On the Refined Open Gromov-Witten Invariants on K3 Surfaces

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Abstract

The paper is a survey of the author's recent work on the correspondence theorem of open Gromov-Witten invariants and the tropical discs counting invariants on K3 surfaces. In particular, both invariants satisfy the Kontsevich-Soibelman wall-crossing formula. One can furthermore prove that the tropical discs counting invariants admits a q-deformation and satisfy the q-deformed wall-crossing formula. In particular, this suggests that the open Gromov-Witten invariants might admit a q-deformation as well.

Keywords: Open Gromov-Witten invariants, K3 surfaces, Refined generalized Donaldson-Thomas invariants

1 Introduction

The story starts from the Strominger-Yau-Zaslow conjecture [26] which predicts that a Calabi-Yau manifold X would admit a special Lagrangian torus fibration near large complex limits. The mirror $\check{X}$ can be constructed by topologically taking the dual torus fibration and the complex structure of $\check{X}$ is quantum corrected by the holomorphic discs in X. The recipe of constructing the mirror is successfully realized in non-Archimedean geometry [17][9].

On the other hand, it is conjectured that the Calabi-Yau manifold X will collapse to an affine manifold with singularities near large complex limit [19]. The statement is verified in the case of certain K3 surface [12] and later generalized to certain hyperKähler manifolds [11]. Conjecturely, the holomorphic curves in X collapse to certain 1-skeletons on the affine manifold with singularities known as

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tropical curves. Although it looks like the collapsing lost much of geometric information of X, the enumerative aspects are preserved. Mikhalkin first carried out the picture on the $\mathbb{C}^2$ inside smooth toric surfaces [23]. The different toric compactifications of $\mathbb{C}^2$ correspond to the possible directions of unbounded edges of tropical curves. Moreover, the counting of Riemann surfaces in a smooth toric surface with constraints is equal to the weighted count of trivalent tropical curves on $\mathbb{R}^2$ with constraints. It is later generalized to counting Riemann surface of genus zero in toric manifolds of all dimension by Nishinou-Siebert [25]. The tropical geometry is generalized to degeneration of schemes and a similar statement for counting holomorphic cylinders was derived by Tony Yue Yu [28] in the context of non-Archimedean geometry. However, due to the lack of explicit expression of Ricci-flat metric of Calabi-Yau manifolds, there is not much known in this aspects in Calabi-Yau manifolds over $\mathbb{C}$.

The survey is arranged as follows: we first review the preliminary knowledge of hyperKähler geometry in Section 2. We review the tropical geometry of K3 surface in Section 3. In particular, we explain the what are "admissible" tropical discs and the associated weights which lead to a tropical discs counting invariant. In Section 3, we define the open Gromov-Witten invariants on K3 surfaces, naively counting holomorphic discs with boundaries on special Lagrangian torus fibres. In Section 4, the local geometry of a singular single nodal fibre in a K3 and the two invariants coincides when the boundary condition is near the singular fibre. In Section 5, we discuss the correspondence theorem between open Gromov-Witten invariants and tropical discs counting invariants. Finally, inspired by the q-deformed wall-crossing formula, one can get a q-deformed tropical discs counting invariants in Section 6. In particular, we expect a q-deformation of open Gromov-Witten invariants as well.

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2 HyperKähler Geometry

Definition 2.1. *1. A holomorphic symplectic form Ω on a complex manifold X is a d-closed, non-degenerate holomorphic 2-form.*

2. *A Kähler manifold X of $\dim_{\mathbb{C}} X = 2n$ is a hyperKähler manifold if it admits a holomorphic symplectic 2-form Ω and a Kähler form ω such that*

$$\omega^{2n} = c\Omega^n \wedge \bar{\Omega}^n, \tag{1}$$

for some constant $c > 0$.

Example 2.2. *By Yau's theorem* [27], *every compact Kähler manifold admits a holomorphic symplectic 2-form is hyperKähler. In particular, every K3 surface is hyperKähler.*

Given a hyperKähler manifold X and its Kähler form ω, holomorphic symplectic form Ω satisfying (1), the pair (ω, Ω) induces an S^2-family of hyperKähler structures on the underlying space of X given by

$$\begin{aligned}
\omega_\zeta &= \frac{i(-\zeta + \bar{\zeta})\mathrm{Re}\Omega - (\zeta + \bar{\zeta})\mathrm{Im}\Omega + (1 - |\zeta|^2)\omega}{1 + |\zeta|^2}, \\
\Omega_\zeta &= -\frac{i}{2\zeta}\Omega + \omega - \frac{i}{2}\zeta\bar{\Omega},
\end{aligned}$$

for each $\zeta \in \mathbb{P}^1$. The d-closedness of Ω_ζ together with the Newlander-Nirenberg theorem imply that the almost complex structure determined by Ω_ζ is actually integrable. In particular, when $\zeta = e^{i\vartheta}, \vartheta \in S^1$, we have

$$\begin{aligned}
\omega_\vartheta &:= \omega_{e^{i\vartheta}} = -\mathrm{Im}(e^{-i\vartheta}\Omega) \\
\Omega_\vartheta &:= \Omega_{e^{i\vartheta}} = \omega - \mathrm{Re}(e^{-i\vartheta}\Omega).
\end{aligned}$$

We will denote the hyperKähler manifold with Kähler form ω_ϑ and holomorphic symplectic 2-form Ω_ϑ by X_ϑ. A smooth holomorphic Lagrangian L in X is a half dimensional complex submanifold with the restriction of holomorphic symplectic form vanishes, $\Omega|_L = 0$. Equivalently, we have $\omega_\vartheta|_L = \mathrm{Im}\Omega_\vartheta|_L = 0$ and thus L is a special Lagrangian submanifold in $X_\vartheta, \vartheta \in S^1$. To sum up, the holomorphic Lagrangians in a hyperKähler manifold (X, ω, Ω) become special Lagrangian submanifolds with respect to another hyperKähler structure $(X_\vartheta, \omega_\vartheta, \Omega_\vartheta), \vartheta \in S^1$ on the same underlying space and vice versa. In particular, if $X \to B$ is a holomorphic Lagrangian fibration then $X_\vartheta \to B$ is a special Lagrangian fibration. This is known as the hyperKähler rotation trick. It worth noticing that the roles of holomorphic Lagrangian and special Lagrangian are not completely symmetric.

