Mumford degeneration and Givental's construction

toric monoid P: (commutative) monoid s.t. P^{gp} is fin. gen. free abelian gp. and $P = P^{gp} \cap \sigma_P$, $\sigma_P \subset P^{gp} \otimes_{\mathbb{Z}} \mathbb{R}$ is convex rat'l polyhedral cone.

$M = \mathbb{Z}^n$, $B = |\underline{\underline{\Sigma}}|$ affine mfd

$\Sigma \longrightarrow$ convex support

Σ_{max} set of maximal cones

piecewise linear fcn $|\Sigma| \longrightarrow \mathbb{R}$

generalize to $|\Sigma| \longrightarrow \underline{\underline{P^{gp}_{\mathbb{R}}}}$

Def: A Σ-piecewise linear fcn. $\varphi : |\Sigma| \longrightarrow P^{gp}_{\mathbb{R}}$ is a continuous fcn. s.t. $\forall\ \sigma \in \Sigma_{max}$ $\varphi|_\sigma$ is given by an elmt

$\varphi_\sigma \in \mathrm{Hom}_{\mathbb{Z}}(M, P^{gp}) = \underline{N \otimes_{\mathbb{Z}} P^{gp}}$

for each cod. 1 cone $\rho \in \Sigma$ contained $\sigma_+, \sigma_- \in \Sigma_{max}$, we can write

$$\varphi_{\sigma_+} - \varphi_{\sigma_-} = n_\rho \otimes \kappa_{\rho,\varphi}$$

n_ρ is the unique element annihilating ρ and positive on σ_+, $\underline{\kappa_{\rho,\varphi} \in P^{gp}}$

bending parameter

$\varphi : |\Sigma| \longrightarrow P^{gp}$ is P-convex (strictly P-conv.) if $\forall$ cod. 1 cone $\rho \in \Sigma$, $\kappa_{\rho,\varphi} \in P \subset P^{gp}$ ($\kappa_{\rho,\varphi} \in P \setminus P^\times$ where $P^\times$ is the gp of invertible elmts. of P)

<u>Example</u>: Complete fan Σ in $M_{\mathbb{R}}$

$\Rightarrow$ toric variety $Y := Y_\Sigma$

(assume Y nonsing.)

Let $P \subset P^{gp}$ be the cone of eff. curves

$$NE(Y) \subset A_1(Y, \mathbb{Z})$$

- cod. 1 cone $\rho \in \Sigma \rightsquigarrow [D_\rho] \in NE(Y) = P$
- dim 1 cone $\omega \in \Sigma(1) \rightsquigarrow$ divisor D_ω

<u>Lem</u>: We have a short exact seq.

$$0 \longrightarrow A_1(Y,\mathbb{Z}) \longrightarrow T_\Sigma := \mathbb{Z}^{\Sigma(1)} \longrightarrow M \longrightarrow 0$$

$\underline{\underline{t_\omega}}$ $(\omega \in \Sigma(1)) \longmapsto$ the first lattice pt. on ω

generator

$$\beta \longmapsto \sum_{\omega \in \Sigma(1)} (D_\omega \cdot \beta)\, t_\omega$$

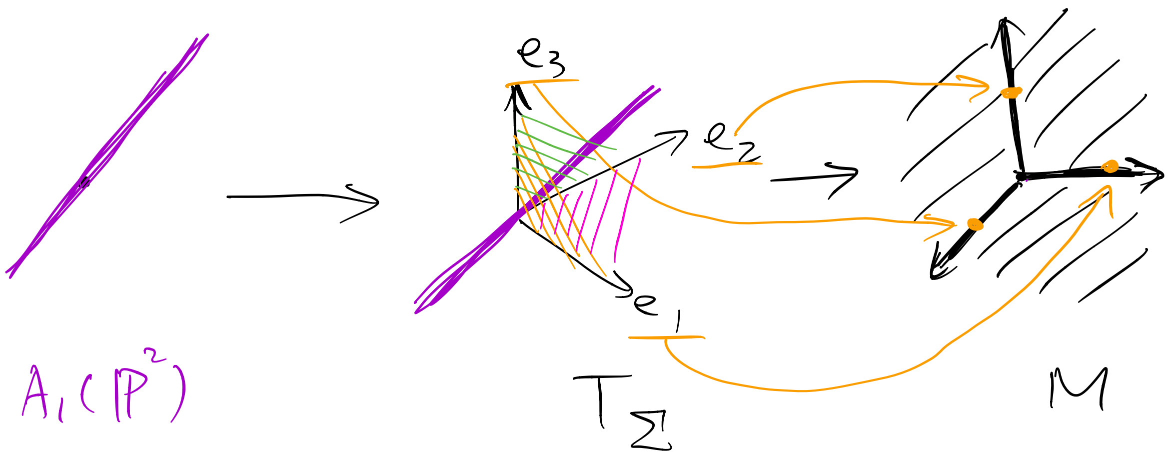
There is a unique Σ-piecewise linear section $\tilde{\varphi} : M \longrightarrow T_\Sigma$ satisfying $\tilde{\varphi}(\underline{m_\omega}) = t_\omega$

