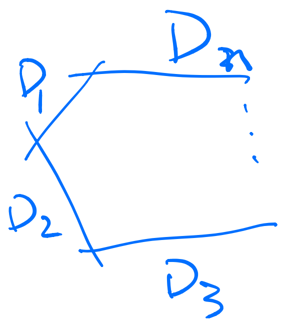
Section 6.1

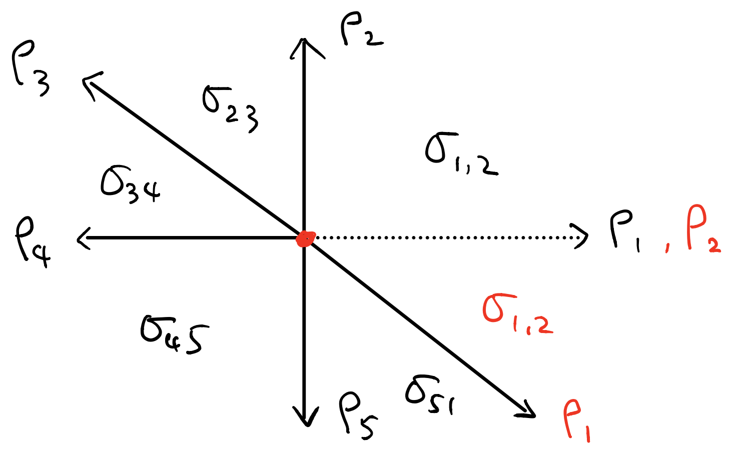
$(Y, D = D_1 + D_2 + \cdots + D_n)$: Looijenga pair

$n \geq 3$



local toric construction $\Longrightarrow$ $B(\mathbb{Z}) = B_0(\mathbb{Z}) \cup \{0\}$

Ex of DP5



- Choose

strictly convex rational poly.

$$\begin{tikzcd}
NE(Y)_{\mathbb{R}} \arrow[r, phantom, "\subset"] & \sigma_P \arrow[r, phantom, "\subset"] & A_1(Y,\mathbb{R}) \\
NE(Y) \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\overset{\eta}{\subset}"] & P \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] & A_1(Y,\mathbb{Z}) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$R = \Bbbk[P], \quad m = (P)$$

- Choose $\varphi = \{ \varphi_i |_{\tau_i = \mathrm{Int}(\sigma_{i-1,i} \cup \sigma_{i,i+1})} \}$

s.t. $\kappa_{\rho_i,\varphi_i} = [D_i]$

$$\Rightarrow 0 \to P^{gp} \to \mathcal{P} \to \underbrace{\overset{TB_0(\mathbb{Z})}{\overset{\shortparallel}{\Lambda}}}_{\text{locally has a fan structure from } B_0(\mathbb{Z})} \to 0$$

$\Rightarrow$ Mumford degeneration:

$$\begin{tikzcd}
V/n\setminus\{0\} \arrow[d, equal] & \\
X^{\circ}_{m} \arrow[r] \arrow[d, phantom, "\cap"] & \mathrm{Spec}\, R/m \arrow[d, phantom, "\cap"] \\
{} & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{missing the } \partial \text{ section} \arrow[r, squiggly] & X^\circ \arrow[r] & \operatorname{Spec} R \\
& X_I^\circ \arrow[u, hook] \arrow[r] & \operatorname{Spec} R/I \arrow[u]
\end{tikzcd}$$

- The cannonical scattering.

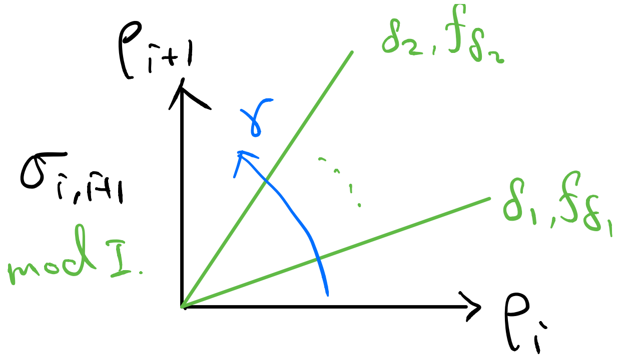
$$D^{\mathrm{can}} = \{ (\delta, f_\delta) \mid \delta \in B(\mathbb{R}) : \underset{\text{slope}}{\text{rational}} \}$$

$$f_\delta := \exp\left[\underbrace{\sum_\beta}_{\mathbb{A}^1\text{- curve class.}} \underbrace{k_\beta}_{\text{contact order}} \underbrace{N_\beta}_{\log GW} z^{[\beta] - \varphi_{\underbrace{\tau_\delta}_{\text{smallest cone containing } \delta}}(k_\beta \underbrace{m_\delta}_{\text{prim. vector } // \delta \text{, point away from } 0.})} \right]$$

