Sep 23

Monday, September 23, 2019 3:36 PM

tropical hypersurface DLT

Balancing condition ($\dim M_{\mathbb{R}} = 2$)

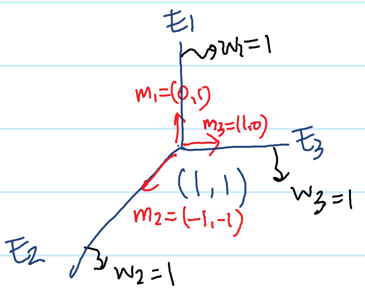
let $v \in \tilde{P}^{[2]}$ be a vertex of $V(f)$ contained in edges $E_1, \dots, E_k \in \check{P}^{[1]}$

let $m_1, \dots, m_k \in M$ be primitive tangent vectors.

Suppose $w(E_i) = w_i \Rightarrow \sum_{i=1}^{k} w_i m_i = 0$

<u>Ex</u> $f(x_1, x_2) = 1 \oplus (0 \odot x_1) \oplus (0 \cdot x_2)$

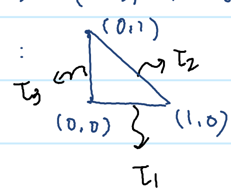
$V(f)$



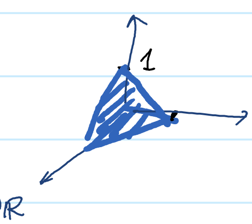
$1(-1,-1) + 1\cdot(1,0) + 1\cdot(0,1) = (0,0)$

<u>Why?</u> $f \Rightarrow (\Delta_S, \varphi, \mathcal{P})$

Δ_S :



$\tilde{S}$



orientation : $\circlearrowleft$

$n_1 = (-1, 0)$, $n_2 = (1, -1)$, $n_3 = (0, 1)$

n_i primitive normal to E_i

Note: $\sum_{i=1}^{3} \mathrm{aff}(\tau_i) \cdot n_i = 0$

last time: $\mathrm{aff}(\tau_i) \overset{\text{corresp.}}{\leftrightsquigarrow} w(\check{\tau}_i = E_i)$

$R\left(\sum_{i=1}^{3} \mathrm{aff}(\tau_i) \cdot n_i\right) = 0 \Rightarrow \sum_{i=1}^{3} w(E_i) \cdot m_i = 0$

$$R\left(\sum_{i=1}^{3} \mathrm{aff}(\tau_i)\cdot n_i\right)=0 \Rightarrow \sum_{i=1}^{3} w(E_i)\cdot m_i=0$$

$\uparrow$ counterclockwise by $\frac{\pi}{2}$

Parametrized tropical curve

comb topo. $\rightsquigarrow$ $\overline{\Gamma}$:= connected graph with no 2-valent vertices

- $\overline{\Gamma}^{[0]}$, $\overline{\Gamma}^{[1]}$
- $\overline{\Gamma}^{[0]}_{\infty}$ set of univalent ver. of $\overline{\Gamma}$

- $\Gamma = \overline{\Gamma} \setminus \overline{\Gamma}^{[0]}_{\infty}$
- $\Gamma^{[1]}_{\infty}$: the set of non-compact edges of Γ.
- flag of Γ : (V,E), $V \in E$
- weight : $\overline{\Gamma}^{[1]} \to \mathbb{N} = \{0,1,\dots\}$
- marked graph: $(\Gamma, x_1,\dots, x_k)$ where $\{x_1 \dots, x_k\} \hookrightarrow \Gamma$ non cpt edges

<u>Def</u> A marked parametrized tropical curve:

$$h:(\Gamma, x_1,\dots, x_k) \to M_{\mathbb{R}} \text{ cts s.t.}$$

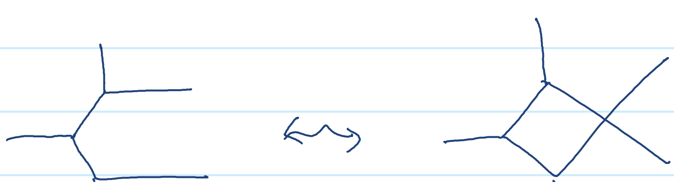
1) if E is marked edge, $h|_E$ constant
2) if E is non-marked edge, $h|_E$ is a line of rational slope.
3) balancing condition: $V\in\Gamma^{[0]}$, $V\in E_1,\dots E_l \in \Gamma^{[1]}$

