Counting Holomorphic Discs in K3 Surfaces

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Abstract

In the survey, we review the recent development of the reduced open Gromov-Witten invariants of K3 surfaces defined by the author. We study the wall-crossing formula, multiple cover formula for the invariants and its relation to tropical geometry.

1 Introduction

In 1987, Yau proved the Calabi conjecture and particularly showed the existence of Ricci-flat metric in any given Kähler class of a Calabi-Yau manifold [32]. The explicit expression of the Ricci-flat metric is need to write down the Lagrangian of superstring theory. However, not much is known about the behavior nor the explicit expression of the Ricci-flat metrics.

It is a classical enumerative problem to count the number of lines in a quintic threefold. The answer 2875 can be computed via intersection theory and Schubert calculus. One can ask the similar questions with lines replaced by rational curves of general degree and these numbers are later known as the genus zero Gromov-Witten invariants. The problem becomes extremely complicated as the degree increases and not solved until 1990. Candelas-de la Ossa-Green-Parkes [6] computed the genus zero Gromov-Witten invariants via mirror symmetry: surprisingly these enumerative problems can be solved by the study of variation of Hodge structures of another family of Calabi-Yau 3-folds, called the mirror of quintic 3-folds. As more and more examples of mirror symmetry are found, the enumerative problems become how to find mirrors effectively.

The Strominger-Yau-Zaslow conjecture (SYZ conjecture for short)[28] tries to answer both problems and connect the Ricci-flat metrics and the enumerative problems on Calabi-Yau manifolds together. The conjecture contains three parts: First of all, SYZ conjecture predicts the existence of special Lagrangian torus fibration on Calabi-Yau manifolds near the large complex limit point. Secondly, the mirror Calabi-Yau manifolds are given by the dual torus fibration. Moreover, it predicts that the Ricci-flat metric is semi-flat metric, which is Ricci-flat and fibrewisely flat on the special Lagrangian tori, with small instanton corrections from (pseudo)-holomorphic discs with boundary on special Lagrangian torus fibre when the Calabi-Yau is near large complex limit.

K3 surfaces are two dimensional Calabi-Yau manifolds and always admit hyperKähler structures due to the dimension reason. To understand the hyperKähler metric when the hyperKähler manifolds go to the complex limit, Gaiotto-Moore-Neitzke proposed a twistor construction of hyperKähler metric (on Hitchin moduli spaces) [4]. The main non-trivial input of the algorithm is an integer-valued invariant called generalized Donaldson-Thomas invariant, satisfying the reality condition and Kontsevich-Soibelman wall-crossing formula. This twistor construction is possible to generalized to abelian fibred hyperKähler manifolds, and it is interesting to understand the mathematical meaning of the generalized Donaldson-Thomas invariants and the relation to hyperKähler metric.

In this survey we will develop an open analogue Gromov-Witten invariants, which count holomorphic discs with boundaries on special Lagrangians in K3 surfaces. There are two difficulties at first glance: on one hand, the relevant moduli spaces have real codimension one boundary in general and the virtual fundamental class will not be well-defined. On the other hand, the virtual dimension of the relevant moduli spaces is negative and thus there is no (pseudo-)holomorhpic discs with respect to a generic almost complex structures on K3 surfaces. The key is to use hyperKähler geometry to introduce an auxiliary S^1-family of complex structures and count the holomorphic discs in the total space instead. This idea is similar to the changing tangent-obstruction theory of defining reduced Gromov-Witten invariants in algebraic geometry [26][4]. The ambiguity caused by the real codimension one boundaries of the moduli spaces can be interpreted as the wall-crossing phenomenon of the reduced open Gromov-Witten invariants as certain parameters change. Depending on whether the boundary of the holomorphic discs are homologous to zero or not, there are two corresponding wall-crossing formulas. The two different wall-crossing formula have their own applications in tropical geometry and the multiple cover formula for holomorphic discs respectively.

Tropical geometry also arises naturally in the picture of Strominger-Yau-Zaslow conjecture [28][18]. One part of Strominger-Yau-Zaslow conjecture says that the special Lagrangian fibration of a Calabi-Yau manifold will collapse to an integral affine manifold with singularities near the large complex limit. It is expected that the holomorphic curves in the Calabi-Yau manifold will converge to some 1-skeletons, known as tropical curves, of the affine manifold with singularities in the sense of Gromov-Hausdorff. The holomorphic conditions will translate into the so-called "balancing condition" of tropical curves. One advantage of the tropical geometry is that the complicated enumerative geometry problems on Calabi-Yau manifolds can broke down to combinatorics once certain corresponding theorems are established. The reduced open Gromov-Witten invariants have a tropical counter part as well. Moreover, using the invariants and its wall-crossing formula, we establish a correspondence between holomorphic discs and tropical discs.

Gromov-Witten invariants are usually interpreted as the counting of holomorphic curves with certain incidence conditions. However, this is only true if the curve classes are primitive. When the multiple covers of holomorphic curves appears, the Gromov-Witten type invariants have values in rational numbers due to the existence of automorphisms. It is interesting to ask the enumerative meaning of Gromov-Witten invariants in general. The Gopakumar-Vafa conjecture asserts that after suitable rearranging the generating functions of Gromov-Witten invariants and one can recover another sets of invariants which are integer-valued! The new integer-value invariants are equivalent to the Gromov-Witten invariants and the multiple cover formula coverts one to the other. The wall-crossing formula for the holomorphic discs with boundaries homologous to zero will lead to an open analogue of the multiple cover formula for holomorphic discs.

The survey is arranged as follows: In Section 2, we review some standard facts about hyperKähler manifolds and discuss the properties of holomorphic discs in a particular S^1-family of K3 surfaces induced from hyperKähler geometry. We will define the reduced open Gromov-Witten invariants in Section 3 and give some examples in Section 4. We will talk about the wall-crossing phenomenon of the reduced open Gromov-Witten invariants in Section 5. In section 6, we will see some applications of the wall-crossing formula, including a tropical/holomorphic discs correspondence and the multiple cover formula for reduced open Gromov-Witten invariants.

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2 Holomorphic Discs in K3 Surfaces

2.1 HyperKähler Rotation

Definition 2.1. *A K3 surface is a compact complex surface with trivial first fundamental group and trivial first Chern class.*

It is showed that every K3 surface is Kähler [27]. Together with the vanishing of first Chern class, the holonomy of a K3 surface falls in $SU(2) = Sp(1)$. Thus, every K3 surface is hyperKähler. More precisely, let X be a K3 surface and Ω be a nowhere vanishing holomorphic $(2,0)$-form guaranteed by the vanishing of first Chern class. Given any Kähler class $[\omega]$, there exists a Ricci-flat metric ω in the Kähler class $[\omega]$ such that

$$\omega^2 = c\Omega \wedge \bar{\Omega},$$

where $c \in \mathbb{R}_{>0}$ from the Calabi conjecture [32]. We will scale Ω (but still denote it Ω) with an overall constant such that $c = \frac{1}{2}$. We will call the pair (ω, Ω) a hyperKähler triple. It is a standard fact that there exists integrable complex structures J_1, J_2, J_3 satisfying quaternion relation, such that

$$\omega(\cdot,\cdot) = g(J_3\cdot,\cdot)$$

is a Kähler form and

$$\Omega(\cdot,\cdot) = g(J_1\cdot,\cdot) + ig(J_2\cdot,\cdot)$$

is a holomorphic 2-form with respect to the complex structure J_3. Moreover, the underlying space $\underline{X}$ of X admits a family of complex structures parametrized by $\mathbb{P}^1$, called twistor line. Explicitly, they are given by

$$J_\zeta = \frac{i(-\zeta+\bar{\zeta})J_1 - (\zeta+\bar{\zeta})J_2 + (1-|\zeta|^2)J_3}{1+|\zeta|^2}, \quad \zeta \in \mathbb{P}^1.$$

