Tropical disks and LG models

Classical mirror symmetry à la Givental

Let X_Σ be a smooth Fano toric variety corresponding to a fan $\Sigma \subset N$. There is an exact sequence

(*) $\quad 0 \to K_\Sigma \to T_\Sigma \to N \to 0$

so after dualising and taking $\otimes \mathbb{C}^*$ one has

$$0 \to M \otimes \mathbb{C}^* \to \mathrm{Hom}(T_\Sigma, \mathbb{C}^*) \xrightarrow{K} \mathrm{Pic}\, X_\Sigma \otimes \mathbb{C}^* \to 0$$

$\mathrm{Pic}\, X_\Sigma \otimes \mathbb{C}^*$ is then the dual of the Mori Cone and one can pass to the universal cover

$$\tilde{\mathcal{M}}_\Sigma = \mathrm{Pic}\, X_\Sigma \otimes \mathbb{C} \quad \text{"Kahler Moduli"}$$

and pull back:

$$\tilde{X}_\Sigma = \mathrm{Hom}(T_\Sigma, \mathbb{C}^*) \times_{\mathrm{Pic} X_\Sigma \otimes \mathbb{C}^*} \tilde{\mathcal{M}}_\Sigma$$

with projection induced by K, W_0

Eg for $\mathbb{P}^2$:

(*) $\quad 0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \xrightarrow{(1,1,1)} \mathbb{Z} \to 0$

and one has the family $(\mathbb{C}^*)^3 \to \mathbb{C} \quad t \to xyz$

$W_0 = x + y + z = x + y + \frac{e^t}{xy}$

Various statements about the Jacobian ring etc. Main statement for us:

Oscillatory integrals on $\check{X}_t$ $\int e^{-W_0}$

$||$

Gromov Witten counts on X (with descendants)

Cho-Oh: This is a first order approximation. The superpotential should count weighted Maslov index 2 disks with boundary on an SYZ fibre

Key word "universal family".

We reinterpret this statement tropically

Defn: (Tropical disk) Ryan defined a stable tropical curve, and most of the rotation etc lines up. Γ a weighted connected finite graph with no bivalent vertices. Choose a vertex V_{out} and edge E_{out} such that E_{out} is the unique edge adjacent to V_{out} and $\Gamma' = (\bar{\Gamma} \setminus \Gamma_\infty^{[0]}) \cup \{V_{out}\}$ is genus zero.

A parametrised d-pointed stable map tropical disk is:

$$\begin{tikzcd} \Gamma'_\infty \setminus (E_{out}) \arrow[r, hook] & \Gamma \arrow[r, "h"] & N_{\mathbb{R}} \\ \bullet \arrow[u, bend left, "p_1"] \arrow[u, bend right, "p_d"'] & & \end{tikzcd}$$

With the stable map conditions Ryan mentioned except no balancing at V_{out}.

If we restrict to the case where all the outgoing edges are parallel to some p_i case we can define degree.

We define multiplicity by ignoring the contribution from V_{out}.

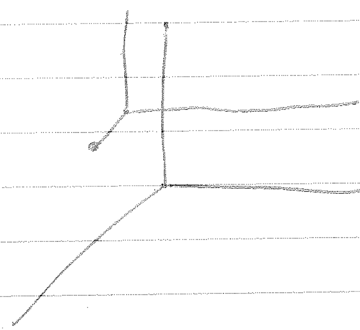
This is the definition of a tropical disk.

eg for $\mathbb{P}^2$

(+)

one for each outgoing ray (W_i)

but also



also valid.

We need to tropicalise Maslov index and form a moduli space.

$$MI(h) := 2(|\Delta(h)| - d)$$

so the above examples are index 2 and 4

Choose $P_1 \dots P_k$ in general position and $1 \le i_1 \le \dots i_d \le k$ a labelling and Γ' a type of the disk.

