$\begin{array}{ccccccc}
\cdots \to & H^1(\mathcal{O}_Y) & \to & H^1(\mathcal{O}_D) & \to & H^2(\mathcal{O}(-D)) & \to \cdots \\
& \| & & \downarrow & & \cong & \\
& 0 & & \leq 1 \text{ dimensional} & & H^0(\mathcal{O})^\vee \to 1 \text{ dimensional} &
\end{array}$$

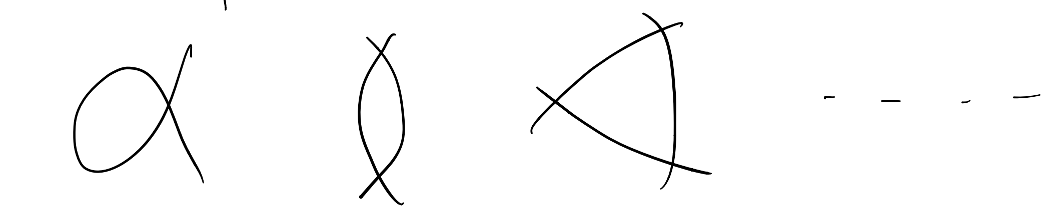
$\Rightarrow D$ is connected



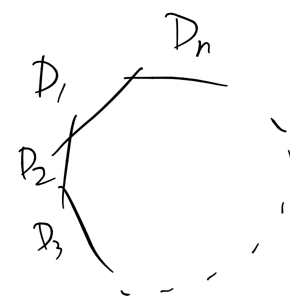
Adjunction: $\omega_C = \mathcal{O}(-2)$

$$C.(D-C) = 2$$

$\Rightarrow$ the possible configurations are:



Write $D = D_1 + \cdots + D_n$



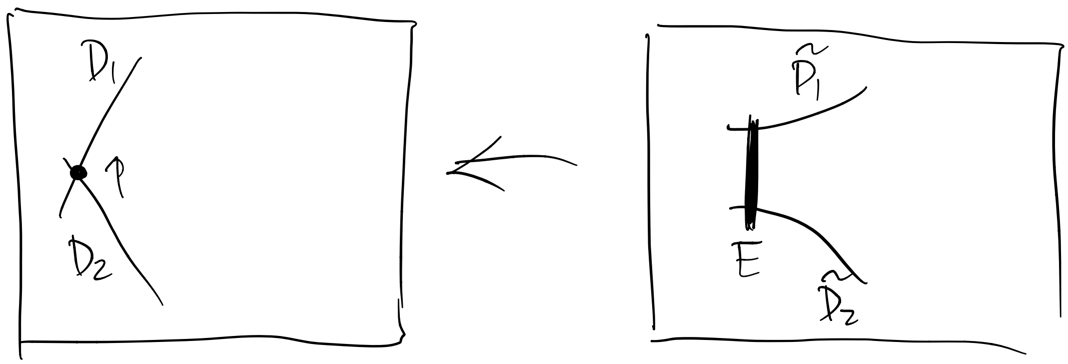
Def: Let (Y, D) be a Looijenga pair

(1) A toric blowup of (Y, D) is a bir. morphism $\pi : \tilde{Y} \to Y$ s.t. $\tilde{D}$ is the reduced scheme of $\pi^{-1}(D)$ then $(\tilde{Y}, \tilde{D})$ is a Looijenga pair.

(2) A toric model of (Y, D) is a bir. mor. $(Y, D) \to (\bar{Y}, \bar{D})$ to a sm toric surf. $\bar{Y}$ w/ its toric boundary $\bar{D}$ s.t. $D \to \bar{D}$ is an isomorphism.

Rmk:

$D = D_1 + D_2$ ($D_1 \cup D_2$ are normal crsing)



$$\tilde{D}_1 + \tilde{D}_2 + E = (\pi^* D_1 - E) + (\pi^* D_2 - E) + E$$
$$= \pi^* D_1 + \pi^* D_2 - E$$
$$= -\pi^* K_Y - E = -K_{\tilde{Y}}$$

$\Rightarrow (\tilde{Y}, \tilde{D}_1 + \tilde{D}_2 + E)$ is a Looijenga pair

but if we blow up points other than nodes, $(\tilde{Y}, \tilde{D})$ will not be a Looijenga pair.