let $m_i \in M$ be primitive tangent to $h(E_i)$

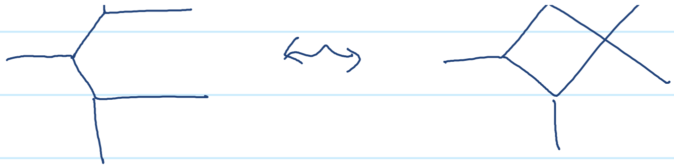
$$\sum_{i=1}^{l} w(E_i)\, m_i = 0$$

$h:(\Gamma, x_1 \dots x_k) \to M_{\mathbb{R}}$ & $h':(\Gamma', x_1' \dots x_k') \to M_{\mathbb{R}}$ are equiv.

if $\exists\, \varphi : \Gamma \to \Gamma'$ s.t. $h\circ\varphi = h'$

e.g. 

e.g.



Called the equiv. class as <u>marked tropical curve</u>.

genus, degree, & weight on $h(\Gamma)$.
h

- genus : $h : (\Gamma, x_1 \ldots x_k) \to M_{\mathbb{R}}$ is $b_1(\Gamma)$
- degree : fix a fan Σ

Define T_Σ as free abelian gp. of $\Sigma^{[1]}$

$$\rho \in \Sigma^{[1]} \rightsquigarrow t_\rho \in T_\Sigma$$

Define $\quad r : T_\Sigma \to M$

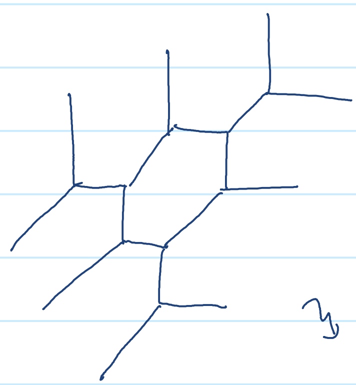
$$t_\rho \mapsto m_\rho$$

h is said to be in X_Σ if $E \in \Gamma_\infty^{[1]}$ nonmarked non-compact edge $h(E)$ is translate of $\rho \in \Sigma^{[1]}$

<u>Def</u> If h is a curve in X_Σ,

$$T_\Sigma \ni \Delta(h) = \sum_{\rho \in \Sigma^{[1]}} d_\rho t_\rho$$ where $d_\rho = \#$ of nonmarked non-cpt edge counted with wgt.

$\mathbb{P}^2$



wgt 1 on each edge

$\rightsquigarrow \quad 3(t_{\rho_1} + t_{\rho_2} + t_{\rho_3})$

Note $r(\Delta(h)) = 0$ by balancing condition

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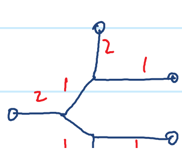
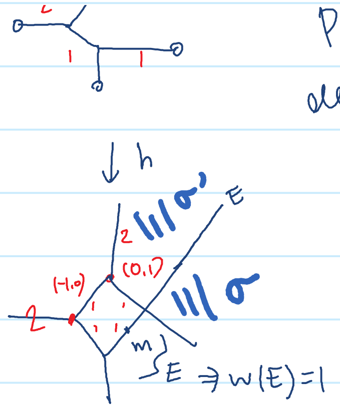
weight:



Let E be an edge of $h(\Gamma)$

Pick $m \in E$.



Pick $m \in E$.

define $w_{h(\Gamma)}(E) := \sum_{\substack{E' \in \Gamma^{[1]} \\ E' \cap h^{-1}(m) \neq \phi}} w_\Gamma(E')$

defined indep of choice of $m \in E$
(by balancing condition)

Prop If $h: \Gamma \to M_{\mathbb{R}}$ is a trop curve, $\dim M_{\mathbb{R}} = 2$
then $\exists$ trop poly f st. $h(\Gamma) = V(f)$

Pf: (sketch) ① $\exists$ $\check{P}$ of $M_{\mathbb{R}}$ s.t. $M_{\mathbb{R}} \setminus h(\Gamma)$

σ = max domain. define $f_\sigma \equiv 0$ (base)

Define slope of $f_{\sigma'} = (\text{slope of } f_{\sigma}) - w(E) \cdot n_E$

⇒ can define $f_{\sigma'}$

∴ done by induction

well-defined since loop around is the same by balancing cond.