The holomorphic symplectic 2-forms Ω_ζ with respect to the compatible complex structure J_ζ are given by

$$\Omega_\zeta = -\frac{i}{2\zeta}\Omega + \omega - \frac{i}{2}\zeta\bar{\Omega}. \tag{1}$$

In particular, straightforward computation from (1) gives

Proposition 2.2. *Assume $\zeta = e^{i\vartheta}$, then we have*

$$\omega_\vartheta := \omega_\zeta = -Im(e^{-i\vartheta}\Omega), \tag{2}$$

$$\Omega_\vartheta := \Omega_\zeta = \omega - iRe(e^{-i\vartheta}\Omega). \tag{3}$$

Remark 2.3. *Let L be a holomorphic Lagrangian in $(\underline{X}, \omega, \Omega)$, namely, a complex submanifold of X with $dim_{\mathbb{C}} L = \frac{1}{2} dim_{\mathbb{C}} X$ and $\Omega|_L = 0$. Assume that the north and south pole of the twistor line are given by (ω, Ω) and $(-\omega, \bar{\Omega})$ respectively, making L a holomorphic Lagrangian. The hyperKähler structures corresponding to the equator $\{\zeta = e^{i\vartheta} : |\zeta| = 1\}$ make L a special Lagrangian in $X_\vartheta = (\underline{X}, \omega_\vartheta, \Omega_\vartheta)$, i.e. $\omega_\vartheta|_L = Im\Omega_\vartheta|_L = 0$ by Proposition 2.2. In particular, if $(\underline{X}, \omega, \Omega)$ admits a holomorphic Lagrangian fibration, then it induces a special Lagrangian fibrations on X_ϑ for each $\vartheta \in S^1$. This is the so-called hyperKähler rotation trick.*

2.2 Holomorphic Discs in Twistor Family

Let (X, ω, Ω) be a K3 surface X with a choice of holomorphic volume form Ω and a Ricci-flat metric ω satisfying $2\omega^2 = \Omega \wedge \bar{\Omega}$. Let L be a holomorphic Lagrangian submanifold in X. From Remark 2.3, the hyperKähler triple (ω, Ω) induces an S^1-family of hyperKähler manifolds $\mathfrak{X}^{[\omega]} = \{X_\vartheta\}$[1] containing L as a special Lagrangian submanifold. For any relative class $\gamma \in H_2(X, L)$[2], we define $\mathcal{M}_\gamma(X_\vartheta, L)$ to be the moduli space of stable discs holomorphic with respect to the complex structure of X_ϑ, with boundary on L and relative class γ. The standard index calculation shows that the virtual dimension of the moduli space is minus one. This suggests that respect to a generic almost complex structure, there is no pseudo-holomorphic discs in a K3 surface with special Lagrangian boundary condition. Since we start with the data (X, ω, Ω, L), there is no favorable $\vartheta \in S^1$ than others. It is more natural to consider the following family version of moduli space

$$\mathcal{M}_\gamma(\mathfrak{X}^{[\omega]}, L) := \bigcup_{\vartheta \in S^1} \mathcal{M}_\gamma(X_\vartheta, L),$$

which is the moduli space of the stable discs holomorphic with respect to the complex structures of X_ϑ for some $\vartheta \in S^1$. A priori, the new moduli space $\mathcal{M}_\gamma(\mathfrak{X}^{[\omega]}, L)$ may be complicated. Assume that $\mathcal{M}_\gamma(\mathfrak{X}^{[\omega]}, L) \neq \emptyset$, then there exist a holomorphic disc in X_ϑ of relative class γ for some ϑ. In particular, we have

$$\begin{aligned} 0 &= \int_\gamma \mathrm{Im}\Omega_\vartheta = -\int_\gamma \mathrm{Re}(e^{-i\vartheta}\Omega) \\ 0 &< \int_\gamma \omega_\vartheta = -\int_\gamma \mathrm{Im}(e^{-i\vartheta}\Omega). \end{aligned}$$

[1]Notice that the family $\{X_\vartheta\}$ does depend on the choice of $[\omega]$. However, we will just omit the subindex $[\omega]$ for simplicity.

[2]All the homology or cohomology in the paper are $\mathbb{Z}$-coefficient unless mentioned

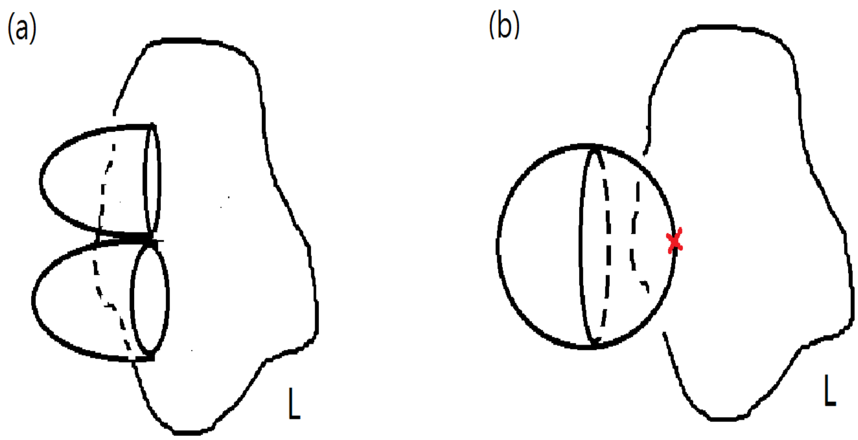

Figure 1: (a)boundary of type I (b)boundary of type II

In other words, the only X_ϑ admits holomorphic discs in the relative class γ is given by $\vartheta = \text{Arg}\int_\gamma \Omega + \frac{\pi}{2}$ if the moduli space $\mathcal{M}_\gamma(\mathfrak{X}^{[\omega]}, L) \neq \emptyset$. Although $\mathcal{M}_\gamma(\mathfrak{X}^{[\omega]}, L) \neq \emptyset$ topologically is the same as $\mathcal{M}_\gamma(X_\vartheta, L)$ for some $\vartheta \in S^1$, its virtual dimension is zero and equipped naturally with a different Kuranishi structure [20]. This observation also motivates the definition of the central charge in the next section.

In general, the moduli space $\mathcal{M}^{[\omega]}_\gamma(\mathfrak{X}, L)$ will have two kinds of real codimension one boundary (see also Figure 2.2):

1. A holomorphic disc of relative class γ can degenerate into two discs (holomorhpic to the same complex structure) of relative classes γ_1, γ_2 such that $\gamma = \gamma_1 + \gamma_2$. We will call this the real codimension one boundary of type I.

2. If $\partial\gamma$ is homologous to zero then a holomorphic disc of relative class γ without marked points can degenerate to a rational curve with one marked point on L. We will call this the real codimension one boundary of type II.

The real codimension one boundary strata will play the important role in the wall-crossing formula of the reduced open Gromov-Witten invariants we defined later. The two different kinds of real codimension one boundaries correspond to the two wall-crossing formula in Section 5.