<u>Prop</u>: Given (Y, D) there exists a toric blowup $(\tilde{Y}, \tilde{D})$ which has a toric model

$$(\tilde{Y}, \tilde{D}) \longrightarrow (\bar{Y}, \bar{D})$$

pf:

- $Y \xrightarrow{p} Y'$ blow down of a (-1)-curve not contained in D

Prop for $(Y', p_* D) \Rightarrow$ Prop for (Y, D)

- $Y'' \longrightarrow Y$ blow up at a node of D

Prop for $(Y'', D'') \Leftrightarrow$ Prop for (Y, D)

reduce to $Y \cong \mathbb{F}_e$ or $\mathbb{P}^2$

Tropical Looijenga pair

$M \cong \mathbb{Z}^n$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$,

$M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

$\mathrm{Aff}(M)$ = the group of linear transformation of the lattice M

Def: An integral affine mfd B is a real mfd with an atlas of charts $\{\psi_i : U_i \to M_{\mathbb{R}}\}$ s.t. $\psi_i \circ \psi_j^{-1} \in \mathrm{Aff}(M)$ for all i, j

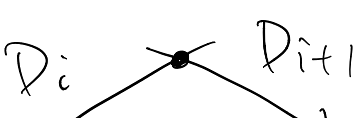
An integral affine mfd w/ singularities B is a mfd w/ open subset $B_0 \subset B$ which carries the str. of integral affine mfd, and $\Delta := B \setminus B_0$ the singular locus with $\Delta := B \setminus B_0$ the singular locus of B is locally finite union of locally closed submfd of cod. at least two.

($\dim B = 2$, Δ are pts)

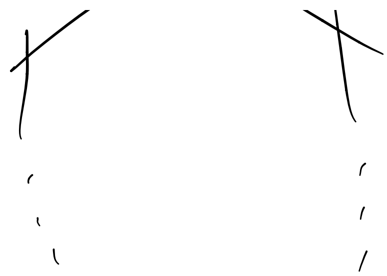
- We associate a Looijenga pair (Y, D) with a pair (B, Σ) where $B \cong \mathbb{R}^2$, integral aff. mfd w/ one sing. at 0, Σ is a decomp. of B into cones

(idea: to pretend (Y, D) is toric)

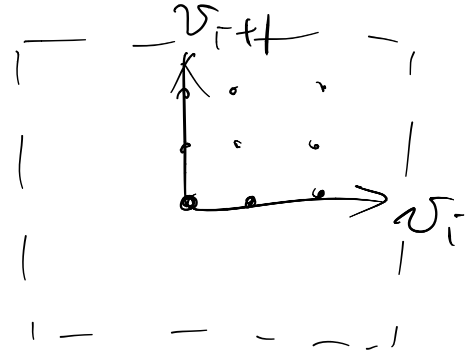
a node



rank two lattice



$\rightsquigarrow M_{i,i+1}$, with basis v_i, v_{i+1}
cone $\sigma_{i,i+1} \subset M_{i,i+1} \otimes_{\mathbb{Z}} \mathbb{R}$
generated by v_i, v_{i+1}



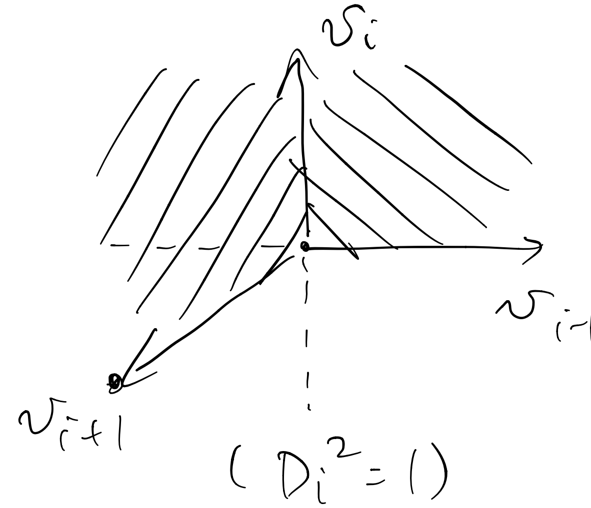
Glue $\sigma_{i,i+1}$ to $\sigma_{i-1,i}$ along the rays $\rho_i := \mathbb{R}_{\geq 0} v_i$ to obtain B

