Canonical Scattering Diagrams

$\tilde{Y}$: rational surface, $\tilde{D}$: anti-canonical cycle

C: irreducible component of $\tilde{D}$

$\beta \in A_1(\tilde{Y}, \mathbb{Z})$ s.t $\quad \beta . \tilde{D}_i = \begin{cases} k_\beta, & \tilde{D}_i = C \\ 0, & \tilde{D}_i \neq C \end{cases}$, $k_\beta > 0$

$F = \overline{\tilde{D} \setminus C}$, $\quad \tilde{Y}^\circ := \tilde{Y} \setminus F$, $\quad C^\circ := C \setminus F$

$\overline{\mathcal{M}}(\tilde{Y}^\circ / C^\circ, \beta)$ = moduli space of stable maps of genus 0 representing β, w/ tangency order k_β at a non-specific point at C°.

A^1-class

$$\text{vir. dim}\, \overline{\mathcal{M}}(\tilde{Y}^\circ / C^\circ, \beta) = \underbrace{-K_{\tilde{Y}} \cdot \beta + (\dim \tilde{Y} - 3)}_{= \text{vir. dim}\, \overline{\mathcal{M}}(\tilde{Y}, \beta)} - (k_\beta - 1)$$

Lemma 1: $\overline{\mathcal{M}}(\tilde{Y}^\circ / C^\circ, \beta)$ is proper / $\mathbb{K}$.

$\rightsquigarrow N_\beta := \int_{[\overline{\mathcal{M}}(\tilde{Y}^\circ / C^\circ, \beta)]^{vir}} 1.$

Q: difference w/ logarithmic Gromov-Witten invariants?

$(Y, D) \rightsquigarrow (B, \Sigma)$, $\varphi = (U_i, \varphi_i)$ s.t $\kappa_{\rho_i, \varphi_i} = \eta([D_i])$

$\eta : NE(Y) \longrightarrow P$

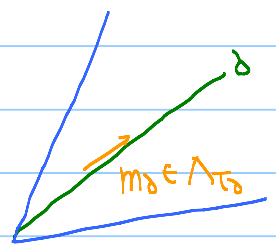
Fix $\delta \in B$ w/ rational slope.

- $\delta \in \Sigma^{[1]}$, set $\Sigma' = \Sigma$
- $\delta \notin \Sigma^{[1]}$ $\rightsquigarrow$ Σ' refinement of Σ by adding δ and other rays s.t

each chamber is integral affine isomorphic to the first quadrant.

$\rightsquigarrow$ toric blow-up $\tilde{Y} \xrightarrow{\pi} Y$

$C \subseteq \pi^{-1}(D)$ be the irreducible component corresponds to γ.



τ_γ: smallest cone containing γ

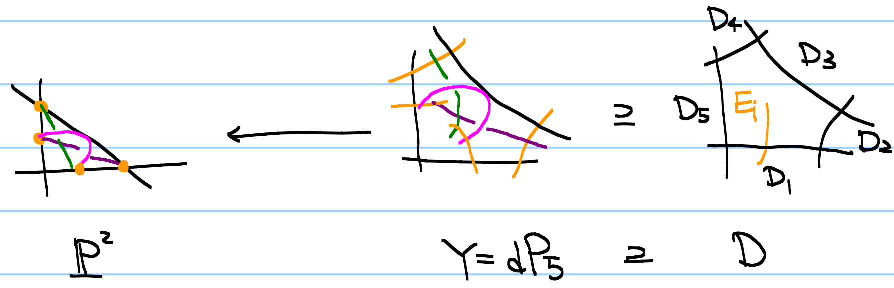
$$f_\gamma := \exp\left[\sum_\beta k_\beta N_\beta z^{\eta(\pi_* \beta) - \varphi_{\tau_\gamma}(k_\beta m_\gamma)} \right]$$

doesn't depend on the choice of refinement since Y° / C° is unchanged.

$\rightsquigarrow \mathfrak{D}^{can} := \{ (\gamma, f_\gamma) \mid \gamma \subseteq B \text{ ray of rational slope} \}$

canonical scattering diagram

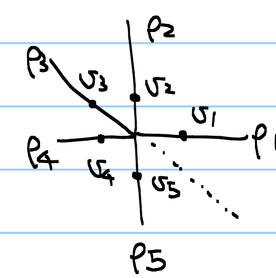
ex.



E_i : $\mathbb{A}^1$-curve intersecting D_i

$NE(Y) \xrightarrow{\eta = id} P = NE(Y)$, $J = m \triangleleft P$, $I \triangleleft P$ s.t $\sqrt{I} = J$

$$\mathfrak{D}^{can} = \{ (\rho_i, 1 + z^{[E_i] - \varphi_{\rho_i}(v_i)}) \mid 1 \le i \le 5 \}$$



- E_i are the only $\mathbb{A}^1$-curves.

