# Broken Lines & Algebraic Structures of Theta Functions

$(Y, D) \rightsquigarrow (B, \Sigma)$, $P$ (monoid), $\varphi$ (multi-valued convex function), $J \subseteq P$ (ideal w/ $\sqrt{J} = J$), $\mathfrak{D}$ (scattering diagram)

**Definition:** Given $q \in B_0(\mathbb{Z})$, $Q \in B_0$. A <u>broken line</u> is

$\beta : (-\infty, 0] \longrightarrow B_0$ piecewise integral affine bending finitely many times.

$\cup$ $L$ : maximal domain of linearity

$\rightsquigarrow$ $m_L = c_L z^{q_L}$, $c_L \in \mathbb{k}^*$, $q_L \in T(L, \beta^{-1}(P)|_L)$

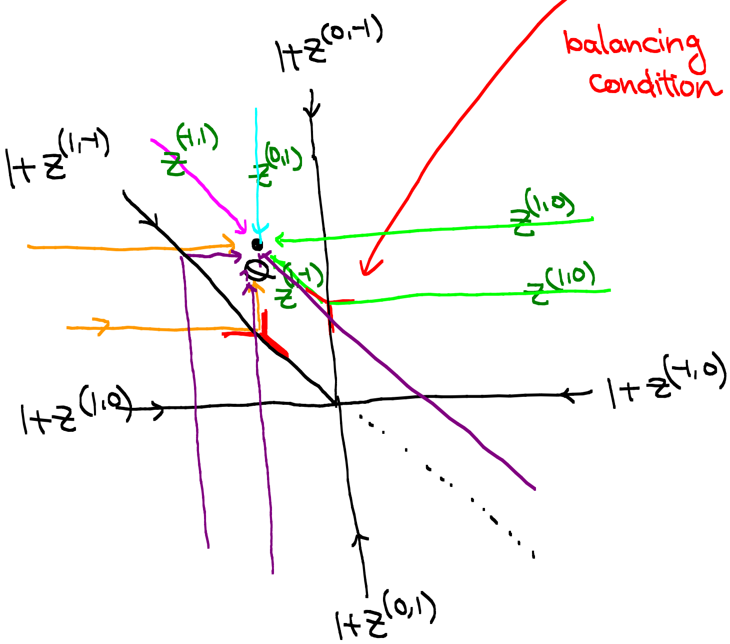
s.t.

- $L$ : unbounded w/ $\beta(L) \subseteq \sigma \in \Sigma_{max}$ (simply connected), then $m_L = z^{\underline{\varphi_\sigma(q)}}$ (parallel transport)
- $\beta'(t) = -r(q_L)$, $\beta(0) = Q$
- $L_{i+1}$, $t_i$, $L_i$, $f_{\delta_i}$: then $m_{L_{i+1}} = m_{L_i} \cdot \underline{cz^s}$ a term in $\prod_j f_{\delta_j}^{\langle n, r(q_L) \rangle}$ ($n = n_{\delta_j}$), $> 0$

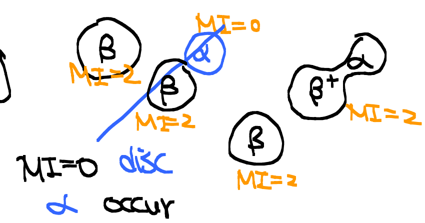
rays in $\mathfrak{D}$ crossing $\beta(t_i)$

balancing condition

ex.

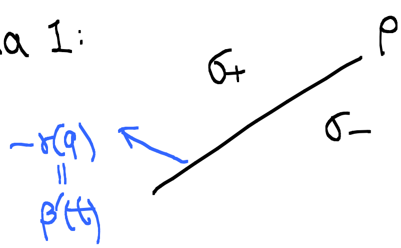


Dictionary:

| | GHK | SG |
|---|---|---|
| Intuitively, | $B_{GHK}$ | base of SYZ fibration of Y\D |
| | rays in scattering diagram | MI=0 discs |
| | broken lines | MI=2 discs |
| | $0 \to P^{gp} \to P \xrightarrow{r} \Lambda_{\mathbb{Z}} \to 0$ <br> $q_0 \longmapsto r(q_0)$ | $0 \to H_2(X)/H_2(L) \to H_2(X, L_u) \to H_1(L_u) \to 0$ <br> disc class $\to$ boundary class <br> up to a sign!! |
| | $\mathrm{Mono}(\beta)$ last monomial $m_L$ of $\beta$ | Open GW of $\beta$ |
| | bending of broken lines | bubbling  |



Lemma 1:





$q \in P_{\varphi_{\sigma_-}}$ w/ $-r(q) \in \mathrm{Int}(\rho^{\perp}\sigma_+)$

then $q \in P_{\varphi_\rho} = P_{\varphi_{\sigma_+}} \cap P_{\varphi_{\sigma_-}}$

pf: $q \in P_{\varphi_{\sigma_-}} \iff$ ① $q = \varphi_{\sigma_-}(r(q)) + p$ , $p \in P$

By definition $\varphi_{\sigma_+} - \varphi_{\sigma_-} = \underbrace{n_\rho}_{\substack{\cap \\ N}} \otimes \underbrace{\kappa_{\rho,\varphi}}_{\in P \setminus P^*}$

anihilate $\rho$
positive on $\sigma_+$

take value on $-r(q) \Rightarrow$ ② $\varphi_{\sigma_+}(-r(q)) - \varphi_{\sigma_-}(-r(q)) = \underbrace{\langle n_\rho, -r(q)\rangle}_{\substack{\vee \\ 0}} \kappa_{\rho,\varphi}$ $\because -r(q) \in \mathrm{Int}(\rho^{\perp}\sigma_+)$

① , ② $\Longrightarrow$ $q = \varphi_{\sigma_+}(r(q)) + \underbrace{p + \langle n_\rho, -r(q)\rangle \kappa_{\rho,\varphi}}_{\hat{p}} \in P_{\varphi_{\sigma_+}}$ $\square$

Corollary 1: $\beta$ is a broken line, $t \in L$ maximal domain of linearity

$$\beta(t) \in \tau \Rightarrow q_L \in P_{\varphi_\tau} \subseteq \mathcal{P}_\tau = \mathcal{P}_{\beta(t)}.$$

pf: • $L$ : unbounded $\Longrightarrow$ $\beta(t) \in \sigma \in \Sigma_{max}$

$m_L := z^{\varphi_\sigma(q)}$ or $q_L := \varphi_\sigma(q) \in P_{\varphi_\sigma}$

• The apply above lemma & by induction. $\square$

This motivates the notion of "order".

Definition: (1) $J \subseteq P$ proper monoid ideal

$$\mathrm{ord}_J(p) = \begin{cases} \max\{k \mid p = p_1 + \cdots + p_k,\ p_i \in J\}, & p \in J \\ 0, & p \in P \setminus J \end{cases}$$

(2) $x \in \tau$, $q \in P_{\varphi_\tau}$, define $\underline{\mathrm{ord}_{J,x}(q)} := \mathrm{ord}_J(q - \varphi_\tau(r(q)))$

measure how much $q$ is above the graph of $\varphi_\tau$.

(3) $\beta$ : broken line, $t \in L$ as before

$$\mathrm{ord}_{J,\beta}(t) := \mathrm{ord}_{J,\beta(t)}(q_L)$$

Corollary 1 $\Longrightarrow$ $\beta$ : broken line, $t \in L$, then $\mathrm{ord}_{J,\beta}(t) \nearrow$.

If $\varphi$ strictly convex, then $\mathrm{ord}_{J,\beta}(t)$ jumps at those bendings.

Definition: $I$ = ideal of $P$ w/ $\sqrt{I} = J$

$\underset{\substack{\uparrow \\ \tau}}{Q} \in B$ generic $\quad Q \in B \setminus \underbrace{\mathrm{Supp}_I(\mathfrak{D})}_{= \bigcup \mathfrak{d}}$

$\mathfrak{d}$: $f_{\mathfrak{d}} \not\equiv 1 \pmod{I\,\mathbb{k}[P_{\varphi_{\tau_{\mathfrak{d}}}}]}$

$q \in B_0(\mathbb{Z})$

$$\mathrm{Lift}_Q(q) := \sum_\beta \mathrm{Mono}(\beta) \in \mathbb{k}[P_{\varphi_\tau}] / I \cdot \mathbb{k}[P_{\varphi_\tau}]$$

last monomial of $\beta$

lift from $X^\circ_{J,\mathfrak{D}}$ to $X^\circ_{I,\mathfrak{D}}$

Lemma 2: $Q \in \sigma \in \Sigma_{max}$, $q \in B_0(\mathbb{Z})$, $\sqrt{I} = J$

