Restrict to the case $\dim M_{\mathbb{R}} = 2$

Def A marked tropical curve $h : (\Gamma, x_1, \cdots, x_k) \to M_{\mathbb{R}}$ is *simple* if:

(1) Γ is trivalent

(2) h is injective on the set of vertices

(3) there are no disjoint edges E_1, E_2 with a common vertex V for which $h|_{E_1}$ and $h|_{E_2}$ are non-constant and $h(E_1) \subseteq h(E_2)$

(4) Each noncpct unmarked edge of Γ has weight one.

Lemma Fix a fan Σ in $M_{\mathbb{R}}$, $\dim M_{\mathbb{R}} = 2$ and a degree $\Delta \in T_\Sigma$

Let $P_1, \cdots, P_{|\Delta|-1} \in M_{\mathbb{R}}$ be general pts. Then there are a finite number of marked genus zero tropical curves. $h : (\Gamma, x_1, \cdots, x_{|\Delta|-1}) \to M_{\mathbb{R}}$ in X_Σ with $h(x_i) = P_i$ for all i. Furthermore, these curves are simple, and there is at most one such curve of any given combinatorial type.

Interpretation:

Fix Σ a fan in $M_{\mathbb{R}}$, in $\dim M_{\mathbb{R}} = 2$ and a degree $\Delta \in T_\Sigma$

we can find a dense open subset $U \subseteq M_{\mathbb{R}}^{|\Delta|-1}$ such that for all $(P_1, \cdots, P_{|\Delta|-1}) \in U$ there are only finitely many marked genus 0 tropical curve. $h : (\Gamma, x_1, \cdots, x_{|\Delta|-1}) \to M_{\mathbb{R}}$ in X_Σ with $h(x_i) = P_i$ for all i.

Furthermore, we can choose $U \subseteq M_{\mathbb{R}}^{|\Delta|-1}$ so that all such curves are simple and there is at most one such curve of any given combinatorial type.

(proof) consists of 8 parts.

① Show that there are only a finite number of combinatorial types of degree Δ in X_Σ.

② Show that with a suitable choice of $U' \subseteq M_{\mathbb{R}}^{|\Delta|-1}$ (dense open), for each $(P_1, \cdots, P_{|\Delta|-1}) \in U$ if we fix a combinatorial type $[h]$ there can be at most one marked genus 0 tropical curve passing through $P_1, \cdots, P_{|\Delta|-1}$ at the marked edges.

③ Show that with a suitable choice of $U \subseteq M_{\mathbb{R}}^{|\Delta|-1}$ (open dense) s.t $U \subseteq U'$ all such curves are simple.

(real proof)

Step 1

(a) Show that there can be only finitely many topological type for Γ

$\Rightarrow$ · Since Γ has genus zero, it's topologically a tree.

· Δ gives the upperbound for the # of leaves of Γ ($\leq |\Delta|$)

· there are only finitely many trees with fixed # of leaves.

(b). Show that for each $E \in \Gamma^{[1]}$, $V \in E$. there can be only finitely many choices of $(m_{(V,E)}, \omega(E))$.

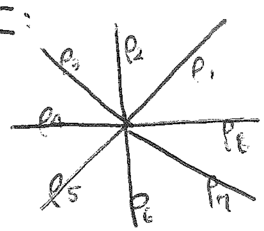
※ $m_{(V,E)} \in \mathbb{Z}^k$, $\omega(E) \in \mathbb{N}$.

$$|\omega(E)\, m_{(V,E)}| \leq \sum_{\rho \in \Sigma^{[1]}} d_\rho |m_\rho| \quad \text{— } (*)$$

↳ standard Euclidean norm on $M_{\mathbb{R}}$.