2.3 Central Charges and the Wall of Marginal Stability

Fix a choice of holomorphic $(2,0)$-form Ω of the K3 surface X. Let $\mathrm{Def}(L)$ be the smooth deformation space of the holomorphic Lagrangian L in X. For each $u \in \mathrm{Def}(L)$, we will denote the corresponding holomorphic Lagrangian by L_u. Then there is an exact sequence of local system of lattices

$$\bigcup_{u\in \mathrm{Def}(L)} H_2(X,\mathbb{Z}) \to \Gamma := \bigcup_{u\in \mathrm{Def}(L)} H_2(X,L_u) \to \bigcup_{u\in \mathrm{Def}(L)} H_1(L_u) \to 0$$

over $\mathrm{Def}(L)$. There exists a natural function defined on Γ called the central charge:

$$\begin{aligned} Z :&\Gamma \longrightarrow \mathbb{C} \\ &\gamma_u \to Z_{\gamma_u} = \int_{\gamma_u} \Omega, \end{aligned}$$

for $\gamma_u \in H_2(X, L_u)$. The holomorphic Lagrangian condition of L guarantees that the central charge Z to be well-defined. It worth noticing that the monodromy will change sheets of Γ, so the notation γ is always local in the parameter space. One can identify the sections of Γ and $\mathrm{Def}(L)$ locally and thus induce a natural complex structure on Γ. The following lemmas followed from straightforward calculations.

Proposition 2.4. [20] *The central charge $Z : \Gamma \to \mathbb{C}$ is a holomorphic function.*

Proposition 2.5. [20] *If $\mathcal{M}_\gamma^{[\omega]}(\mathfrak{X}, L)$ is non-empty, then*

1. *$|Z_\gamma|$ is the symplectic area of the corresponding holomorphic discs, and*
2. *$\vartheta = ArgZ_\gamma + \frac{\pi}{2}$ denotes the unique $\vartheta \in S^1$ such that $\mathcal{M}_\gamma(X_\vartheta, L)$ is non-empty.*

Let $\gamma_1, \gamma_2 \in H_2(X, L)$ and

$$W'_{\gamma_1,\gamma_2} := \{u \in B_0 | \mathrm{Arg}Z_{\gamma_1}(u) = \mathrm{Arg}Z_{\gamma_2}(u)\}. \tag{4}$$

Then we define the wall of marginal stability associated to a relative class γ locally to be

$$W'_\gamma = \bigcup_{\substack{\gamma_1+\gamma_2=\gamma \\ \langle\gamma_1,\gamma_2\rangle\neq 0}} W'_{\gamma_1,\gamma_2}. \tag{5}$$

By Gromov compactness theorem, the wall of marginal stability W'_γ is a locally finite union of real codimension one and real analytic subsets of $\mathrm{Def}(L)$.

Definition 2.6. *Let $\gamma \in H_2(X, L_u)$ be a relative class, then we say γ is strongly primitive if γ can not be written in the form*

$$k\gamma' + \gamma'',$$

such that $\gamma' \in H_2(X, L_u)$, $\gamma'' \in \iota(H_2(X))$ and $\int_{\gamma''} \Omega = 0$.

Then we have the following theorem:

Theorem 2.7. [22] *Let $\gamma \in H_2(X, L_u)$ is strongly primitive. Assume that*

1. *$u \notin W'_\gamma$, and*
2. *$\langle [\omega], \tilde{\gamma} \rangle \neq 0$, for every lifting $\tilde{\gamma} \in H_2(X)$ of γ with $\tilde{\gamma}^2 \geq -2$.*

then the moduli space $\mathcal{M}^{[\omega]}_\gamma(\mathfrak{X}, L_u)$ has no boundary.

Proof. Assume that the moduli space $\mathcal{M}_{\gamma_u}(\mathfrak{X}, L_u)$ has non-empty real codimension one boundary of type I. Then there exists $\gamma_{1,u}, \gamma_{2,u} \in H_2(X, L_u)$ such that

$$\gamma_u = \gamma_{1,u} + \gamma_{2,u}$$

and $\mathcal{M}_{\gamma_{1,u}}(\mathfrak{X}, L_u)\ {}_{ev_\vartheta}\times_{ev_\vartheta} \mathcal{M}_{\gamma_{2,u}}(\mathfrak{X}, L_u)$ is non-empty, where ev_ϑ is the evaluation map to S^1. In particular, we have $Z_{\gamma_1}(u) Z_{\gamma_2}(u) \neq 0$ from Proposition 2.5 and

$$\mathrm{Arg} Z_\gamma(u) = \mathrm{Arg} Z_{\gamma_1}(u) = \mathrm{Arg}_{\gamma_2}(u) = \vartheta + \pi/2.$$

The interesting implication is that we may not always have bubbling phenomenon of the moduli space $\mathcal{M}_\gamma(\mathfrak{X}, L_u)$ unless the torus fibre L_u sits over the locus characterized by

$$\mathrm{Arg} Z_{\gamma_1} = \mathrm{Arg} Z_{\gamma_2}. \tag{6}$$

Assume that Z_{γ_1} is not a multiple of Z_{γ_2}. Since the central charges are holomorphic functions, the equation (6) locally is harmonic and defines a real analytic hypersurface. In particular, the mean value property of harmonic functions implies that locally this hypersurface divides the base into chambers. If $Z_{\gamma_1} = kZ_{\gamma_2}$, then $Z_{\gamma_1 - k\gamma_2} = 0$. In particular, $dZ_{\gamma_1 - k\gamma_2} = 0$ implies

$$\partial\gamma_1 - k\partial\gamma_2 = 0 \in H_1(L, \mathbb{Z}) \cong \mathbb{Z}^2$$

Thus, there exists positive integers $k_1 = kk_2, k_2$, and $\partial\gamma' \in H_1(L, \mathbb{Z})$, such that we have

$$\partial\gamma_i = k_i\partial\gamma' \in H_1(L, \mathbb{Z})),\ i = 1, 2.$$

and

$$\partial\gamma = \partial\gamma_1 + \partial\gamma_2 = (k_1 + k_2)\partial\gamma'.$$

Thus, $\partial\gamma$ is not primitive.

If the moduli space $\mathcal{M}_\gamma(\mathfrak{X}, L_u)$ admits a real codimension one boundary of type II, then there exists a lifting $\tilde{\gamma} \in H_2(X)$ of γ which holomorphic discs of relative class γ (with no marked points) degenerate to rational curves in class $\tilde{\gamma}$ with one marked point on L. Assume the rational curve is realized in X_ϑ, then

$$\langle [\omega], \tilde{\gamma} \rangle = \langle \mathrm{Re}\Omega_\vartheta, \tilde{\gamma} \rangle = 0.$$

$\square$

For preparation of the definition of the reduced open Gromov-Witten invariant in the next section, we need to orient the moduli space $\mathcal{M}_\gamma^{[\omega]}(\mathfrak{X}, L)$ coherently. We first recall the following theorem for coherent orientation of the moduli spaces of holomorphic discs:

Theorem 2.8. [11] *Let X be a symplectic manifold and L be its Lagrangian submanifold. Given a choice of relative spin structure of L will naturally determine an orientation of the moduli space of stable discs $\mathcal{M}_\gamma(X, L)$ for all $\gamma \in H_2(X, L)$.*

The moduli space $\mathcal{M}_\gamma^{[\omega]}(\mathfrak{X}, L)$ is virtually $\mathbb{R} \times \mathcal{M}_\gamma(X_\vartheta, L)$ for some $\vartheta \in S^1$ and thus admits a natural orientation if both factors do. The $\mathbb{R}$ factor is the linearization of $S^1 \subseteq \mathbb{C} \subseteq \mathbb{P}^1$ and thus admits a natural orientation. By Theorem 2.8, the later factor admits a natural orientation by choosing a relative spin structure of L. Notice that L is a special Lagrangian and thus is oriented. Let V be the normal bundle of L and $TL \oplus V$ is a trivial real rank 4 bundle over L and admits a trivia spin structure. These together give the data for orientation of the moduli space of the later factor. Thus, there exists a coherent orientation for the moduli space $\mathcal{M}_\gamma(\mathfrak{X}, L)$, for all $\gamma \in H_2(X, L)$.