$$\Sigma = \{\sigma_{i,i+1}\} \cup \{\rho_i\} \cup \{0\}$$

B is int. aff. mfd. given by charts

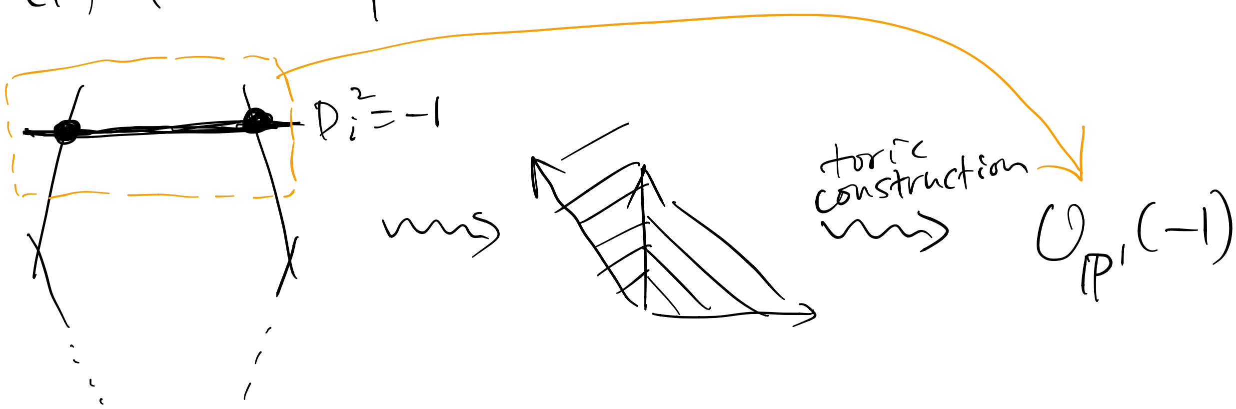
$$\begin{aligned} \psi_i : \mathrm{Int}(\sigma_{i-1,i} \cup \sigma_{i,i+1}) &\longrightarrow M_{\mathbb{R}} \\ v_{i-1} &\longmapsto (1,0) \\ v_i &\longmapsto (0,1) \end{aligned}$$

$v_{i+1} \longmapsto (-1, -\underline{D_i^2})$

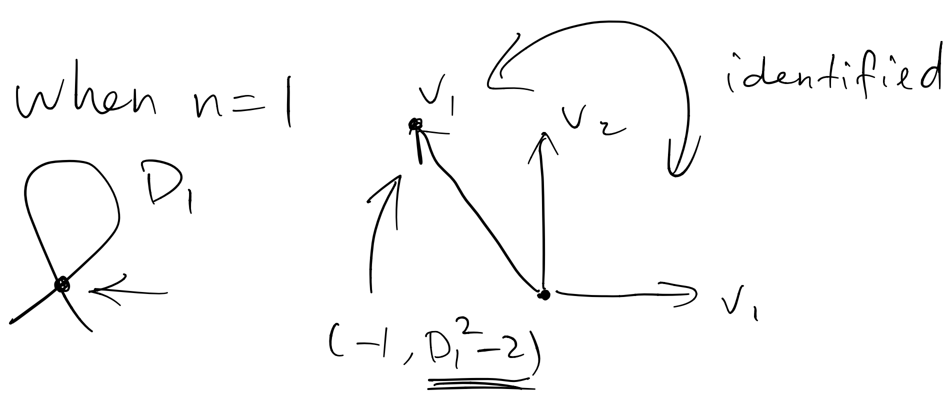


Rmks:

(1) for example



(2) when $n = 1$



(3) $v_{i-1} + (D_i^2) v_i + v_{i+1} = 0$

Def: (B, Σ) a refinement is a pair $(B, \tilde{\Sigma})$ where $\tilde{\Sigma}$ is a decomp. of B refining Σ. each cone of $\tilde{\Sigma}$ is int. aff isom. to the first quadrant of $\mathbb{R}^2$

Lemma: There is 1-1 correspondence

$$\{\text{toric blowups of } (Y, D)\} \Longleftrightarrow \{\text{refinements of } (B, \Sigma)\}$$

Furthermore

$$\begin{array}{ccc} (\tilde{Y}, \tilde{D}) & \longrightarrow & (Y, D) \quad \text{toric blowup} \\ \wr\downarrow & & \wr\downarrow \\ (\tilde{B}, \tilde{\Sigma}) & & (B, \Sigma) \end{array}$$

$\Rightarrow$ $\tilde{B} \cong B$ as int. aff mfd

$\tilde{\Sigma}$ is a refinement of Σ.