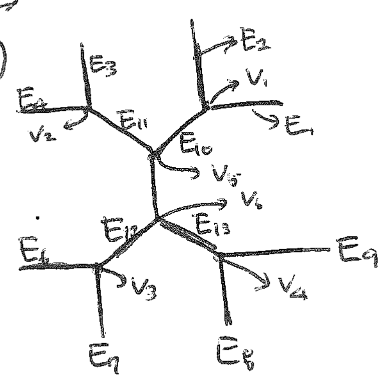
(*) Example.

Σ:



Sps $h|_{\Gamma \setminus \Gamma^{[1]}_\infty}$ is injective.

$h(\Gamma)$



$$\Delta = [\omega(E_2)+\omega(E_3)]\rho_2 + [\omega(E_4)+\omega(E_6)]\rho_4 + [\omega(E_7)+\omega(E_8)]\rho_6 + [\omega(E_1)+\omega(E_9)]\rho_8.$$

* apply triangle inequality to balancing conditions going outward until we get to non-compact edges.

$$\begin{aligned}
|\omega(E_{14})\cdot m_{(V_5,E_{14})}| &\leq |\omega(E_{11})\cdot m_{(V_5,E_{11})}| + |\omega(E_{10}) m_{(V_5,E_{10})}| \\
&= |\omega(E_{11}) m_{(V_2,E_{11})}| + |\omega(E_{10})\cdot m_{(V_1,E_{10})}| \\
&\leq \left(|\omega(E_4) m_{(V_2,E_4)}| + |\omega(E_3) m_{(V_2,E_3)}|\right) \\
&\quad + \left(|\omega(E_2) m_{(V_1,E_2)}| + |\omega(E_1) m_{(V_1,E_1)}|\right) \\
&= \omega(E_4)|m_{\rho_4}| + (\omega(E_2)+\omega(E_3))|m_{\rho_2}| \\
&\quad + \omega(E_1)|m_{\rho_8}| \\
&\leq [\omega(E_2)+\omega(E_3)]\,|m_{\rho_2}| \\
&\quad + [\omega(E_4)+\omega(E_6)]\,|m_{\rho_4}| \\
&\quad + [\omega(E_7)+\omega(E_8)]\,|m_{\rho_6}| \\
&\quad + [\omega(E_1)+\omega(E_9)]\,|m_{\rho_8}|.
\end{aligned}$$

Step 2

Recall from the theorem we proved last time in genus zero case.

$$M^{[h]}_{0,|\Delta|-1}(\Sigma,\Delta) \cong \underbrace{M_{\mathbb{R}}}_{\text{choosing where our reference vertex maps to}} \times \underbrace{\mathbb{R}_{>0}^{e+|\Delta|-4-ov(\Gamma)}}_{\text{assigning affine lengths to compact edges}}$$

evaluation map (value: where the markings $x_1 \sim x_{|\Delta|-1}$ go) the points.

$$ev : M^{[h]}_{0,|\Delta|-1}(\Sigma,\Delta) \longrightarrow (M_{\mathbb{R}})^{|\Delta|-1}$$

$$h \longmapsto (h(x_1), \cdots, h(x_{|\Delta|-1}))$$

(✱) h passes through $h(x_1),\cdots,h(x_{|\Delta|-1})$ at $x_1,\cdots,x_{|\Delta|-1}$

Explicitly $h(x_i)$'s can be obtained as below.

- Fix a path from our reference vertex V to the unique vertex attached to x_i, say,

$$V = V_0 \xrightarrow{E_1} V_1 \xrightarrow{E_2} \cdots \xrightarrow{E_n} V_n$$

※ all the E_i's are compact edges.

then $h(x_i) = h(V) + \sum_{i=1}^{n} l_{E_i}\, m_{(V_{i-1},E_i)}$.

∴ evaluation map depends affine linearly on $h(V)$ and affine lengths of cpct edges.

(i) If $\dim M^{[h]}_{0,|\Delta|-1}(\Sigma,\Delta) < \dim((M_{\mathbb{R}})^{|\Delta|-1})$

⇒ image of the evaluation map has codimension at least 1.