$\kappa_{\rho,\varphi} \in J$ for at least 1 ray $\rho \in \Sigma$

$\Longrightarrow$ (1) $\exists$ finite $\beta$ s.t. $\mathrm{Mono}(\beta) \notin I \cdot \mathbb{k}[P_{\varphi_\sigma}]$

so $\mathrm{Lift}_Q(q)$ is well-defined

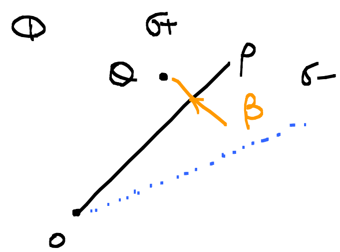
(2) One boundary of the connected component of $B \setminus \mathrm{Supp}_I(\mathfrak{D})$ containing $Q$ is a ray $\rho \in \Sigma$, then $\mathrm{Mono}(\beta) \in \mathbb{k}[P_{\varphi_\rho}]$, only finitely many $\beta$ s.t. $\mathrm{Mono}(\beta) \notin I \cdot \mathbb{k}[P_{\varphi_\rho}]$

pf: (1) $\mathbb{k}[P]$ Noetherian $\Longrightarrow \exists k \in \mathbb{N}$ s.t. $J^k \subseteq I$

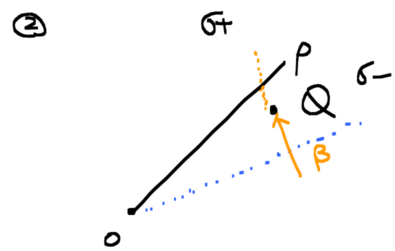
Corollary 1 $\Longrightarrow$ $\beta$ broken line s.t. $\mathrm{Mono}(\beta) \notin I \cdot \mathbb{k}[P_{\varphi_\sigma}]$ then bending of $\beta$ is uniformly bounded

Starting from $Q$ and tracing back $\beta$, there are only finitely many such.

(2) Just need to show $\mathrm{Mono}(\beta) \in \Bbbk[P_{\varphi_\rho}]$




use Lemma 1




use Lemma 1

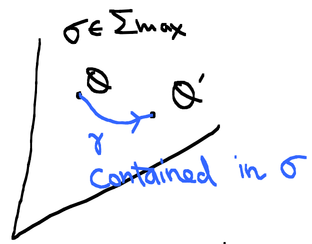
Definition: (Consistency of scattering diagram)

$K_{\rho,\varphi} \in J$ for any $\rho \in \Sigma$.

The scattering diagram $\mathfrak{D}$ is consistent if $\forall\, I \subseteq P$, $\sqrt{I} = J$

$q \in B_0(\mathbb{Z})$, $Q, Q' \in B_0$ generic st

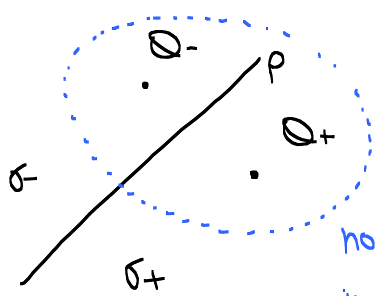
(1)




$$\mathrm{Lift}_Q(q) \in \Bbbk[P_{\varphi_\sigma}] / I \cdot \Bbbk[P_{\varphi_\sigma}] =: R_{\sigma,I}$$
$$\ni \mathrm{Lift}_{Q'}(q)$$

then $\mathrm{Lift}_{Q'}(q) = \theta_{\gamma,\mathfrak{D}}(\mathrm{Lift}_Q(q))$.

(2)




no rays of $\mathfrak{D}_I$ in this neighborhood

$\exists\, \mathrm{Lift}_\rho(q)$

$$\begin{tikzcd}
& R_{\rho,I} \ni \mathrm{Lift}_\rho(q) \arrow[dl, "\psi_{\rho,-}"'] \arrow[dr, "\psi_{\rho,+}"] & \\
\mathrm{Lift}_{Q_-}(q) \in R_{\sigma_-,I} & & R_{\sigma_+,I} \ni \mathrm{Lift}_{Q_+}(q)
\end{tikzcd}$$

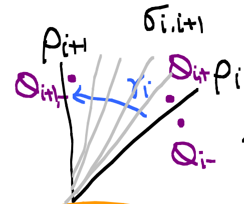
Notice that $f_{\rho_i}$ is hidden in $\psi_{\rho,\pm}$.