Let C be a simple $\mathbb{A}^1$-curve

① C is exceptional $\Rightarrow$ $C = E_1, E_5$

② $Y \xrightarrow{\pi} \mathbb{P}^2$

$C \longmapsto \pi(C) \subseteq \mathbb{P}^2$ curve, then $\underline{\pi(C).D_1 = \pi(C).D_3 = \pi(C).D_5 > 0}$

$\deg \pi(C) = 1 \Rightarrow C = E_2, E_3, E_4$

$\deg \pi(C) \geq 2$ can't happen

$\pi(C)$ needs to pass through at least 2 blow-up points

- A k-fold multiple cover contributes to $\frac{(-1)^{k-1}}{k^2}$.

$$\therefore \; f_i = \exp\Big[\underbrace{\sum_{k\geq 1} k \frac{(-1)^{k-1}}{k^2}}\, z^{kE_i - \varphi_{P_i}(kv_i)} \Big] = 1 + z^{E_i - \varphi_{P_i}(v_i)}$$

$$\sum_{k\geq 1} k \frac{(-1)^{k-1}}{k^2} z^k = \log(1+z)$$

- If we choose $q, Q \in \sigma \in \Sigma_{max}$, then $\mathrm{Lift}_Q(q) = z^{\varphi_\sigma(q)}$.

$$\therefore \; \vartheta_q \Big|_{\substack{\text{open set of } X^{\circ}_{I,\mathrm{Bcan}} \\ \text{corresponding to } \sigma}} = z^{\varphi_\sigma(q)}$$

$$\Longrightarrow \quad \vartheta_{v_i}^a \, \vartheta_{v_{i+1}}^b = \vartheta_{av_i + bv_{i+1}}, \quad a, b \geq 0$$

In particular, $\{\vartheta_{v_i}\}_{i=1,\cdots,5}$ generates $\Gamma(X_I, \mathcal{O}_{X_I})$ as $k[P]/I$-algebra.

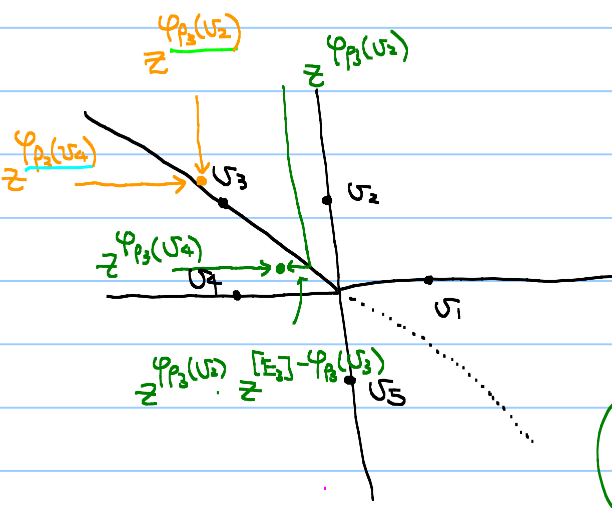
- To determine the algebra structure of $\Gamma(X_I, \mathcal{O}_{X_I})$,

it suffices to compute $\vartheta_{v_{i-1}} \vartheta_{v_{i+1}}$, $i = 1, \cdots, 5$

$$\vartheta_{v_{i-1}}\cdot\vartheta_{v_{i+1}} = z^{[D_i]}\left(\vartheta_{v_i} + z^{[E_i]}\right)$$

Recall $\quad \vartheta_{q_1}\cdot\vartheta_{q_2} = \sum_{q\in B(\mathbb{Z})} \alpha_q \vartheta_q ,$

where $\quad \alpha_q = \sum_{\substack{(\beta_1,\beta_2)\\ \mathrm{Limits}(\beta_i)=(q_i,z)\\ \delta(\beta_1)+\delta(\beta_2)=q}} C(\beta_1)\,C(\beta_2), \quad z$ close to q.