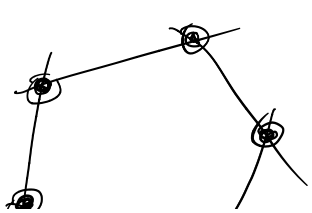
omit the proof.

(Y, D) if Y is a nonsing. toric, D = toric boundary, then the aff. structure extends across the origin.

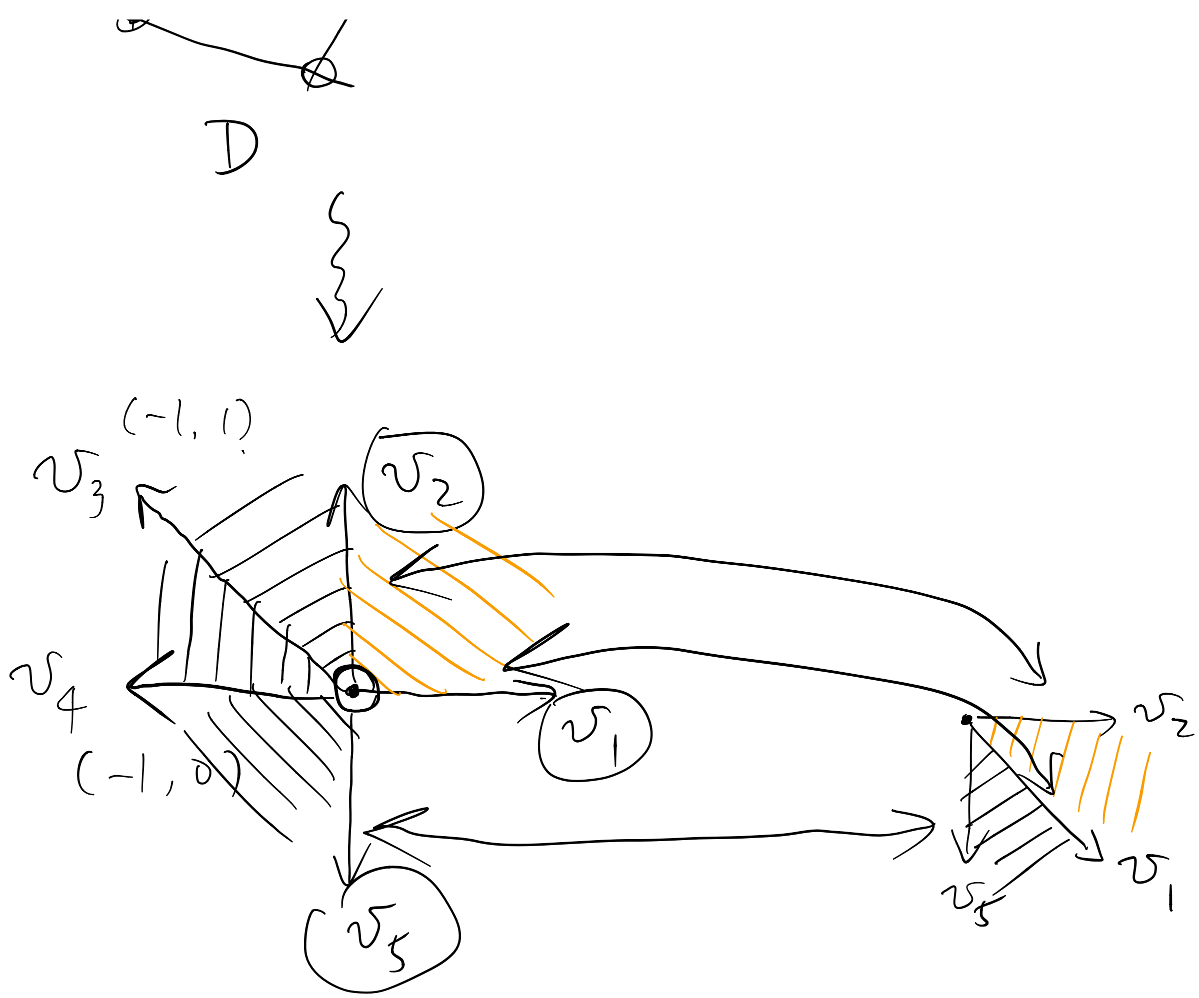
<u>Lemma</u>: Converse is true.

omit the proof.