Moduli of tropical disks with $p_j \mapsto P_{i_j}$ is

dim $\frac{1}{2} MI(h) - 1$

Specifying $V_{out} \to Q$ the dim is the

$\frac{1}{2}\mathrm{MI}(Ch) - 1$

ie index 2 disks $\Rightarrow$ finite count.

Pf: Same as Ryan's

Define $R_k = \mathbb{C}[u_1 \ldots u_k] / \langle u_1^2 \ldots u_k^2 \rangle$

a handy coefficient ring. Working to first order like this solves some combinatorial issues (cte).

Given this h define $\mathrm{Mono}(h) = \mathrm{Mult}(h)\, z^{\partial(h)}\, u_{I(h)}$

$\prod$ over endpoints w_i

and set $W_k(Q) = y_0 + \sum_h \mathrm{Mono}(h) \in \mathbb{C}[T_s] \otimes R_k[[y_0]]$

[Why $[[y_0]]$? The universal unfolding of this potential can have critical values coming in from infinity. for us this is not an issue so long as we stay infinitely close to the central fibre, ie we take formal power series.]

$W_0(Q)$

index $2 \Rightarrow |\Delta(h)| = d+1$, so no P_i's

so we can only have the curves in (†)

$W_0(Q) = y_0 + \sum z^{t_p}$

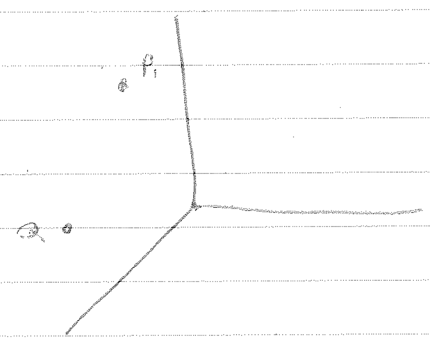
taking k to infinity we get a formal family

$K: \mathcal{X}_{\Sigma,\infty} \to \mathcal{M}_{\Sigma,\infty}$

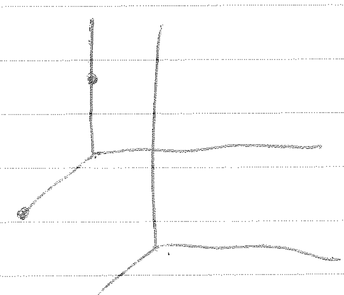
and so long as every curve is tonally transverse the u_i are flat coordinates. (e.g. for $\mathbb{P}^2$)

$(\tilde{\mathcal{M}}_{\Sigma,k})^{rel} = \mathbb{C}$, $\tilde{\mathcal{X}}_\Sigma$ is $x_0 x_1 x_2 = e^{y_1} \subset \mathbb{C}^3 \times \tilde{\mathcal{M}}_{\Sigma,k}$

Given

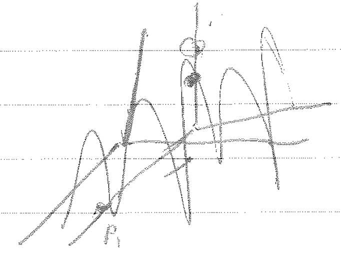


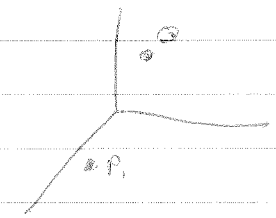
We have contributions to W_1



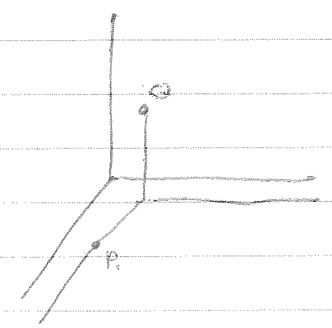
$W_1(Q) = y_0 + x_0 + x_1 + x_2 + a_1 x_1 x_2$

But for





we have



and $W_1(Q) = y_0 + x_0 + x_1 + x_2 + a_1 x_0 x_1$

So different positions of the P_i and Q differ by wall crossing formulae (combinatorial gluings of Maslov index zero disks).