3 Reduced Open Gromov-Witten Invariants in K3 Surfaces

Under the same assumption of Theorem 2.7, the moduli space $\mathcal{M}_\gamma(\mathfrak{X}, L_u)$ has no boundary. In particular, there is no real codimension one boundary and we are back to the situation of defining closed Gromov-Witten invariants. There exists a virtual fundamental class $[\mathcal{M}_\gamma^{[\omega]}(\mathfrak{X}, L_u)]^{vir} \in H_0(\mathcal{M}_\gamma^{[\omega]}(\mathfrak{X}, L_u), \mathbb{Q})$ [10]. We will define the reduced open Gromov-Witten invariants by

$$\tilde{\Omega}(\gamma; u) := \int_{[\mathcal{M}_\gamma^{[\omega]}(\mathfrak{X}, L_u)]^{vir}} 1. \tag{7}$$

For the general case, the invariant is defined via the de Rham model developed in [8].

Theorem 3.1. [20] *Let $\gamma \in H_2(X, L_u)$ be a relative class. Assume that $u \notin W'_\gamma$, then the invariant $\tilde{\Omega}(\gamma; u)$ is well-defined.*

Notice that the assumption in the Theorem 3.1 is an open condition. Using the cobordism argument, one has the follow basic property of the reduced open Gromov-Witten invariants.

Theorem 3.2. [20] *Let $u_0 \in B_0$ and $\gamma_{u_0} \in H_2(X, L_{u_0})$ such that $\partial\gamma \neq 0$. Assume that $u \notin W'_\gamma$, then $\tilde{\Omega}(\gamma; u)$ is well-defined in a neighborhood of u_0 and is locally constant.*

Although the S^1-family $\mathcal{M}_\gamma^{[\omega]}(\mathfrak{X}, L)$ depends on the choices of the Kähler forms, the invariant $\tilde{\Omega}(\gamma; u)$ actually does not depend on such choice. Indeed, for any two choices of Kähler class $[\omega_0], [\omega_1]$, one can find a path of Kähler classes $[\omega_t]$ connecting them because the Kähler cone is path connected. Then the natural family of moduli spaces $\mathcal{M}_\gamma(\mathfrak{X}_t, L_u)$ naturally give a cobordism between $\mathcal{M}_\gamma(\mathfrak{X}_0, L_u)$ and $\mathcal{M}_\gamma(\mathfrak{X}_1, L_u)$. Therefore, we have

Theorem 3.3. *Assume that $\partial\gamma \neq 0 \in H_1(L_u)$, then the invariant $\tilde{\Omega}(\gamma; u)$ is independent of the choice of Kähler classes.*

Any holomorphic disc in X_ϑ would also be a holomorphic disc in $X_{-\vartheta}$ after the orientation is reversed. In particular, the moduli space $\mathcal{M}_{-\gamma}(\mathfrak{X}, L_u)$ topologically is the same as $\mathcal{M}_{\gamma}(\mathfrak{X}, L_u)$. By comparing the Kuranishi structure and the orientation of the above two moduli spaces, we have the "reality condition"[3].

Theorem 3.4. [20] *Assume that $\gamma \in H_2(X, L_u)$ and $u \notin W'_\gamma$, then*

$$\tilde{\Omega}(-\gamma; u) = \tilde{\Omega}(\gamma; u). \tag{8}$$

4 Examples

4.1 Ooguri-Vafa Spaces

In this section, we will study a local model called Ooguri-Vafa space. The Ooguri-Vafa space X_{OV} is an elliptic fibration over a disc in $\mathbb{C}$ with only one singular fibre, a one-nodal rational curve, over the origin. Let $\mathcal{B} \in \mathbb{C}$ be a disc centered at the origin and Λ is a lattice in $T^*(\mathcal{B}\backslash\{0\})$ with monodromy conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ around the origin. Then X_{OV} can be realized as a partial compactification of $T^*(\mathcal{B}\backslash\{0\})/\Lambda$. The canonical symplectic 2-form of the cotangent bundle descend to the quotient and can be extended to a nowhere vanishing holomorphic $(2,0)$-form Ω_{OV} on X_{OV}.

There exists a holomorphic S^1-action on X_{OV}. Moreover, there exists an S^1-equivariant Ricci-flat metric ω_{OV} which can be written down explicitly [24]. Using the hyperKähler rotation in Remark 2.3, we derive an S^1-family of K3 surfaces $\{X_\vartheta\}$ and each K3 surface X_ϑ admits a special Lagrangian fibration. Let L_u denote the special Lagrangian torus over $u \in \mathcal{B}$. Then by the standard maximal principle argument, L_u can bound a holomorphic disc in X_ϑ if and only if u falls on the unique affine ray emanating from the origin respect to the complex affine coordinate [1][5]. Moreover, there is only a unique simple holomorphic which is the union of vanishing cycles along the affine line segment from the origin to u, which is usually known as the Lefschetz thimble (see Figure 4.1 below). All the other holomorphic discs are the multiple cover of it. The open Gromov-Witten invariants associated to the Lefschetz thimble are calculated using localization in [20].

[3] This is one of the two important properties of the generalized Donaldson-Thomas invariant for constructing hyperKähler metric in [12]

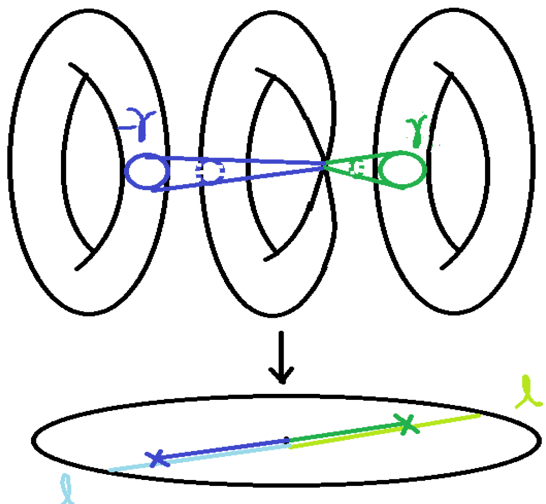

Figure 2: The Ooguri-Vafa space and its unique simple holomorphic disc.

Theorem 4.1. [20] *Let* $\gamma_e \in H_2(X, L_u)$ *denote the relative class of Lefschetz thimble, then*

$$\tilde{\Omega}(\gamma, u) = \begin{cases} \frac{(-1)^{d-1}}{d^2} & , \text{ if } \gamma = d\gamma_e,\ d \in \mathbb{Z} \\ 0 & , \text{ otherwise.} \end{cases}$$

Geometrically the part that $\tilde{\Omega}(\gamma; u) = 1$ can be understand as follows: We know that only the torus fibre over the affine line through the origin can bound holomorphic discs in X_ϑ. As ϑ varies in S^1, the affine ray also rotates and every point in $\mathcal{B}$ is swept exactly once. In other words, every torus fibre bounds a unique simple holomorphic disc (in the relative class of Lefschetz thimble) with respect to some (different) complex structure in the S^1-family $\mathfrak{X}_{OV}^{[\omega_{OV}]}$. One can write down the holomorphic disc explicitly and it is smooth. Thus the unique holomorphic disc is Fredholm regular in the family. The rest of the Theorem 4.1 is the multiple cover formula for this simple holomorphic disc.

Remark 4.2. *Notice that the invariant computed above indeed satisfies Con-*

jecture 6.9. Moreover, it suggests that

$$\Omega(\gamma,u)=\begin{cases}1 & \text{if } \gamma=\pm\gamma_e,\\ 0 & \text{otherwise,}\end{cases}$$

which gives rise to a geometric meaning of the BPS counting for the Ooguri-Vafa space [12].