Choose $I \subset P$ s.t. $\sqrt{I} = m$. then

$$D_I^{\mathrm{can}} := \{ (\delta, f_\delta \bmod I) \mid f_\delta \not\equiv 1 \bmod I \}$$

is finite



$\Rightarrow$ Automorphism $\Theta_{\gamma, I}$

$\Rightarrow$ locally Modified gluing of X_I° using $\Theta_{\gamma,I}$

☆ · D_I^{can} is consistent

$\Longleftrightarrow$ The local modification can be glued globally

$\Rightarrow$

①

$$\begin{tikzcd}
V/n \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X_I = \operatorname{Spec} A_I \arrow[r, hook] \arrow[d, "f_I"] \arrow[dr, phantom, "\square"] & \mathfrak{X}_m \arrow[d] \\
\operatorname{Spec} R/m \in & \operatorname{Spec} R_I \arrow[r, hook, "\varprojlim_{\sqrt{I}=m}"'] & \operatorname{Spf} \hat{R}
\end{tikzcd}$$

for $\sqrt{I} = m$

② **In Section 4** : f is a formal smooth of $V/_n$

③ **In Section 5** : f_I is flat and is $T^D \simeq T^n$ equivariant

Thm 0.1 : ① ② ③

Next Replace $\sqrt{I} = m$, by $\sqrt{I} = J \subset P$ in the above construction for ① ② ③.

Thm 0.2 There is a unique smallest $J \subset P$ does this :

(i) If the intersect matrix $(D_i, D_j) \not\leq 0$ then $J = 0$

(ii) In general, $V(J) \subset \operatorname{Spec} R$ contains the union of toric strata corresponding to $F \underset{\text{face}}{\subset} \sigma_P$ s.t. $[D_i] \notin F$

i.e. $V(\sigma_\sigma \setminus F) \subset V(J)$

$$\begin{array}{ccccc} NE(Y)_{\mathbb{R}} & \subset & \overset{\substack{\text{strictly conve}\\ \text{rational poly.}\\ \downarrow}}{\sigma_P} & \subset & A_1(Y,\mathbb{R}) \\ \cup & & \cup & & \cup \\ NE(Y) & \overset{\eta}{\subset} & P & \subset & A_1(Y,\mathbb{Z}) \end{array}$$

Next Prove Thm 0.2 assum $n \geq 3$

key : theta function.

From Section 2

Fix $I \subset P$ s.t. $\sqrt{I} = m$

$$A_I = \bigoplus_{q \in B(\mathbb{Z})} R_I \cdot \theta_q$$

Alg. Str. :

$$\theta_{q_1} \cdot \theta_{q_2} = \sum_{q \in B(\mathbb{Z})} \alpha_q \, \theta_q$$

$$\alpha_q = \sum_{(\gamma_1,\gamma_2)} c(\gamma_1)\cdot c(\gamma_2)$$

- Take $\underset{\hookrightarrow\ \text{finite set}}{\Gamma} = \{ v_i \mid \text{generator for } \rho_i \}$

  Then $\{ \theta_q \mid q \in \Gamma \}$ generates A_I mod m.

  hence generates A_I for $\sqrt{I} = m$

- $\mathcal{A}$: the collection of $J \subset P$ s.t.

  (1) There is an R_J-alg str. on

$$A_J = \overset{R_J\text{-module.}\ \swarrow}{\bigoplus_{q \in B(Z)}} R_J \cdot \theta_q$$

  s.t. A_J mod $(I+J)$ has the canonical R_{I+J}-alg str. for $\sqrt{I} = m$.

  (2) $\{ \theta_q \mid q \in \Gamma \}$ generates A_J as an R_J-alg.

Lemma. There is a unique, minimal, radical monomial ideal $I_{min} \subset P$ s.t. (1) & (2) holds for any $I \subset P$ with

$$I_{min} \subset \sqrt{I}$$

Sketch:

- $J \subset P$ is a thickening of the toric strata defined by $\sqrt{J}$.
- (1) & (2) hold for $\sqrt{J}$, then they hold for J

$\Rightarrow$ So we only need to find the strata that (1) & (2) hold.

- $m \in A \neq \phi$
- If (1) & (2) hold over $\sqrt{J}$ and $\sqrt{J'}$

then they hold over $\sqrt{J \cap J'}$

$\hookrightarrow$ union of strata.

- Only finitely many toric strata. ■

- Another $(1') \Leftrightarrow (1)$.

$\wedge$: completion along m.

$$\hat{A} = \varprojlim_{\sqrt{I}=m} A_I \subset \prod_{q\in B(\mathbb{Z})} \hat{R}\cdot\theta_q$$

$$\begin{array}{ccc} \hat{A}_J = \varprojlim_{\sqrt{I}=m} A_{I+J} & \subset & \prod_{q\in B\mathbb{Z}} \hat{R}_J\cdot\theta_q \\ & & \cup \\ & A_J := & \bigoplus_{q\in B(\mathbb{Z})} R_J\cdot\theta_q \end{array}$$

$$\Rightarrow \quad A_J \subset \hat{A}_J$$

$(1) \Longleftrightarrow (1')$:

In $\hat{A}_J$,

$$\Theta_{q_1} \cdot \Theta_{q_2} = \sum_{q \in B(\mathbb{Z})} \alpha_q \Theta_q$$

$$= \sum_{q} \sum_{c} ?\cdot z^c \Theta_q$$

has finitely many $z^c \Theta_q$ s.t. $c \notin J$

i.e. the product str. is defined over A_J.

