Thm Suppose $n=2$. Then $N^{0,hol}_{\Delta,\Sigma}$ is finite and $N^{0,trop}_{\Delta,\Sigma} = N^{0,hol}_{\Delta,\Sigma}$

Mikhalkin's original proof $\Rightarrow$ works in 2 dimension

✓ Proof by Nishinou & Siebert $\Rightarrow$ works in all dimension.

Idea: ① Construct a 1-parameter degeneration of toric variety X_Σ adapted for the particular choice of pts $P_1, \cdots, P_s \in M_{\mathbb{R}}$ used to define $N^{0,trop}_{\Delta,\Sigma}$ $(s = |\Delta| - 1)$

② Find a correspondence between

(1) Trop curves passing through $P_1, \cdots, P_{|\Delta|-1}$

(2) Log stable curves in the central fibre

(3) Ordinary stable curves in the general fibre.

※ For general points $P_1 \sim P_s \in M_{\mathbb{R}}$, we know that #(genus 0 trop curve passing through $P_1 \sim P_s$) is finite, but yet we do not know this # does not depend on $P_1 \sim P_s$.

For general points $Q_1 \sim Q_s \in X_\Sigma$, we do not know yet that $N^{0,hol}_{\Delta,\Sigma}$ is finite but once we know this is finite we know that $N^{0,hol}_{\Delta,\Sigma}$ does not depend on the choice of $Q_1 \sim Q_s$

we can find $P_1 \sim P_s \in M_{\mathbb{Q}}$ so that there are only finite # of marked genus 0 trop curve passing through $P_1 \sim P_s$ at the marking S. Furthermore, we can make all these curve simple & every curve has different combinatorial type.

($\because$ the same proof of Lemma 1.20, Kansas note, works over $\mathbb{Q}$)

We denote the set of all these curves by $M_{0,s}(\Sigma, \Delta, P_1, \cdots, P_s)$

$\boxed{\text{Def}}$ Given the data of Σ, Δ & $\overset{\text{general}}{P_1 \sim P_s} \in M_{\mathbb{Q}}$ with $\dim M_{\mathbb{Q}} = 2$, a finite polyhedral decomposition $\mathcal{P}$ of $M_{\mathbb{R}}$ is said to be <u>good</u> if it satisfies the following properties.

(1) For $\sigma \in \mathcal{P}$ has faces of rational slopes and vertices in $M_{\mathbb{Q}}$. Furthermore, each σ has at least 1 vertex.

(2) If $\sigma \in \mathcal{P}$, then $\mathrm{Asym}(\sigma)$, the asymptotic cone to σ is an element of the fan Σ. Furthermore, every cone of Σ appears as the asymptotic cone of some $\sigma \in \mathcal{P}$.

(3) The image of any $h \in M_{0,s}(\Sigma, \Delta, P_1, \cdots, P_s)$ lies in one-dimensional skeleton of $\mathcal{P}$.

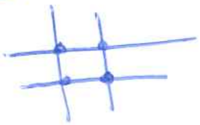
(4) Each P_i is a vertex of $\mathcal{P}$

Prop Given $\Sigma, \Delta, P_1 \sim P_s \in M_{\mathbb{Q}}$ (general). there exists a good polyhedral decomposition for this data.

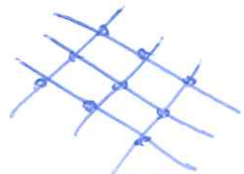
(proof) Define $*$ a binary operation on polyhedral decompositions as

$$\mathcal{P} * \mathcal{P}' := \{\sigma \cap \sigma' \mid \sigma \in \mathcal{P},\ \sigma' \in \mathcal{P}'\}$$

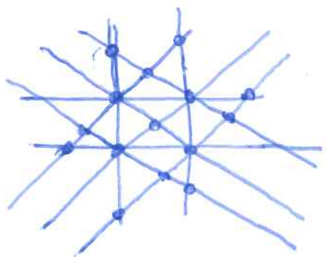
EX $\mathcal{P}$



$*$ $\mathcal{P}'$



$\Rightarrow$ $\mathcal{P} * \mathcal{P}'$



※ ① If $\mathcal{P}, \mathcal{P}'$ both satisfies (1), (2)

$\Rightarrow$ So does $\mathcal{P} * \mathcal{P}'$

② If at least one of $\mathcal{P}, \mathcal{P}'$ satisfies (3) (resp. (4)) then $\mathcal{P} * \mathcal{P}'$ satisfies (3) (resp. (4)).

Strategy (a) Construct $\mathcal{P}$ that satisfies (1), (2), (3)

(b) Construct $\mathcal{P}'$ that satisfies (1), (2), (4)

then $\mathcal{P} * \mathcal{P}'$ satisfies (1), (2), (3), (4).