the first lattice pt on ω

Let $\pi : T_\Sigma \longrightarrow A_1(Y, \mathbb{Z})$ be any splitting, and let $\varphi := \pi \circ \tilde{\varphi}$, then $M \to A_1(Y,\mathbb{Z})$ is Σ-piecewise linear and strictly P-conv. with $\kappa_{\rho,\varphi} = [D_\rho]$ for each codim 1 cone ρ.

It is unique up to a linear fcn.

e.g. $\mathbb{P}^2$



Given a Σ-piecewise linear and P-convex fcn. $\varphi : |\Sigma| \longrightarrow P^{gp}$ we can define a monoid $P_\varphi \subset M \times P^{gp}$

$$\underline{P_\varphi} := \{ (m, \varphi(m) + p) \mid m \in |\Sigma|, \ p \in P \}$$

the set of integral pts lying "above" the graph of φ.

$$\begin{aligned} P &\hookrightarrow P_\varphi \\ p &\longmapsto (0, p) \end{aligned}$$

$\rightsquigarrow \; f : \operatorname{Spec} k[P_\varphi] \longrightarrow \operatorname{Spec} k[P]$

- Describe the fibers.

- $f^{-1}(\operatorname{Spec} k[P^{gp}]) = \underbrace{\operatorname{Spec} k[M]}_{\text{torus}} \times \operatorname{Spec} k[P^{gp}]$

$$\left(\begin{tikzcd} M \times P^{gp} & P^{gp} \arrow[l, hook'] \\ P_\varphi \arrow[u] & P \arrow[l] \arrow[u] \end{tikzcd} \qquad P_\varphi \otimes_P P^{gp} = M \times P^{gp} \right)$$

- Describe fibers over toric strata

$Q < P$ a face $\rightsquigarrow$ toric strata $\ni x$

replacing P by $P - Q$ (inverting elmts in Q), we can assume x is the smallest toric stratum

Consider $\bar{\varphi} : |\Sigma| \xrightarrow{\varphi} P^{gp} \longrightarrow P^{gp}/P^{\times}$