4.2 Elliptic K3 Surface with I_1-Type Singular Fibres

Since we want to study holomorphic discs in K3 surfaces, it is inevitable to answer the existence of such discs. For elliptic K3 surface with I_1-type singular fibres, a tubular neighborhood of such a singular fibre topologically is the same as Ooguri-Vafa space. Thus, it is natural to guess that there is one simple holomorphic disc near each I_1-type singular fibre in the K3 surface which corresponds to the unique simple holomorphic disc in the Ooguri-Vafa space. The answer to the question is confirmative and is proved in [20]: one first constructs an approximate simple special Lagrangian disc (with its boundary on an elliptic fibre) using the behavior of hyperKähler metric on K3 surfaces near the large complex limit [14]. Then use the deformation theory of special Lagrangian with boundaries and quantitative implicit function theorem to deform the approximate special Lagrangian disc to a special Lagrangian disc, which give rise to a holomorphic disc after hyperKähler rotation.

Theorem 4.3. [20] *Let (X,ω,Ω) be an elliptic K3 surface with a holomorphic section* [4]*, where ω is the Ricci-flat metric and Ω is a nowhere vanishing holomorphic volume form. Let u be a point on the above affine ray starting at the singular point p. Assume there is no other singular point of affine structure on the affine segment between u and p. Then there exists $\epsilon_0=\epsilon_0(u)>0$ such that there exists an immersed holomorphic disc in the relative class γ_e and boundary on L_u in X'_ϑ. Here X'_ϑ is a K3 surface with special Lagrangian fibration and $\int_{L_u}\omega'<\epsilon_0$ derived from hyperKähler rotation* [5]*. Moreover, the holomorphic disc geometrically is a perturbation of the vanishing cycles over the affine ray away from the singularity.*

One can construct a family of hyperKähler metric connecting the Ooguri-Vafa metric and the true Ricci-flat metric of K3 surface near the I_1-type

[4] If we only want to know the existence of initial discs near the singular fibre then one can drop this assumption.

[5] One may view $\int_{L_u}\Omega'$ as the distance of the K3 surface X' to the large complex limit point.

singular fibre. In particular, this helps to establish a cobordism the moduli spaces of holomorphic discs with small area. In particular, the reduced open Gromov-Witten invariants of this simple disc is calculated as well as its multiple cover contribution.

Theorem 4.4. [20] *Let γ_e be the relative class of Lefschetz thimble around an I_1-type singular fibre, then given any $d_0 \in \mathbb{N}$, there exists a non-empty neighborhood $\mathcal{U}_{d_0}$ of the singularity such that for each $u \in \mathcal{U}_{d_0}$, we have*

$$\tilde{\Omega}(d\gamma_e; u) = \frac{(-1)^{d-1}}{d^2}, \text{ for every integer } d, \ |d| \leq d_0.$$

Moreover, for u close enough to the singularity, $\pm\gamma_e$ are the only classes support holomorphic discs which achieve minimum symplectic with $\tilde{\Omega}(\gamma) \neq 0$.

Remark 4.5. *We will call the simple holomorphic discs in the Theorem 4.3 the initial discs, which correspond to the initial rays in the Gross-Seibert program* [13].

4.3 A Vanishing Theorem

Let $[L] \in \mathbb{L}_{K3}$, $[L]^2 \geq -2$ and we set $\mathcal{M}'_{[L]}$ be the moduli space of marked K3 surfaces such that the curve class corresponding to $[L]$ can be realized as a smooth rational curve (see Section 5.2 for the definition). We will still denote the corresponding smooth holomorphic curve by L in the K3 surface X in $\mathcal{M}'_{[L]}$. Let $\gamma \in H_2(X, L)$ be a relative class and $\tilde{\gamma} \in H_2(X)$ be a lifting. We will denote the Poincaré dual of $\tilde{\gamma}$ by $\mathrm{PD}(\tilde{\gamma}) \in \mathbb{L}_{K3}$. Assume moreover that X is in $\mathcal{M}'_{[L]} \cap \mathcal{M}_{\mathrm{PD}(\tilde{\gamma})}$. Then for any choice of Kähler class $[\omega]$, the moduli space $\mathcal{M}^{[\omega]}_{\gamma}(\mathfrak{X}, L)$ is empty. Indeed, if there is a holomorphic disc in $\mathfrak{X}^{[\omega]}$ with relative class γ, then its symplectic area is given by

$$|\int_{\gamma} \Omega| = |\int_{\tilde{\gamma}} \Omega| = 0,$$

where Ω is the holomorphic volume form of X. In particular, this proves the vanishing of the reduced open Gromov-Witten invariants.

Theorem 4.6. [21] *With the notation above, then $\tilde{\Omega}^{[\omega]}(\gamma) = 0$.*

4.4 A 2-Elementary Surfaces

Let E be the elliptic curve given by $\mathbb{C}/\langle 1, i\rangle$, which admits a real structure τ and a holomorphic involution induced from $x \mapsto -x$ on $\mathbb{C}$. Let Y be the Kummer K3 surface associated to $E \times E$, namely, the minimal resolution of the $(E \times E)/\mathbb{Z}_2$. Then Y admits a real structure induced from τ (and we will still denote it by the same notation). Let Ω_Y be the holomorphic volume form of Y and ω_Y be a Ricci-flat Kähler form. It is straight-forward to check that $\tau^*\Omega_Y = \bar{\Omega}_Y$ and $\tau^*\omega_Y = -\omega_Y$. The fixed loci of τ are four spheres. Since τ is anti-symplectic and anti-holomorphic, the fixed locus are special Lagrangian spheres. There are in total 16 exceptional rational curves on Y. They form four groups and each group of four intersect one special Lagrangian sphere on an S^1. In other words, each special Lagrangian sphere bounds four pairs of smooth holomorphic discs. Each pair of holomorphic discs is fixed by the involution τ and glue together to an exceptional curve in Y. We will denote one of the special Lagrangian spheres by L and it bounds holomorphic discs of relative classes $\gamma_i \in H_2(Y, L)$, $i = 1, \cdots, 8$.

Let X be the K3 surface derived by hyperKähler rotation from Y such that its holomorphic volume form

$$\Omega_X = \omega_Y + i\mathrm{Im}\Omega_Y$$

and a choice of Ricci-flat Kähler form $\omega_X = \mathrm{Re}\Omega$. Then X is a 2-elementary K3 Surface with the anti-(holomorphic)symplectic involution τ and L is a smooth rational curve in X. Let $\gamma = \gamma_i$ for some $i \in \{1, 2, 3, 4\}$ then any holomorphic disc in the relative class γ in the family $\mathfrak{X}^{[\omega]}$ can be doubled to a (-2)-rational curve under the involution τ. Thus, such holomorphic disc is unique. Moreover, the holomorphic disc is smooth implies that it is Fredholm regular in the family $\mathfrak{X}^{[\omega]}$. In particular, it gives that $\tilde{\Omega}^{[\omega]}(\gamma) = 1$.

5 Wall-Crossing Phenomenon of the Reduced Open Gromov-Witten Invariants

5.1 Wall-Crossing Formula I: $\partial\gamma \neq 0$

Let $X \to B$ be an elliptic K3 surface and $u_0, u_1 \in B_0$ falls on different sides of W'_{γ_1,γ_2}. Here we assume that $\partial\gamma_1, \partial\gamma_2, \partial\gamma_1 + \partial\gamma_2$ are non-zero in $H_1(L_{u_1})$. Choose a path $u(t) \in B_0$ such that $u(0) = u_0$ and $u(1) = u_1$ passing through

W'_{γ_1,γ_2} transversally at a generic point p once. The family of moduli space

$$\bigcup_{t\in[0,1]} \mathcal{M}_{\gamma_1+\gamma_2}(\mathfrak{X}, L_{u(t)}) \tag{9}$$

forms a cobordism between $\mathcal{M}_{\gamma_1+\gamma_2}(\mathfrak{X}, L_{u_0})$ and $\mathcal{M}_{\gamma_1+\gamma_2}(\mathfrak{X}, L_{u_1})$. However, there exists an additional boundary of (9) due to the bubbling phenomenon and induces the wall-crossing formula below.