⇒ For general points $(P_1,\cdots,P_{|\Delta|-1}) \in (M_{\mathbb{R}})^{|\Delta|-1}$ there does not exist a trop curve passing through that points at the markings.

(ii) If $\dim M^{[h]}_{0,|\Delta|-1}(\Sigma,\Delta) \geq \dim((M_{\mathbb{R}})^{|\Delta|-1})$

i.e. $2 + e + |\Delta| - 4 - ov(\Gamma) \geq 2(|\Delta|-1)$

⇒ $e - ov(\Gamma) \geq |\Delta|$

※ $\underbrace{e}_{\#\text{ of unmarked noncpct edges}} \leq \underbrace{|\Delta|}_{\#\text{ of unmarked noncpct edges counted with weights}}$

∴ $e = |\Delta|$, $ov(\Gamma) = 0$.

⇒ every unmarked noncpct edges have weight 1. & Γ is trivalent.

(a) If $\mathrm{Im}(ev)$ has codimension at least 1.

⇒ again there's no trop curve passing through general points at the markings.

(b) If $\dim \mathrm{Im}(ev) = \dim (M_{\mathbb{R}})^{|\Delta|-1}$

⇒ ev is injective.

there can be at most 1 curve of combinatorial type $[h]$ passing through $P_1,\cdots,P_{|\Delta|-1} \in M_{\mathbb{R}}$ at the markings $x_1,\cdots,x_{|\Delta|-1}$.

STEP3

(i) for $h \in M^{[n]}_{0,|\Delta|-1}(\Sigma, \Delta)$

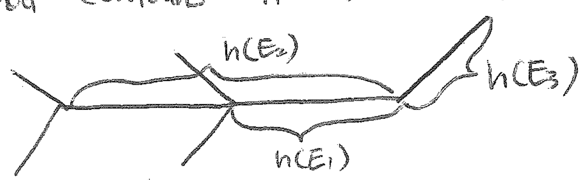
$h|_{V(\Gamma)}$ <u>not</u> injective is a closed condition

$\Rightarrow$ general curves in $M^{[n]}_{0,|\Delta|-1}(\Sigma, \Delta)$ are injective on the set of vertices

(ii) Sps V is a vertex attached to E_1, E_2.
Sps $h(E_1) \subseteq h(E_2)$

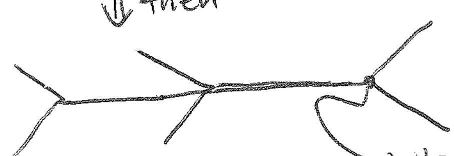
<u>Note</u> we already know that Γ is trivalent.
$\Rightarrow$ this implies for the 3^{rd} edge E_3 attached to V.
$h(E_3)$ should be contained in the same line that contains $h(E_1)$ & $h(E_2)$

o.w.



$\hookrightarrow$ this violates balancing condition.

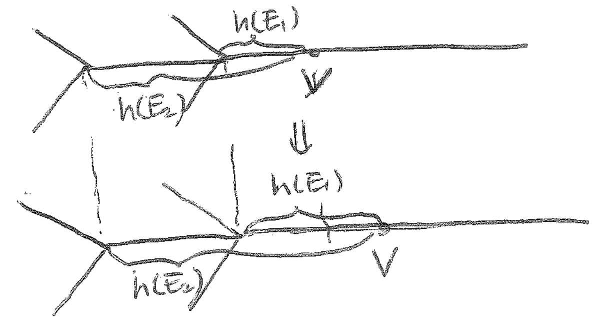
$\Downarrow$ then



$\hookrightarrow$ this vertex becomes 4-valent.
violating trivalency.

So, we can move V back & forth without changing the combinatorial type which violates the uniqueness we proved in <u>Step2</u>

<u>Ex</u>



$\therefore$ h should be simple.