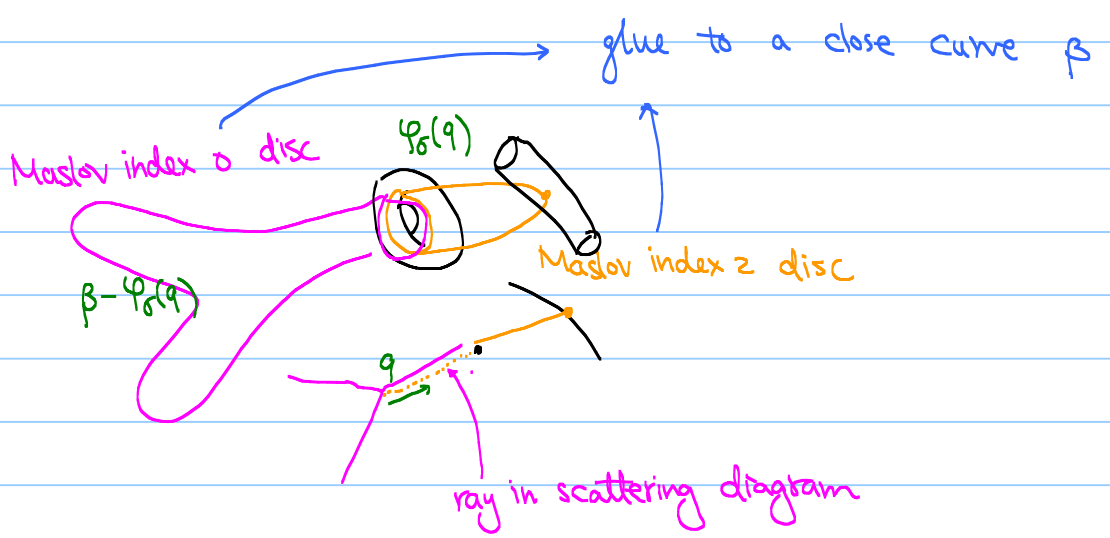


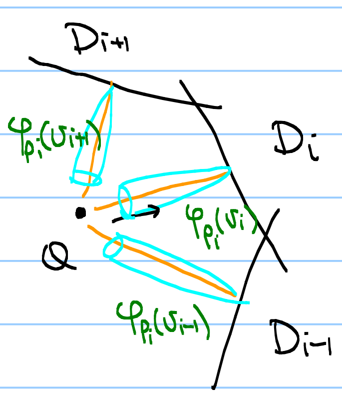
$$\vartheta_{v_2}\cdot\vartheta_{v_4} = z^{[D_3]}\vartheta_{v_3} + z^{[D_3]+[E_3]}$$

$$\varphi_{p_3}(v_2)+\varphi_{p_3}(v_4) = [D_3]+\varphi_{p_3}(v_3)$$

$$\left(\varphi_{p_3}(v_2)+[E_3]-\varphi_{p_3}(v_3)\right)+\varphi_{p_3}(v_4) = [E_3]+[D_3]$$

Remark:





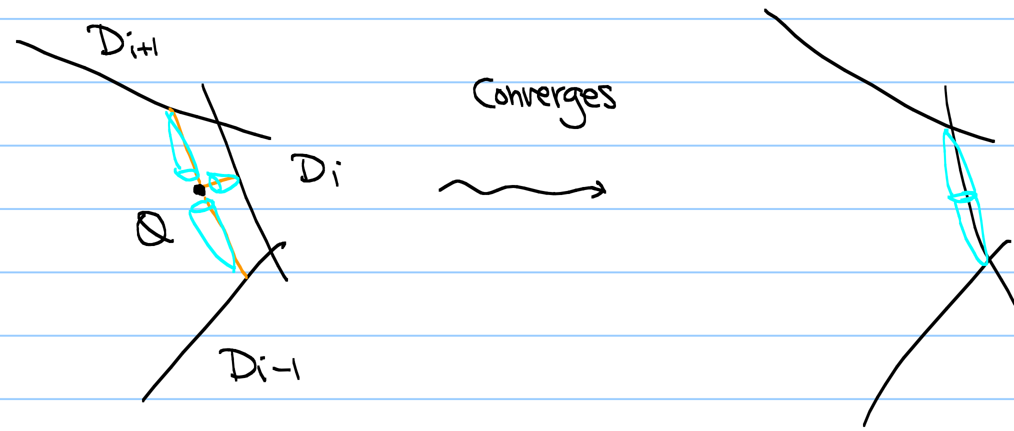
$$\partial\left(\varphi_{p_i}(v_{i+1})+\varphi_{p_i}(v_{i-1})-\varphi_{p_i}(v_i)\right)=0$$

$$\Updownarrow$$

$$v_{i+1}+v_{i-1}+(D_i)^2 v_i = 0$$

So $\varphi_{p_i}(v_{i+1}), \varphi_{p_i}(v_{i-1}), -\varphi_{p_i}(v_i)$ glue to a closed 2-cycle up to a multiple of fibres.

Move Q towards to D_i, notice that $\partial \varphi_{p_i}(v_i)$ is the vanishing cycle near D_i.



i.e $\varphi_{p_i}(v_{i-1}) + \varphi_{p_i}(v_{i+1}) - \varphi_{p_i}(v_i) = [D_i]$

in $H_2(Y,\mathbb{Z}) / H_2(L,\mathbb{Z})$

Main Theorem: Assume that

(I) $\forall$ $\mathbb{A}^1$-class β, $\eta(\pi^*(\beta)) \in J$

(II) $\forall I \triangleleft P$, $\sqrt{I} = J$, $\exists$ only finitely many $\mathbb{A}^1$-class β s.t $\eta(\pi^*(\beta)) \notin I$

(III) $\eta([D_i]) \in J$ for at least one $D_i \subseteq D$.

Then $\mathfrak{D}^{can}$ is a consistent scattering diagram.

Remark: $\mathfrak{D}^{can}$ only depends on the deformation class of (Y,D).