Theorem 5.1. [20] *Assume that*

1. $\gamma_1, \gamma_2 \in H_2(X, L_p)$ *are primitive classes.*
2. *There is only one splitting of the holomorphic discs* γ_1, γ_2, *i.e.,*

$$\mathcal{M}_{1,\gamma_1}(\mathfrak{X}, L_p) \times_{L\times S^1_\vartheta} \mathcal{M}_{1,\gamma_2}(\mathfrak{X}, L_p) = \mathcal{M}_{\gamma_1+\gamma_2}(\mathfrak{X}, L_p) \tag{10}$$

topologically,

then the difference of the reduced open Gromov-Witten invariants on different sides of the wall is given by

$$\tilde{\Omega}(\gamma_1+\gamma_2; u_1) - \tilde{\Omega}(\gamma_1+\gamma_2; u_0) = \pm\langle\gamma_1, \gamma_2\rangle\tilde{\Omega}(\gamma_1; p)\tilde{\Omega}(\gamma_2; p) \tag{11}$$

Here we gives an example that the holomorphic discs that can only smooth out in one side of the wall. In particular, this gives an example of non-trivial wall-crossing phenomenon of holomorphic discs in K3 surfaces.

Example 5.2. [20] *Assume there are two initial rays emanating from two I_1-type singularities of phase ϑ_0 intersect at $p \in B_0$. From Theorem 4.3, there are two initial holomorphic discs of relative classes γ_1, γ_2 corresponding to the initial rays which are Fredholm. Moreover, the two initial holomorphic discs intersect transversally in L_p. Apply automatic transversality* [31] *on K3 surfaces, these two discs cannot be smoothed out in L_p. First, we will prove that these two discs will smooth out when the Lagrangian boundary conditions vary across the wall. Pick two point p_1, p_2 near p but on the different side of wall of marginal stability W_{γ_1,γ_2}. Let $\psi : (-\epsilon, 1+\epsilon)$ be a path on B_0 such that $\psi(0) = p_1$, $\psi(1) = p_2$ and intersect W_{γ_1,γ_2} once transversally at p. Let $\mathcal{X}$ be the total space of twistor space of X with two fibres with elliptic fibration threw away. Then $L_u \times S^1_\vartheta$ is a totally real torus in $\mathcal{X}$. Now consider an complex manifold $\mathcal{X} \times \mathbb{C}$ with a totally real submanifold*

$$\mathcal{L} = \bigcup_t (L_{\psi(t)} \times S^1_\vartheta).$$

By our assumption, there are two regular simple holomorphic discs in $\mathcal{X}$ with boundaries in $L_p \times \{\vartheta_0\} \subseteq \mathcal{L}$ of relative classes again we denoted by γ_1, γ_2. The tangent of evaluation maps for both discs are two dimensional and transversal from the C^1-estimate of the Monge-Amperé equation. By Theorem 4.1.2 [3]*, these two discs can be smoothed out into simple regular discs in $\mathcal{L}$ and the union of initial holomorhpic discs are indeed the codimension one of the boundary of the usual moduli space of holomorphic discs $\mathcal{M}_{0,\gamma_1+\gamma_2}(\mathcal{X},\mathcal{L})$. By maximal principle twice, each of the holomorphic disc falls in $\mathcal{M}_{\gamma_1+\gamma_2}(\mathfrak{X}, L_{\psi(t_0)})$ for some t_0. In particular, we will have $\psi(t_0) = p$ and*

$$\mathcal{M}_{1,\gamma_1}(\mathfrak{X}, L_p) \times_{L\times S^1_\vartheta} \mathcal{M}_{1,\gamma_2}(\mathfrak{X}, L_p) \subseteq \mathcal{M}_{\gamma_1+\gamma_2}(\mathfrak{X}, \{L_t\}) \tag{12}$$

as codimension one boundary. Each point (12) smooth out to a smooth holomorphic disc. Therefore, from the Theorem 5.1

$$\Delta\tilde{\Omega}(\gamma_1+\gamma_2) = \pm\langle\gamma_1,\gamma_2\rangle\tilde{\Omega}(\gamma_1,p)\tilde{\Omega}(\gamma_2,p).$$

Assume moreover that the two I_1-type singularities on the base are closed enough to each other. If any of the smoothed holomorphic discs appear on the side of the wall

$$W_- = \{u \in B_0 || Z_{\gamma_1+\gamma_2}(u)| < |Z_{\gamma_1+\gamma_2}(p)|\}, \tag{13}$$

then $\tilde{\Omega}(\gamma) \neq 0$ on the side W_-. There exists an affine line l passing through p which $Z_{\gamma_1+\gamma_2}$ has constant phase ϑ_0 on l and $|Z_{\gamma_1+\gamma_2}|$ strictly decreasing in towards W_-. There will be a point p' on l such that $Z_{\gamma_1+\gamma_2} = 0$ and thus $\tilde{\Omega}(\gamma_1+\gamma_2;p') = 0$. This contradicts to the fact that there is no wall of marginal stability W_{γ_1,γ_2} between p and p' (see Figure 5.2).

Thus the difference of the invariant only appear the other side of the wall. Using the fact that the central charges are holomorphic functions and its Cauchy-Riemann equation, this is equivalent to the side of the wall where

$$\frac{1}{2}\frac{\langle\gamma_1,\gamma_2\rangle|Z(\gamma_1+\gamma_2)|}{Im\big[Z(\gamma_1)\bar{Z}(\gamma_2)\big]} > 0.$$

Remark 5.3. *The general wall-crossing formula later is proved by the author* [22].

Remark 5.4. *The Example 5.2 can also be viewed as the fact that special Lagrangian discs can only be smoothed out in part of the parameter space, the elliptic fibre boundary conditions. Similar structure also appears in smoothing of conical special Lagrangians in Calabi-Yau* 3*-folds* [17].

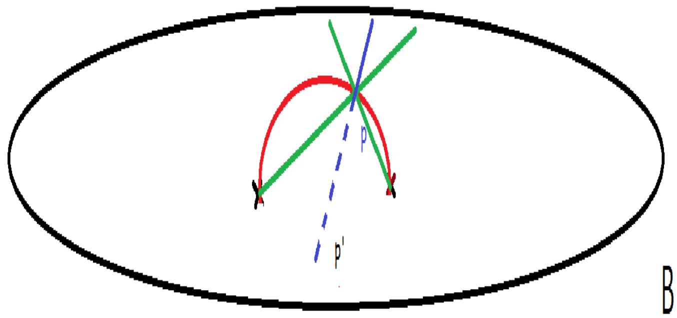

Figure 3: The red curve is the wall of marginal stability and the holomorphic disc

5.2 Wall-Crossing Formula II: $\partial\gamma = 0$

In this section, we will focus on the situation for $L \cong S^2$. In particular, the relative classes always have their boundaries homologous to zero.

In this situation, the holomorphic Lagrangian L is rigid and thus the argument in the first part of the Theorem 2.7 breaks down. Instead of $\mathrm{Def}(L)$, we consider the moduli space of marked K3 surfaces. Let $\mathbb{L}_{K3}$ be the K3 lattice

$$\mathbb{L}_{K3} = (-E_8) \oplus (-E_8) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where E_8 is the unique positive definite, unimodular even lattice of rank 8.