(a) Let $\mathcal{P}$ be a polyhedral decomp whose maximal cells are closure of the connected components of $M_{\mathbb{R}} \setminus S$

where S is the union of images of parametrized trop curves in $M_{0,s}(\Sigma, \Delta, P_1 \sim P_s)$ and elts of $\Sigma^{[1]}$

→ to make $\mathcal{P}$ satisfy the latter part of the condition (2) in the def of good polyhedral decomp.

$\Rightarrow$ this $\mathcal{P}$ satisfies (1), (2), (3)

(b) Let $\mathcal{P}'$ be the polyhedral decomp. whose maximal cells are closures of the connected components of $M_{\mathbb{R}} \setminus S'$

where $S' = \bigcup_{\rho \in \Sigma^{[1]}} \bigcup_{i=1}^{s} (P_i + \rho)$

→ obtained by attaching 1-skeleton of the fan to every $P_1 \sim P_s$.

$\Rightarrow$ this $\mathcal{P}'$ satisfies (1), (2), (4)

$\therefore$ $\mathcal{P} * \mathcal{P}'$ satisfies (1) $\sim$ (4).

If φ is the common denominator of coordinates of all the vertices of $\mathcal{P}$
by enlargening everything by $(\times \varphi)$
we make $\mathcal{P}$ a lattice polyhedral decomp.
(in particular, all the $P_1 \sim P_s$ are in M).
without changing $\mathcal{N}^{0,\text{trop}}_{\Delta,\Sigma}$.

So wlog we can assume $P_1, \cdots, P_s \in M$.
and good polyhedral decomp for $\Delta, \Sigma, P_1 \sim P_s$ is a lattice polyhedral decomp.

---

we introduce some notations.

$$\tilde{M} := M \oplus \mathbb{Z} \quad , \quad \tilde{N} = \mathrm{Hom}_{\mathbb{Z}}(\tilde{M}, \mathbb{Z})$$

For the good polyhedral decomposition $\mathcal{P}$ we fixed, we define a fan $\Sigma_{\mathcal{P}}$ in $\tilde{M}_{\mathbb{R}}$ as follows.

For $\sigma \in \mathcal{P}$, let $\underbrace{C(\sigma)}_{\text{cone over } \sigma} := \overline{\{(rm, r) \mid r \geq 0, m \in \sigma\}}$

then $\Sigma_{\mathcal{P}} := \{\text{all the faces of } C(\sigma) \mid \sigma \in \mathcal{P}\}$.

↳ this is indeed a fan in $\tilde{M}_{\mathbb{R}}$.

※ $C(\sigma) \cap (M_{\mathbb{R}} \oplus \{0\}) = \mathrm{Asym}(\sigma) \in \Sigma$
and every cone in Σ arises in this way
(by the def of good polyhed. decomp.)
$\Rightarrow \Sigma = \{\tau \in \Sigma_{\mathcal{P}} \mid \tau \subseteq M_{\mathbb{R}} \oplus \{0\}\}$

So we get a toric variety $X := X_{\Sigma_{\mathcal{P}}}$ given by the fan $\Sigma_{\mathcal{P}}$.

Since every cone in $\Sigma_{\mathcal{P}}$ is contained in the upper half space $M_{\mathbb{R}} \oplus \mathbb{R}_{\geq 0}$.
every dual cone contains $z^{(0,1)}$, $(0,1) \in \tilde{N}$.
$\therefore$ $z^{(0,1)}$ is a global section of X.

So $\begin{array}{l} k[t] \longrightarrow \Gamma(X, \mathcal{O}_X) \\ t \longmapsto z^{(0,1)} \end{array}$ induces.

$$\pi : X \longrightarrow \mathbb{A}^1_k$$

$\boxed{\text{Central fibre}}$

$\pi^{-1}(0) = V(z^{(0,1)})$.
Use the formula.

$$(z^{(0,1)}) = \sum_{\rho \in \Sigma_{\mathcal{P}}^{(1)}} \langle (0,1), \underbrace{m_\rho}_{\text{primitive integral generator of } \rho} \rangle D_\rho$$

$\Rightarrow$ Union of D_ρ's s.t. m_ρ has nonzero last coordinate.
$\Rightarrow$ union of $D_{C(v)}$'s. for $v \in \mathcal{P}$ a vertex

$\therefore$ $\pi^{-1}(0)$ is a union of toric varieties.

General fibre $X \setminus \pi^{-1}(0)$

$\Rightarrow$ Union of X_σ where $z^{(0,1)}$ nonvanishes.

Have to find $\sigma \in \Sigma_P$ s.t. in $k[\sigma^\vee \cap \tilde{N}]$ $z^{(0,1)}$ is invertible.

$\Rightarrow$ $\sigma^\vee$ should contain both $z^{(0,1)}$ & $z^{(0,-1)}$

$\Rightarrow$ $\sigma \in \Sigma$.