Mikhalkin Index

Def Let $h: \Gamma \to M_{\mathbb{R}}$ be a simple (esp. trivalent) trop curve with $\dim M_{\mathbb{R}} = 2$.

For $V \in \Gamma^{[0]}$ with adjacent edges E_1, E_2, E_3, we define the multiplicity of h at V to be.

(i) If none of E_1, E_2, E_3 are marked edges

$$\begin{aligned} \mathrm{Mult}_V(h) &:= \omega_\Gamma(E_1)\omega_\Gamma(E_2)\,|m_{(V,E_1)} \wedge m_{(V,E_2)}| \\ &= \omega_\Gamma(E_2)\omega_\Gamma(E_3)\,|m_{(V,E_2)} \wedge m_{(V,E_3)}| \\ &= \omega_\Gamma(E_3)\omega_\Gamma(E_1)\,|m_{(V,E_3)} \wedge m_{(V,E_1)}| \end{aligned}$$

(ii) o.w. $\mathrm{Mult}_V(h) = 1$.

※ • by $|m_1 \wedge m_2|$, we are using the isomorph" $\Lambda^2 M \cong \mathbb{Z}$. Because we are taking the absolute value, choice of isomorphism does not matter.

• Equalities in (i) follows from balancing condition.

Def

• The Mikhalkin multiplicity of h:

$$\mathrm{Mult}(h) := \prod_{V \in \Gamma^{[0]}} \mathrm{Mult}_V(h)$$

• $N^{0,\mathrm{trop}}_{\Delta,\Sigma}$: For a given fan Σ, degree Δ

$$N^{0,\mathrm{trop}}_{\Delta,\Sigma} := \sum_h \mathrm{Mult}(h)$$

where the sum is taken all $M_{0,|\Delta|-1}(\Sigma,\Delta)$ passing through $|\Delta|-1$ general points in $M_{\mathbb{R}}$.

Note $N^{0,\mathrm{trop}}_{\Delta,\Sigma}$ is not well-defined yet

we do not know it does not depend on the choice of $|\Delta|-1$ general points.

Mikhalkin's Curve counting formula

Fix a fan Σ, degree $\Delta \in \ker(r)$
where $r: T_\Sigma \to M$
$t_\rho \mapsto m_\rho$ (the primitive generator of ρ)

then by the natural isomorphism

$$\begin{aligned} H_2(X_\Sigma, \mathbb{Z}) &\cong \ker(r) \\ \beta &\mapsto \sum_{\rho \in \Sigma^{(1)}} (\beta \cdot D_\rho) t_\rho. \end{aligned}$$

fixing degree amounts to fixing a homology class of X_Σ.

<u>Def</u> Sps $Q_1, \cdots, Q_{|\Delta|-1} \in X_\Sigma$ are general points.

$$N^{0,hol}_{\Delta,\Sigma} := \# \left\{ f \in \overline{M}_{0,|\Delta|-1}(X_\Sigma, \Delta) \,\middle|\, \begin{array}{l} f: (C, x_1, \cdots, x_{|\Delta|-1}) \to X_\Sigma \\ \text{is toric\-ally transverse.} \\ \text{and} \\ f(x_i) = Q_i \ \forall 1 \le i \le |\Delta|-1 \end{array} \right\}$$

- we will show, in the course of proving the curve counting formula, that $N^{0,hol}_{\Delta,\Sigma}$ is finite.

- Once we know that $N^{0,hol}_{\Delta,\Sigma} < \infty$ we can see that it does not depend on the choice of general points $Q_1 \sim Q_{|\Delta|-1}$ as the set being counted fits into one algebraic family as $Q_1, \cdots, Q_{|\Delta|-1}$ vary.

Th$\underline{\text{m}}$. Sps $\dim M_{\mathbb{R}} = 2$. Then $N^{0,hol}_{\Delta,\Sigma}$ is finite and

$$N^{0,trop}_{\Delta,\Sigma} = N^{0,hol}_{\Delta,\Sigma}$$