Definition 5.5. *A pair (X, α) is called a marked K3 surface if X is a K3 surface and $\alpha : H^2(X) \to \mathbb{L}_{K3}$ is an isomorphism of lattices.*

Now given $[L] \in \mathbb{L}_{K3}$ with $[L]^2 = -2$, we set $\mathcal{M}_{[L]}$ be the moduli space of marked K3 surfaces such that the homology class corresponds to $[L]$ can be realized as holomorphic cycles. To be precise, we set

$$\mathcal{M}_{[L]} := \{(X, \alpha) \text{ marked K3 surface} | \alpha^{-1}[L] \wedge \Omega_X = 0\} / \sim, \qquad (14)$$

where Ω_X is a holomorphic volume form of the K3 surface X. Two marked K3 surfaces (X,α) (X',α') are equivalent if and only if there exists a diffeomorphism $f: X \to X'$ such that $\alpha \circ f^* = \alpha'$. There is a Zariski open subset of $\mathcal{M}_{[L]}$, denoted by $\mathcal{M}'_{[L]}$, parametrized the marked K3 surfaces such that there exists a smooth rational curve (we will still denote it by L) in the homology class correspond to $[L]$. Then we can define the charge lattice

$$\Gamma := \bigcup_{s \in \mathcal{M}'_{[L]}} H_2(X_s, L) \cong \mathcal{M}'_{[L]} \times \big(H_2(X)/\mathrm{Im} H_2(L)\big)$$

is a trivial local system over $\mathcal{M}'_{[L]}$ and the central charge locally to be

$$\begin{aligned} Z : \Gamma \cong \mathcal{M}'_{[L]} \times \big(H_2(X)/\mathrm{Im} H_2(L)\big) &\to \mathbb{C} \\ (s,\gamma) \longmapsto Z_\gamma(s) = \int_\gamma &\Omega_s \end{aligned}$$

by choosing a local section Ω_s of holomorphic volume forms over $\mathcal{M}'_{[L]}$. Although the definition of the central charge depends the choices of local section but the locus defined by equation

$$\mathrm{Arg} Z_{\gamma_1} = \mathrm{Arg} Z_{\gamma_2}$$

is well-defined.

The dependence of the invariant on the choice of the Kähler class $[\omega]$ is given by the following theorem.

Theorem 5.6. [21] *Assume that there is a 1-parameter family of Kähler classes $[\omega_t], t \in [-\epsilon, \epsilon]$ goes across a single valid hyperplane labeled by $\tilde{\gamma}$ at $t = 0$. Then the wall-crossing formula for crossing the hyperplane labeled by $\tilde{\gamma}$ is given by*

$$\Delta\tilde{\Omega}(\gamma) := \tilde{\Omega}^{[\omega_\epsilon]}(s_\epsilon, \gamma) - \tilde{\Omega}^{[\omega_{-\epsilon}]}(s_{-\epsilon}, \gamma) = \pm([L]\cdot\tilde{\gamma}) GW_{red}(\tilde{\gamma}), \tag{15}$$

where $([L]\cdot\tilde{\gamma})$ is the intersection pairing in $\mathbb{L}^\vee_{K3}$ and $GW_{red}(\tilde{\gamma})$ denotes the reduced Gromov-Witten invariants associated to $\tilde{\gamma}$. The sign in (15) is given by

$$\frac{1}{2}\big(sgn(\omega_{-\epsilon}\cdot\tilde{\gamma}) - sgn(\omega_\epsilon\cdot\tilde{\gamma})\big). \tag{16}$$

The Theorem 4.6 together with Theorem 5.6 then provide a closed formula to compute the reduced open Gromov-Witten invariants.

Theorem 5.7. [21] *Let $\gamma \in H_2(X, L)$, then*

$$\tilde{\Omega}^{[\omega]}(\tilde{\gamma}) = \sum_{\tilde{\gamma}:\iota(\tilde{\gamma})=\gamma} \pm([L]\cdot\tilde{\gamma})GW_{red}(\tilde{\gamma}), \tag{17}$$

*where the sign in given in (**??**).*

A direct application of Theorem 5.7 is another proof of the "reality condition". Indeed, if we replace γ by $-\gamma$, then exactly the $\pm$ and $([L]\cdot\tilde{\gamma})$ change the sign and all the other terms remain the same.

6 Applications

6.1 Correspondence between Tropical Discs and Holomorphic Discs

In this section, we will illustrate the application of the reduced open Gromov-Witten invariant in tropical geometry. We will show that for any relative class with non-trivial reduced open Gromov-Witten invariants, there exists a corresponding tropical disc. Moreover, the tropical discs counting and the reduced Gromov-Witten invariants are the same is equivalent to the Kontsevich-Soibelman wall-crossing formula for the reduced open Gromov-Witten invariants.

We first review the definition of tropical curves.

Definition 6.1. *A manifold B of dimension n is called an integral affine manifold if it admits a collection of coordinate charts such that its transition functions falls in $GL_n(\mathbb{Z}) \ltimes \mathbb{R}^2$. A manifold B is called an integral affine manifold with singularities if there exists a subset Δ of codimension at least two such that $B\backslash\Delta$ is an integral affine manifold.*

Example 6.2. [16] *Let $X \to B$ be a special Lagrangian fibration (without singular fibres), then there exists two affine structures on B , known as the symplectic affine structure and the complex affine structure.*

Definition 6.3. *Let B be a two dimensional integral affine manifold with singularity and B_0 be the complement of the singularities Δ. A tropical curve (tropical disc) on B is a 3-tuple (ϕ, G, w) where G is a rooted connected graph (with a root x). We denote the set of vertices and edges by $G^{[0]}$ and $G^{[1]}$ respectively, with a weight function $w : G^{[1]} \to \mathbb{N}$. And $\phi : G \to B$ is a continuous map such that*

1. *We allow G to have unbounded edges only when B is non-compact.*

2. *For each $e \in G^{[1]}$, $\phi|_e$ is either an embedding of affine segment on B_0 or $\phi|_e$ is a constant map. In the late case, e is associated with a non-zero tangent direction (up to sign).*

3. *For the root x, $\phi(x) \in B_0$.*

4. *For each $v \in G^{[0]}$, $v \neq x$ and $val(v) = 1$, we have $\phi(v) \in \Delta$. Moreover, if the monodromy of the affine structure at $\phi(v)$ is conjugate to $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), n \in \mathbb{Z}$* [6]*, then the image of edge adjacent to v is in the monodromy invariant direction.*

5. *For each $v \in G^{[0]}$, $val(v) \geq 1$, we have the following assumption:*

   (balancing condition) Each outgoing tangent at u along the image of each edge adjacent to v is rational with respect to the natural integral structure on $T_{\phi(v)}B$. Denote the outgoing primitive tangent vectors by v_i, then

$$\sum_i w_i v_i = 0.$$

Remark 6.4. [20] *Given an elliptic K3 surface $X \to B$ and a choice of Kähler class, the hyperKähler rotation provides an S^1-family of K3 surfaces with special Lagrangian fibration. From the Example 6.2, the above data induce an S^1-family of integral affine structures with singularities on B. Let $u \in B_0$ and denote the fibre over u by L_u. Then there exists an associate relative class $[\phi]$ in $H_2(X, L_u)$ for every tropical discs ϕ with stop at u. There is an tropical discs counting invariant $\tilde{\Omega}^{trop}(\gamma; u)$, which is a weighted counts*[7] *of tropical discs with stop at u and with respect to one of the above S^1-family of affine structures on B, for each relative class $\gamma \in H_2(X, L_u)$. Moreover, the tropical discs counting invariants satisfy the Kontsevich-Soibelman wall-crossing formula.*

Now we can state the correspondence theorem in [20]:

Theorem 6.5. *Let X be an elliptic K3 surface (singular fibres not necessarily of I_1-type). For every relative class $\gamma \in H_2(X, L_u)$ with $\tilde{\Omega}(\gamma; u) \neq 0$,*

[6] Straightforward computation shows that the monodromy invariant direction of the affine structure is rational. There might be other constraints for other kind of singularities. In other words, there might be tropical curve in this definition which is not coming from geometry.