$\therefore$ toric variety corresponding to Σ as a fan not in $M_{\mathbb{R}}$ but in $\tilde{M}_{\mathbb{R}}$.

So for all $\sigma \in \Sigma$, Spec$[\sigma^\vee \cap \tilde{N}]$ we glue

dual is taken in $\tilde{N}_{\mathbb{R}}$.

$\sigma^\vee \cap \tilde{N} = \{(n,0) \mid n \in \sigma^\vee \cap N\} \oplus \mathbb{Z}\cdot(0,1)$.

dual is taken in $\tilde{N}$

dual is taken in N.

$\Rightarrow$ Spec$[\sigma^\vee \cap \tilde{N}] \cong$ Spec$[\sigma^\vee \cap N] \times \mathbb{G}_m$

$\therefore$ after gluing we get

$$X \setminus \pi^{-1}(0) \cong X_\Sigma \times \mathbb{G}_m.$$

? Toric Strata

- we denote $X_0 := \pi^{-1}(0)$.
- For each $\tau \in P \rightarrow C(\tau) \in \Sigma_P \Rightarrow$ toric stratum $D_{C(\tau)}$

for simplicity we denote it D_τ.

The fan $\Sigma_\tau := \{T_\tau\sigma / T_\tau\tau \mid \sigma \in P \text{ s.t } \tau \subseteq \sigma\}$ in $M_{\mathbb{R}}/T_\tau\tau$ defines D_τ.

- we denote ∂X_0 to be the union of 1-dim toric strata of X_0 not contained in Sing(X_0)

$\Rightarrow$ 1-dim toric strata of X_0 not contained in two irreducible components of X_0.

$\Rightarrow$ D_ω where $\omega \in P^{[1]}$ ($\because$ since $C(\omega) \in \Sigma_P^{[2]} \Rightarrow D_\omega$ is codim 2 in dim $\tilde{M}_{\mathbb{R}}=3$) where ω is attached to only one vertex

($\because$ there's an inclusion reversing correspondence btw cones & toric strata)

$$\therefore \quad \partial X_0 = \bigcup_{\substack{\omega \in P^{[1]} \\ \omega:\text{noncpct}}} D_\omega$$

Alternatively, $\partial X_0 = \overline{\partial X \setminus X_0} \cap X_0$

Now we specify points to the degeneration.
I.e. sections $\sigma_1, \cdots, \sigma_3 : \mathbb{A}^1_k \to X$.

Let $L_i := \mathbb{Z}(P_i, 1) \subseteq \tilde{M}$ be the rank 1 -
sublattice generated by $(P_i, 1)$.

$\Rightarrow \tilde{N} = \mathrm{Hom}_{\mathbb{Z}}(\tilde{M}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(L_i, \mathbb{Z})$

$\Rightarrow k[\tilde{N}] \longrightarrow k[\mathrm{Hom}_{\mathbb{Z}}(L_i, \mathbb{Z})]$

$\Rightarrow \mathbb{G}(L_i) \subseteq \mathbb{G}(\tilde{M})$

Now pick $Q_1 \sim Q_3$ in the big torus orbit of X.

Lets consider $\overline{G(L_i) \cdot Q_i}$

Let's define α, β, γ

$$
\begin{array}{c}
k[\mathrm{Hom}_{\mathbb{Z}}(L_i, \mathbb{Z})] \otimes_k k \\
\uparrow \alpha \\
k[\tilde{N}] \otimes_k k[\tilde{N}] \\
\uparrow \beta \\
k[\tilde{N}] \\
\uparrow \gamma \\
k[\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot (0,1), \mathbb{Z})]
\end{array}
$$

So that if we take $\mathrm{Spec}(-)$

$$
\begin{array}{c}
\mathbb{G}(L_i) \times_k \{Q_i\} \\
\downarrow \alpha^* \\
\mathbb{G}(\tilde{M}) \times_k \mathbb{G}(\tilde{M}) \\
\downarrow \beta^* \\
\mathbb{G}(\tilde{M}) \\
\downarrow \gamma^* \\
\mathbb{G}(\mathbb{Z})
\end{array}
$$

we get $\alpha^* =$ the product of inclusion maps
$\mathbb{G}(L_i) \hookrightarrow \mathbb{G}(\tilde{M})$
$\{Q_i\} \hookrightarrow \mathbb{G}(\tilde{M})$

$\beta^* =$ the torus action map.

$\gamma^* = \pi$ (restricted).

Then we get that $\alpha \circ \beta \circ \gamma$ is an isomorphism of k-algebras.