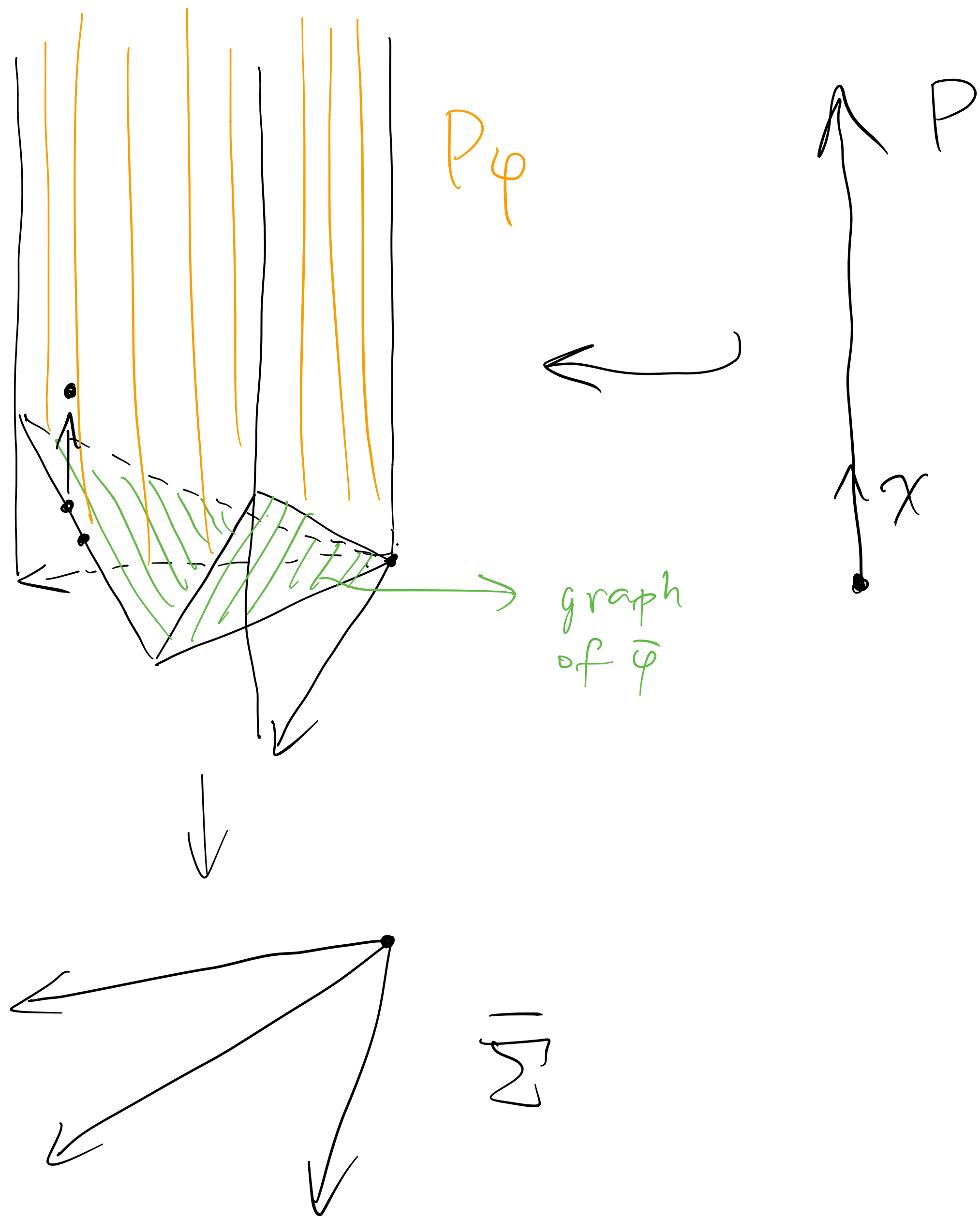
$\bar{\varphi}$ is also piecewise linear.

Let $\bar{\Sigma}$ be the fan whose maximal cones are the maximal domains of linearity of $\bar{\varphi}$, then $f^{-1}(x) = \operatorname{spec} k[\bar{\Sigma}]$ where

$$k[\bar{\Sigma}] = \bigoplus_{m \in M \cap |\Sigma|} k \cdot z^m$$

$$z^m \cdot z^{m'} = \begin{cases} z^{m+m'} & \text{if } m, m' \text{ lie in the same cone of } \bar{\Sigma} \\ 0 & \text{otherwise} \end{cases}$$

Irreducible components of $f^{-1}(x)$ are $\operatorname{spec} k[\sigma \cap M]$ for $\sigma \in \bar{\Sigma}_{max}$



P_φ
P
χ
graph of $\bar{\varphi}$
$\bar{\Sigma}$

rank $M = 2$, toric boundary has n components

φ strictly convex, if x is a point of the smallest toric stratum of $\operatorname{Spec} k[P]$

then $f^{-1}(x) = \mathbb{V}_n \subset \mathbb{A}^n$ where

$$\underbrace{\mathbb{V}_n}_{\text{the } n\text{-vertex}} = \mathbb{A}^2_{x_1,x_2} \cup \mathbb{A}^2_{x_2,x_3} \cup \cdots \cup \mathbb{A}^2_{x_n,x_1} \subset \mathbb{A}^n_{\underbrace{x_1,\dots,x_n}_{\text{coordinates}}}$$

when $n=2$

$$\mathbb{V}_2 = \operatorname{Spec} k[x_1, x_2, y]/(y^2 - x_1^2 x_2^2) = \mathbb{A}^2_{x_1 x_2} \cup \mathbb{A}^2_{x_2, x_1}$$

when $n=1$

$$\mathbb{V} = \operatorname{Spec} k[x, y, z]/(xyz - x^2 - z^3)$$