## Construction of theta function $\vartheta_q$

want $\vartheta_q \in \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$

Recall $X^\circ_{I,\mathfrak{D}} = \coprod_i U_{\rho_i,I} \big/ \sim$ ,

$\parallel$ Spec $R_{\rho_i,I}$




$\rightsquigarrow \theta_{\gamma_i,\mathfrak{D}} : R_{\sigma_{i,i+1},I} \to R_{\sigma_{i,i+1},I}$

$$\begin{tikzcd}
U_{\rho_i,\sigma_{i-1,i},I} \arrow[r, phantom, "\subseteq"] & U_{\rho_i,I} & U_{\rho_{i+1},I} \\
 & U_{\rho_i,\sigma_{i,i+1},I} \arrow[u, phantom, "\cup\mid"] & U_{\rho_i,\sigma_{i,i+1},I} \arrow[u, phantom, "\cup\mid"] \arrow[l, "\theta_{\gamma_i,\mathfrak{D}}"', "\cong"] \\
\mathrm{Lift}_{Q_{i,-}}(q) \arrow[uu, squiggly] & \mathrm{Spec}\, R_{\sigma_{i,i+1},I} \arrow[u, phantom, "\parallel"] & \\
 & \mathrm{Lift}_{Q_{i,+}}(q) \arrow[u, squiggly] \arrow[r, "(1)"] & \mathrm{Lift}_{Q_{i+1,-}}(q) \arrow[uu, squiggly]
\end{tikzcd}$$

(2)

$\vartheta_0 := 1$.

Therefore, consistency of scattering diagram implies $\vartheta_q$ is regular function on $X^\circ_{I,\mathfrak{D}}$.

Theorem 1: $X_I := \mathrm{Spec}\, \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}}) \xrightarrow{f_I} \mathrm{Spec}\, R_I$

$$\begin{tikzcd}
 & \mathrm{Spec}\, R_I \\
 & X^\circ_{I,\mathfrak{D}} \arrow[u]
\end{tikzcd}$$

Then (1) $X^\circ_{I,\mathfrak{D}} \underset{\text{open}}{\subseteq} X_{I,\mathfrak{D}} \supseteq \mathbb{V}_n = \mathbb{A}^2_{x_1,x_2} \cup \cdots \cup \mathbb{A}^2_{x_{n-1},x_n} \cup \mathbb{A}^2_{n,1} \subseteq \mathbb{A}^n$

$$\begin{tikzcd}
X_{I,\mathfrak{D}} \arrow[d, "f_I \ \text{flat}"] & \mathbb{V}_n \arrow[d] \\
\mathrm{Spec}\, R_I & \ni x
\end{tikzcd}$$

smallest toric stratum

$R_I = \mathbb{k}[P]/I$

(2) $\forall q \in B_0(\mathbb{Z})$, $\exists\, \vartheta_q \in \Gamma(X_I, \mathcal{O}_{X_I})$

s.t $\{\vartheta_q \mid q \in B(\mathbb{Z})\}$ free $R_I$-basis of $\Gamma(X_I, \mathcal{O}_{X_I})$.

pf: • The gluing of $X^\circ_J := \coprod_i U_{\rho_i,J} \big/ \sim$ are trivial.

$\because$ ① $K_{\rho,\varphi} \in J$, $\forall \rho \in \Sigma^{[1]}$, $R_{\rho_i,I} = \dfrac{R_I[X_{i-1}, X_i^{\pm 1}, X_{i+1}]}{(X_{i-1}X_{i+1} - z^{K_{\rho_i,\varphi}} X_i^{-D^2_{\rho_i}} f_{\rho_i})}$

② $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$, $\mathfrak{d} \in \mathrm{Int}(\sigma)$

then $f_{\mathfrak{d}} \equiv 1 \pmod J$, so $U_{\rho_i, \sigma_{i,i+1}, I} \xleftarrow{\theta_{\gamma_i,\mathfrak{D}} = id} U_{\rho_i, \sigma_{i,i+1}, I}$

In other word, $I = J \Leftrightarrow \mathfrak{D} = \text{trivial}$.

In this case, $\underline{X_J^\circ \cong (\mathrm{Spec}\, R_I[\Sigma]) \setminus \mathrm{Spec}\, R_I \times \{0\}}$ last time

$$\bigoplus_{q \in B(\mathbb{Z})} R_J \cdot \vartheta_q \xrightarrow{\cong} \Gamma(X_J^\circ, \mathcal{O}_{X_J^\circ}) \cong R_J[\Sigma]$$

$$\vartheta_q \longmapsto z^q$$

"Hartog extension"

$\exists!$ the unbending broken line

$$X_J := \mathrm{Spec}(\Gamma(X_J^\circ, \mathcal{O}_{X_J^\circ})) = \mathrm{Spec}\, R_J[\Sigma] \longrightarrow \mathrm{Spec}\, R_J$$

$$V_n \longrightarrow x, \quad V_n \subset \mathrm{Spec}\, R_J[\Sigma], \quad x \in \mathrm{Spec}\, R_J$$