Note the fn is convex ⇒ given by tropical poly.

∴ $V(f) = h(\Gamma)$. ∎

Now define moduli of marked tropical curve.

- Combinatorial type of $h: (\Gamma, x_1, \dots, x_k) \to M_{\mathbb{R}}$
  - the data of

① $(\Gamma, x_1 \dots x_k)$

② $w: \Gamma \to \mathbb{N}$

③ for each flag (V, E) of Γ, the primitive tangent vector $m_{(V,E)} \in M$ to $h(E)$ of $h(V)$.

- Combinatorics : equiv. class $[h]$

Def for $g, k \geq 0$, fan Σ, $\Delta \in T_\Sigma$ deg. with $r(\Delta) = 0$.

$$\mathcal{M}_{g,k}(\Sigma, \Delta) := \left\{ \begin{array}{l} \text{trop. curve in } X_\Sigma \text{ of genus } g \text{ and deg } \Delta \\ \text{with } k \text{ marked pt.} \end{array} \right\}$$

$\mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta) \subseteq \mathcal{M}_{g,h}(\Sigma, \Delta)$ of equiv. class of $[h]$

Prop ① $\mathcal{M}_{g,k}(\Sigma, \Delta) = \coprod_h \mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta)$

② For a given $[h]$, $\mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta)$ is the interior of polyhedron of

$\dim \geq \ell + k + (3 - \dim M_{\mathbb{R}})(g-1) - ov(\Gamma)$

where $ov(\Gamma) = \sum_{v \in \Gamma^{[0]}} (\text{valency}(v) - 3)$

$\ell = \#$ of noncompact unmarked edges

fixing $[h] \Rightarrow$ fixing (i) weight of Γ

(ii) $m_{(V,E)}$

fix ref pt. + (i) + (ii) + affine length to cpt edges $\Rightarrow$ fully describe the tropical curves

first, $\chi(\bar{\Gamma}) = b_0(\bar{\Gamma}) - b_1(\bar{\Gamma}) = 1 - g = \#(\bar{\Gamma}^{[0]}) - \#(\bar{\Gamma}^{[1]})$

Note $\#(\bar{\Gamma}^{[0]}_\infty) = \#(\bar{\Gamma}^{[1]}_\infty)$

$\Rightarrow \chi(\bar{\Gamma}) = \#(\Gamma^{[0]}) - \#(\Gamma^{[1]} \setminus \Gamma^{[1]}_\infty)$ $\Bigg\}\ (*)$

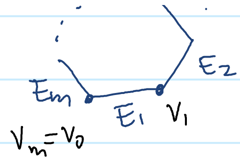
$(**)$: $3(\#\Gamma^{[0]}) + ov(\Gamma) \overset{\text{by def.}}{=} \sum_{v \in \Gamma^{[0]}} \text{valency}(v) = 2(\#\Gamma^{[1]} \setminus \Gamma^{[1]}_\infty) + \#(\Gamma^{[1]}_\infty)$

$3 \cdot (*) + (**) \Rightarrow \#\text{ cpt. edg.} = \#\text{ noncompact edges} + 3g - 3 - ov(\Gamma)$

$= \ell + k + 3(g-1) - ov(\Gamma)$

$$\underset{\text{pick ref pt.}}{M_{\mathbb{R}}} \times \underset{\text{affine length of cpt edge.}}{\mathbb{R}_{>0}^{\ell+k+3g-3-ov(\Gamma)}}$$

If there is a cycle in Γ



$\partial E_i = \{v_{i-1}, v_i\}$

Note $\quad v_i = v_{i-1} + l_{E_i} \cdot m_{(v_{i-1}, E)}$

$\rightsquigarrow \quad v_0 = v_0 + \sum_{i=1}^{m} l_{E_i} m_{(v_{i-1}, E)}$

$\Rightarrow \quad \underbrace{\sum_{i=1}^{m} l_{E_i} \cdot m_{(v_{i-1}, E)} = 0}$

g many of these eqn
($\because$ genus!)

$g \cdot \dim M_{\mathbb{R}}$ many relation

$L \subseteq \mathbb{R}^{l+k+3g-3-ov(\Gamma)}$
linear subsp. defined by the eqn

$\therefore \dim \geq \dim M_{\mathbb{R}} + l + k + 3g - 3 - ov(\Gamma) - g \cdot \dim M_{\mathbb{R}}$