<u>Next</u> : Verify (i) and (ii)

In (i), $F = P$

In (ii). $F \subset \sigma_P$, $[D_1] \notin F$.

In both cases, can find

$$W = \sum_{i \neq 1} a_i D_i$$

s.t. (a). $a_i > 0$

(b) $W.D_j > 0$ for all $[D_j] \in F$.

Set $F' = \overline{NE(Y)}_{\mathbb{R}} \cap F$

$\Rightarrow$ $\overline{NE}(Y)_{\mathbb{R}}$ is polyhedral near F'.

$$\Rightarrow \quad NE(Y)_{\mathbb{R}} \subset \sigma_{P'} \subset \sigma_P$$

s.t. (a) Structure using P comes from P'.

(b) $F' \underset{\text{face}}{\subset} \sigma_{P'}$.

May assume • $F = F'$.

• $W \cap (F \setminus \{0\}) > 0$

<u>Next</u> we verify (1') & (2) for

J s.t. $\sqrt{J} = P \setminus F$.

Key: $T^D \simeq (\mathbb{C}^*)^n$ - equivariance.

Define T^D-action.

- $$\begin{tikzcd} P \arrow[d, phantom, "\cap"] & \\ A_1(Y) \arrow[r, "w"] & X(T^D) = \bigoplus_i \mathbb{Z} e_{D_i} \\ C \arrow[r, mapsto] & \sum (C.D_i)\cdot e_{D_i} \end{tikzcd}$$

- $$w : B = B_0(\mathbb{R}) \cup \{0\} \xrightarrow[\substack{\uparrow \\ \text{piecewise} \\ \text{linea.}}]{} X(T^D)\otimes\mathbb{R}$$

  $w(0) = 0$

  $w(V_i) = e_{D_i}$

- Note $w(f_\delta) = 0$, thus T^D-action is well-defined.

Verify (1') for $\sqrt{J} = P \setminus F$

For each max cone $\sigma \subset B$, it suffices to show that

$$\begin{tikzcd}
w: \sigma(\mathbb{Z}) \times (P \setminus J) \arrow[r] & X(T^P) \\
(q, C) \arrow[r, mapsto] & w(q) + w(C) \arrow[d, equal] \\
 & w(z^C \Theta_q) = w(\Theta_{q_1} \cdot \Theta_{q_2})
\end{tikzcd}$$

has finite fibers.

$$\Leftarrow \quad \mathrm{Ker}\left(\sigma(\mathbb{Z}) \times P \to X(T^P)\right) \cap (\sigma(\mathbb{Z}) \times F) = 0$$

Set $\sigma = \sigma_{i,i+1}$ and $(q, C) \in \sigma(\mathbb{Z}) \times F$

s.t. $\quad w(q) + w(C) = 0$

$$\Rightarrow \quad q = a v_i + b v_{i+1}, \quad a, b \geq 0$$

$$0 = w(q) + w(C) = a e_{D_i} + b e_{D_{i+1}} + \sum_j (C.D_j) e_{D_j}$$

$\Rightarrow \quad C.D_j \leq 0$ for all j.

$\Rightarrow \; C \cdot W \leq 0$

But $W \cap (F \setminus \{0\}) > 0$

$\Rightarrow \; C = 0, \; a = b = 0$

$\Rightarrow \; (1') \; \checkmark$.

<u>Verify (2) for $\sqrt{J} = P \setminus F$</u>.

Set $A_J' = R_J\{\theta_q \mid q \in \Gamma\} \subset A_J$

Want: $A_J' = A_J$

$\Longleftrightarrow$ For any fixed weight w:

$$\overbrace{\left\{ z^c \cdot \theta_q \in A_J \;\middle|\; (q, c) \in B(\mathbb{Z}) \times (P \setminus J) \text{ of } wt = w \right\}}^{\Bbbk\text{-basis of } A_J^w}$$

$$\underset{(*)}{\subset} \; A_J'$$

Prove by decreasing induction on $\mathrm{ord}_m(c)$

(a) (*) is true for very large $\mathrm{ord}_m(c)$

(b) $\{\theta_q \mid q \in P\}$ generates A_J mod $\mathfrak{m}$.

$\Longrightarrow \exists\ a \in A_J'$ s.t.

$$\theta_q = a + m$$

for $m \in \mathfrak{m} \cdot A_J$

Choose a & m homogeneous for T^P-action.

$$\Longrightarrow z^c \theta_q = \underset{\substack{A_J' \\ \text{as } a \in A_J' \\ z^c \in R_J}}{\underset{\cap}{z^c \cdot a}} + \underset{\substack{A_J' \text{ by} \\ \text{induction}}}{\underset{\cap}{z^c \cdot m}} \in A_J'$$

$\boxed{z}$