- Now $I \triangleleft P$, $\sqrt{I} = J$

$$X_J^\circ \overset{i}{\hookrightarrow} X_J \rightsquigarrow X_I' = (X_J, i_* \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$$

$$\mathcal{O}_{X_I'} \to \mathcal{O}_{X_J} = R[\Sigma]$$

$$\vartheta_q \longmapsto z^q$$

Claim: $X_I' / \mathrm{Spec}\, R_I$ is a flat deformation of $X_J / \mathrm{Spec}\, R_J$

$X_J$ affine $\Longrightarrow$ $X_I' \cong X_I := \mathrm{Spec}\, \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$

thus no deformation

$$X_I' \to \mathrm{Spec}\, R_I \leftarrow X_I$$

In particular, $X_I'$ affine

- $R_I \longrightarrow \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$ flat morphism of Noetherian rings
  $A$      $B$

  $S = \{\vartheta_q \mid q \in B(\mathbb{Z})\}$ generates $R[\Sigma] = \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}}) / (J/I)$
  $B/I\cdot B$    $I$ nilpotent

  $I$ nilpotent, $S$ generates $B/IB \Rightarrow S$ generates $B$

$$\therefore \quad 0 \longrightarrow K \longrightarrow \bigoplus_{s\in S} A\cdot s \longrightarrow B \longrightarrow 0$$

$$\otimes A/I \quad \downarrow \quad 0 \to K/IK \longrightarrow \bigoplus_{s\in S} A/I \cdot s \xrightarrow{\cong} B/IB \longrightarrow 0 \quad \because B \text{ flat}/A$$

$$\Downarrow$$

$$K/IK = 0$$

$$\Downarrow \quad \because I \text{ nilpotent}$$

$$K = 0$$

Proposition 1: $X_I / \operatorname{Spec} R_I$ flat family above.

Then the relative dualizing sheaf $\omega_{X_I/\operatorname{Spec} R_I}$ is trivial.

i.e the mirrors are CYs

pf: $X_I = \coprod U_{\rho_i,I} / \sim$

$$U_{\rho_i,I} = \operatorname{Spec} \frac{R_I[X_{i-1}, X_i^{\pm 1}, X_{i+1}]}{(X_{i-1}X_{i+1} - z^{\kappa_{\rho_i,\varphi}} X_i^{-D_i^2} f_{\rho_i})} \subseteq \mathbb{A}^2_{X_{i-1},X_{i+1}} \times \mathbb{G}_{m,X_i} \times \operatorname{Spec} R_I$$

$$X_{i-1}X_{i+1} = cX_i^{-D_i^2} \Longrightarrow \frac{dX_{i-1}}{X_{i-1}} \wedge \frac{dX_i}{X_i} = \frac{dX_i}{X_i} \wedge \frac{dX_{i+1}}{X_{i+1}} =: \Omega$$

non-vanishing section of $\omega_{U_{\rho_i,I}/\operatorname{Spec} R_I}$

$$\vartheta_{\gamma,\mathfrak{D}_I}(\Omega) = \Omega \quad \rightsquigarrow \quad \text{''} \quad \omega_{X^\circ_{I,\mathfrak{D}}/\operatorname{Spec} R_I}$$

Claim: can extend over $X_I$.

Q: How to understand the algebra structure of $A_I = \Gamma(X^\circ_{I,D}, \mathcal{O}_{X^\circ_{I,D}})$ via $\{\vartheta_q \mid q \in B(\mathbb{Z})\}$ ?

Given broken line $\beta$ w.r.t $q \in B_0(\mathbb{Z})$, $Q \in B_0$

$\rightsquigarrow$ $\mathrm{Limit}(\beta) = (q, Q)$, $\mathrm{Mono}(\beta) = c(\beta) z^{\varphi_\tau(s(\beta))}$

Theorem 2: $q_1, q_2 \in B(\mathbb{Z})$, then $\vartheta_{q_1} \cdot \vartheta_{q_2} = \sum_{q \in B(\mathbb{Z})} \alpha_q \vartheta_q$,

where $\alpha_q = \sum_{\substack{(\beta_1, \beta_2) \\ \mathrm{Limits}(\beta_i) = (q_i, z) \\ s(\beta_1) + s(\beta_2) = q}} c(\beta_1) c(\beta_2)$, $z$ close to $q$.

pf: • $z$ close to $q$,

$$\mathrm{Lift}_z(q_1) \cdot \mathrm{Lift}_z(q_2) = \sum_q \alpha_q \, \mathrm{Lift}_z(q)$$

Notice that there exists only one broken line contributing to $\mathrm{Lift}_z(q)$ w/ image $z + q \cdot \mathbb{R}_{\geq 0}$.

• Counting of broken lines are locally constant.