<u>Example</u>: ① $Y = DP5$ blowup of $\mathbb{P}^2$ at 4 pts



(-1)-curve



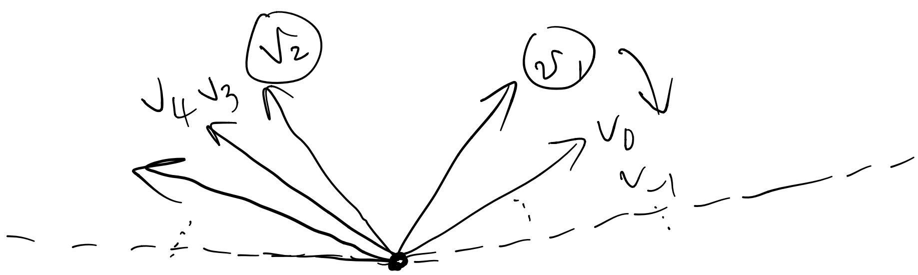
② (Y, D), all $D_i^2 \le -2$, D negative definite
($\Leftrightarrow \exists D_i^2 \le -3$)

$Y \xrightarrow{\text{contract}} \overline{Y}$ with cusp singularity

Construct infinite fan

$i \in \mathbb{Z}$

$$(*) \quad v_{i-1} + (D^2_{i \bmod n}) v_i + v_{i+1} = 0$$



Check: support is a convex cone

$T \in SL(M)$, $T(v_0) = v_n$, $T(v_1) = v_{n+1}$

$(*)$ is preserved under $SL(M)$ action $\Longrightarrow$ $T(v_i) = v_{i+n}$

$\Rightarrow$ boundary rays are eigenvectors of T, $\operatorname{Tr} T > 2$, ...

Mumford degeneration and Givental's construction toric monoid P: commutative monoid s.t. P^{gp} is fin. gen free ab. gp. and $P = P^{gp} \cap \sigma_P$

$\sigma_P \subset P^{gp} \otimes_{\mathbb{Z}} \mathbb{R}$ is a convex rat'l polyh. cone

$M = \mathbb{Z}^n$ $\qquad B = |\Sigma|$ affine mfd,

Σ_{max} set of maximal cones

piecewise linear function $|\Sigma| \longrightarrow \mathbb{R}$

generalize to $|\Sigma| \longrightarrow P^{gp}_{\mathbb{R}}$

<u>Def</u>: A Σ-piecewise linear fcn $\varphi : |\Sigma| \longrightarrow P^{gp}_{\mathbb{R}}$ is a continuous fcn. s.t. for each $\sigma \in \Sigma_{max}$, $\varphi|_\sigma$ is given by an elmt

$\varphi_\sigma \in \mathrm{Hom}_{\mathbb{Z}}(M, P^{gp}) = N \otimes_{\mathbb{Z}} P^{gp}$

for each cod. 1 cone $\rho \in \Sigma$

ρ contained in $\sigma_+, \sigma_- \in \Sigma_{max}$

we can write

$$\varphi_{\sigma_+} - \varphi_{\sigma_-} = n_\rho \otimes \kappa_{\rho,\varphi}$$

$n_\rho \in N$ is the unique primitive elmt annihilating ρ and positive on σ_+. $\underbrace{\kappa_{\rho,\varphi}}_{\text{bending parameters}} \in P^{gp}$

$\varphi : |\Sigma| \longrightarrow P^{gp}$ is P-convex (strict P-convex)

if for every codim one cone

$\rho \in \Sigma$, $\kappa_{\rho,\varphi} \in P$

$(\kappa_{\rho,\varphi} \in P \setminus P^\times$ where $P^\times$ is the gp of invertible elmts of P).

<u>Example</u> complete fan Σ in $M_{\mathbb{R}}$

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