$\Rightarrow \gamma^* \circ \beta^* \circ \alpha^*$ is an isomorphism

$\Rightarrow$ has an inverse σ_i"

So we get

$$\begin{tikzcd}
\mathbb{G}(L_i) \times \{Q_i\} \arrow[d, "\alpha^*"] \\
\mathbb{G}(\tilde{M}) \times \mathbb{G}(\tilde{M}) \arrow[d, "\beta^*"] \\
\mathbb{G}(\tilde{M}) \arrow[d, hook, "i"] \\
\pi^{-1}(\mathbb{A}^1_k \setminus \{0\}) \arrow[d, "\pi"] \\
\mathbb{A}^1_k \setminus \{0\} \arrow[uuuu, bend right=60, "\sigma_i''"']
\end{tikzcd}$$

If we put $\sigma_i' := i \circ \beta^* \circ \alpha^* \circ \sigma_i''$

this is a section of π.

with image $G(L_i) \cdot Q_i$

So we get a diagram.

$$\begin{tikzcd}
\operatorname{Spec} k(x) \arrow[r] \arrow[d] & \mathbb{A}^1_k \setminus \{0\} \arrow[r] \arrow[d, hook] & X \arrow[d, "\pi\ (\text{proper})"] \\
\operatorname{Spec} \mathcal{O}_{\mathbb{A}^1_k, 0} \arrow[r] \arrow[urr, dashed] & \mathbb{A}^1_k \arrow[r, "id"'] & \mathbb{A}^1_k
\end{tikzcd}$$

Since π is proper, by the valuative criterion. we get the dotted map.

Then we can explicitly construct a map.

$\sigma_i : \mathbb{A}^1 \to X$ that lifts. both

$\mathbb{A}^1_k \setminus \{0\} \to X$ and

$\operatorname{Spec} \mathcal{O}_{\mathbb{A}^1_k, 0} \dashrightarrow X$.

So we get sections $\sigma_i : \mathbb{A}^1_k \to X$.

for π.

Since. $id = \pi \circ \sigma_i$ is proper.

& π is separated.

$\Rightarrow$ σ_i's are proper.

Then we can show that

$\operatorname{Im} \sigma_i = \overline{\mathbb{G}(L_i) \cdot Q_i}$.

$\because$ $\sigma_i(\overline{\mathbb{A}^1_k \setminus \{0\}}) \subseteq \overline{\sigma_i(\mathbb{A}^1_k \setminus \{0\})}$

and $\sigma_i(\mathbb{A}^1_k \setminus \{0\})$, $\sigma_i(\mathbb{A}^1_k)$

differ by 1pt

and

$\sigma_i(\mathbb{A}^1_k)$ is closed.

Objects in 3 worlds.

① <u>Tropical world</u>: Consists of marked parametrized tropical curves of genus 0 through $P_1 \sim P_s$ in $M_{\mathbb{R}}$.

② <u>Log world</u>: Consists of log morphisms $f : (C^\dagger, x_1, \cdots, x_s) \longrightarrow X_0^\dagger$ from genus 0 log curves $C^\dagger$ with marked points $x_1, \cdots, x_s$ such that $f(x_i) = \sigma_i(0) \in X_0$ & X_0 is given the log structure induced by the divisorial log structure coming from the inclusion.

③ <u>The classical world</u>

$N_{\Delta, \Sigma}^{0, hol}$ make sense over any algebraically closed field of characteristic zero. & do not depend on the choice of field.

Consider $K = \overline{k((t))}$ the algebraic closure of the field of formal Laurent series.

Then the inclusion

$$k[t] \hookrightarrow k[t]_t \hookrightarrow K$$

induces

$$\operatorname{Spec} K \to \mathbb{A}^1_k \setminus \{0\} \hookrightarrow \mathbb{A}^1_k$$

$$X \times_{\mathbb{A}^1_k} K \cong \left(X \times_{\mathbb{A}^1_k} (\mathbb{A}^1_k \setminus \{0\}) \right) \times_{\mathbb{A}^1 \setminus \{0\}} K$$

$$\cong \left(X_\Sigma \times_k \mathbb{G}_m \right) \times_{\mathbb{A}^1_k \setminus \{0\}} K$$

$$\cong X_\Sigma \times_k K$$

Also, if we base change sections $\sigma_1, \cdots, \sigma_s : \mathbb{A}^1_k \to X$ we get points $\sigma_1, \cdots, \sigma_s : \operatorname{Spec} K \to X_\Sigma \times_k K$ using these points we count torically transverse curve over K passing through $\sigma_1, \cdots, \sigma_s$ at the marking to get $N_{\Delta, \Sigma}^{0, hol}$.

Diagram of the Proof

(log deformation theory)
each log curve deforms
to give rise to a torically transverse curve over K.

Log world

Classical world

torically transverse curve over K
gives rise to a log curve.

Show that
for a given tropical curve h
there is a $\mathrm{Mult}(h)$ log curves
associated to it

Show
how certain
log curves give
rise to tropical curves
via "dual intersection cplx" construction.

tropical world.