[7] It worth noticing that the weight is related to the weight introduced by Milkalkin [25] but not the same one

there exists a tropical disc ϕ such that $[\phi] = \gamma$. Moreover, the symplectic area of the holomorphic disc is just the total affine length of the corresponding tropical disc.

Proof. (Sketch) Assume that L_{u_0} is a torus fibre bounds a holomorphic discs in the relative class $\gamma \in H_2(X, L_{u_0})$ and $\tilde{\Omega}(\gamma; u_0) \neq 0$. From Proposition 2.5, the holomorphic disc is realized in X_ϑ, $\vartheta = \mathrm{Arg}Z_\gamma(u)$. There exists an affine ray l on B_ϑ emanate from u_0 such that

1. $|Z_\gamma|$ is decreasing along l and
2. $\mathrm{Arg}Z_\gamma$ remains constant along l.

There is some point u' on l the function $|Z_\gamma|$ decreased to zero. Then there are two situation:

1. The invariant $\tilde{\Omega}(\gamma; u)$ is constant along the line. In this situation, there are holomorphic discs of arbitrary small symplectic area of boundary on L_u, where u is closed enough to u'. From gradient estimate, the point u' is a singularity of the affine structure and the relative γ (up to parallel transport) is the Lefschetz thimble.
2. If the invariant $\tilde{\Omega}(\gamma; u)$ jumps somewhere along l, say first jump at $u' \in B_0$, then $u' \in W_\gamma$. Moreover, one can show that there exists $\gamma_i \in H_2(X, L_{u'})$ and integers n_i such that
$$\gamma = \sum_i n_i \gamma_i \tag{18}$$
and $\tilde{\Omega}(\gamma_i; u') \neq 0$. Then we can replace the holomorphic discs of relative class γ by each γ_i and run the same procedure.

From Gromov compactness theorem, the procedure will end with finitely many splittings. The union of the affine rays give the tropical disc with stop at u and the equation (18) translates into the balancing condition of the tropical discs [8]. □

It is natural to conjecture that the tropical discs counting invariants coincide with the reduced open Gromov-Witten invariants.

Conjecture 6.6. *Given any relative class $\gamma \in H_2(X, L_u)$, then*
$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}^{trop}(\gamma; u).$$

[8]It worth mentioning that the similar mechanism also appears in the (split) attractor flows of black holes [7]

Since the tropical discs counting invariant $\tilde{\Omega}^{trop}(\gamma; u)$ satisfies the Kontsevich-Soibelman wall-crossing formula, the Conjecture 6.6 is equivalent to the Kontsevich-Soibelman wall-crossing formula for the reduced open Gromov-Witten invariants $\tilde{\Omega}(\gamma; u)$.

6.2 Multiple Cover Formula for Holomorphic Discs

In the section, we will prove the multiple cover formula for the reduced Gromov-Witten invariants using the closed formula developed in Theorem 5.7.

Gromov-Witten invariants naively are the counting of (pseudo-)holomorphic curves with certain incidence conditions in a target symplectic manifold. When the curve class is primitive, one can slightly perturb the almost complex structures such that the Gromov-Witten invariants are the true enumerative counting. However, the Gromov-Witten invariants might become rational numbers when the curve class is not primitive due to the presence of the automorphism of the relevant moduli spaces. Therefore, it is a natural question to ask whether there is a geometrically enumerative interpretation for Gromov-Witten invariants. The Gopakumar-Vafa conjecture suggests that one might want to consider another equivalent invariants to answer the question.

Conjecture 6.7. [15] *Let X be a Calabi-Yau 3-fold. Let N_d^g denote the Gromov-Witten invariants of genus g curves of degree d. If we define the sets of number n_d^g by the formula*

$$\sum_{\beta\neq 0}\sum_{g\geq 0} N_d^g t^{2g-2} q^d = \sum_{\beta\neq 0}\sum_{g\geq 0} n_d^g \sum_{k>0} \frac{1}{k}\big(2\sin\frac{kt}{2}\big)^{2g-2} q^{kd}, \tag{19}$$

then $n_d^g\in\mathbb{Z}$. In particular, we consider only the genus zero Gromov-Witten invariants, we have

$$N_d^0 = \sum_{d|k} n^0_{\frac{d}{k}} k^{-3}. \tag{20}$$

The equation (20) is known as the Aspinwall-Morrison formula [2][30][23]. It worth to mention that Taubes also had the similar point of view connecting Gromov invariants to the solutions of Seiberg-Witten equations [29].

Although the Gromov-Witten invariants of K3 surfaces vanish for all genus, one can change the tangent-obstruction theory to define the reduced Gromov-Witten invariants [33][4][19][26]. Let G_d be number of rational curves of with d nodes in a generic algebraic K3 of genus d. Then the multiple cover formula for genus zero reduced Gromov-Witten invariants is similar to that of Gromov-Witten invariants for Calabi-Yau 3-folds.

Theorem 6.8. [15][26] *Let $\mathbb{L}_{K3}$ be the K3 lattice. Given a class $\beta \in \mathbb{L}_{K3}$ and denote the genus zero reduced Gromov-Witten invariant associated to (the Poincaré dual of) β by n_β, then*

$$n_\beta = \sum_d \frac{1}{d^3} G_{\frac{1}{2}(\frac{\beta}{d})^2+1}. \tag{21}$$

Here we set $G_d = 0$ if d is not an integer.

There is a similar multiple cover formula for holomorphic discs conjectured by [11][9] with $\frac{1}{d^3}$ replaced by $\pm\frac{1}{d^2}$. Since the philosophy that the reduced theory of K3 surfaces is similar to the original theory of Calabi-Yau 3-folds, we have the following conjecture for multiple cover formula for reduced open Gromov-Witten invariants on K3 surfaces:

Conjecture 6.9. *For any choice of the Kähler form, there exists a collection of integers $\{\Omega^{[\omega]}(\gamma) \in \mathbb{Z}\}$ such that*

$$\tilde{\Omega}^{[\omega]} = \sum_{d|\gamma} \pm \frac{1}{d^2} \Omega^{[\omega]}(\gamma/d). \tag{22}$$

In the previous section, it is shown that the conjecture holds for the Ooguri-Vafa space, which is a local model for K3 surfaces. Now we are ready for the main theorem of this section:

Theorem 6.10. *The Conjecture 6.9 holds for the case when L is a sphere with all $\pm$ in (22) are taken to be positive.*

Proof. Since for any $\gamma \in H_2(X, L)$, there exists a Kähler class $[\omega]$ (with respect to some complex structure such that L is holomorphic Lagrangian) such that $\tilde{\Omega}^{[\omega]}(\gamma) = 0$, which obviously obeys the multiple cover formula holds. It suffices to prove that all the wall-crossing terms in Theorem 5.6 also satisfy (22). Indeed, from Theorem 5.6 and (21), we have

$$\begin{aligned}
\Delta\tilde{\Omega}(\gamma) &= \pm([L].\tilde{\gamma})n_{\tilde{\gamma}} \\
&= \pm([L].\tilde{\gamma}) \sum_k \frac{1}{k^3} G_{\frac{1}{2}(\frac{\tilde{\gamma}}{k})^2+1} \\
&= \pm \sum_k \frac{1}{k^2}([L].\frac{\tilde{\gamma}}{k}) G_{\frac{1}{2}(\frac{\tilde{\gamma}}{k})^2+1}.
\end{aligned}$$

Therefore,

$$\Delta\Omega(\gamma) = \pm([L].\frac{\tilde{\gamma}}{k})G_{\frac{1}{2}(\frac{\tilde{\gamma}}{k})^2+1} \tag{23}$$

are integers from Yau-Zaslow formula. □

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