Remark 2.3. *For a compact hyperKähler manifold, the holomorphic symplectic 2-form is unique up to a $\mathbb{C}^*$-scaling. Given a fixed choice of the Kähler class $[\omega]$, the S^1-family of hyperKähler manifold $\{X_\vartheta\}_{\vartheta \in S^1}$ does not depend on the choice of* *the holomorphic symplectic* 2*-form* Ω*. Although, the* S^1*-family does depend on the choice of the Kähler class* $[\omega]$*, the invariants we defined later in the paper does not (See Theorem 4.2).*

In the rest part of the paper, we will always assume that $X \to B$ is an elliptic K3 surface and B_0 is the complement of the discriminant locus in B. For each $u \in B$, we denote the fibre over u by L_u.

3 Tropical Geometry on K3 Surfaces

To talk about tropical geometry for elliptic K3 surface, one need an integral affine structure on the base of the fibration. For each $\vartheta \in S^1$, the special Lagrangian fibration on $X_\vartheta \to B$ induces the complex affine structure on B_0 [14]. Explicitly, the affine coordinates can be understood via introducing the central charge function:

Definition 3.1. *The central charge Z is a* $\mathbb{C}$*-valued function on the local system of lattices*

$$\begin{aligned} Z : \Gamma = \bigcup_{u \in B_0} H_2(X, L_u; \mathbb{Z}) &\to \mathbb{C} \\ \gamma_u &\longmapsto Z_\gamma(u) := \int_\gamma \Omega. \end{aligned}$$

The integral in Definition 3.1 is well-defined because $\Omega|_{L_u} = 0$.

Proposition 3.2. [20]

1. *The central charge Z is a holomorphic function on* Γ[1]*.*

2. *If* $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$ *is non-empty, then topologically it is* $\mathscr{M}_\gamma(X_\vartheta, L_u)$ *and* $\vartheta = ArgZ_\gamma(u)$*.*

Given $u_0 \in B_0$ and choose $\gamma_1, \gamma_2 \in H_2(X, L_{u_0})$ such that $\partial\gamma_1, \partial\gamma_2$ generate $H_1(L_{u_0})$. Then $\{f_i(u) = \mathrm{Re}(e^{-i\vartheta} Z_{\gamma_i}(u))\}_{i=1,2}$ give integral affine coordinates near u_0. We will denote the base with this integral affine structure by B_ϑ. One observation is the following:

Proposition 3.3. [20] *If* $L_{u(t)}$ *is a* 1*-parameter family of special Lagrangian torus bounding holomorphic discs of relative class* γ *in* X_ϑ*, then* $u(t)$ *is characterized by an affine line in* B_ϑ*. In other words, the locus of special Lagrangian tori bounding holomorphic discs of a fixed relative class locally form an affine line.*

[1] Here we identify Γ and B_0 locally

Proof. Assume that $L_{u(t)}$ bounds a holomorphic disc in the relative class γ. Write $\gamma = a\gamma_1 + b\gamma_2 + \gamma_0$, where $\gamma_0 \in H_2(X,\mathbb{Z})$. Then we have

$$\begin{aligned}
0 &= \int_{\gamma_{u(t)}} \mathrm{Im}(\Omega_\vartheta) \\
&= \int_{a\gamma_{1,u(t)}+b\gamma_{2,u(t)}+\gamma_0} \mathrm{Re}(e^{-i\vartheta}\Omega) \\
&= af_1(u(t)) + bf_2(u(t)) + \int_{\gamma_0} \mathrm{Re}(e^{-i\vartheta}\Omega).
\end{aligned}$$

The last term is a constant and thus the proposition follows. □

Assume that the elliptic K3 surface X admits only I_1-type singular fibres, now we will define tropical discs on B_ϑ, modified from [6][24]:

Definition 3.4. *Let B be an affine 2-manifold with at singularities Δ and with integral structure on TB. In other words, there exists an integral affine structures on $B\backslash\Delta$. Assume that around each singularity of the affine structure the monodromy is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let B_0 be the complement of the singularities Δ. A tropical curve (with stop) on B is a 3-tuple (ϕ, T, w) where G is a rooted connected graph (with a root x). We denote the set of vertices and edges by $C_0(T)$ and $C_1(T)$ respectively. Then the weight function $w : C_1(T) \to \mathbb{N}$ and the continuous map $\phi : T \to B$ satisfy the following:*

1. *We allow G to have unbounded edges only when B is non-compact.*
2. *For any vertex $v \in C_0(T)$, the unique edge e_v closest to the stop is called the outgoing edge of v and $w_v := w(e_v)$.*
3. *For each $e \in C_1(T)$, $\phi|_e$ is either an embedding of affine segment on B_0 or $\phi|_e$ is a constant map. In the later case, e is associated with an integral primitive tangent vector at $\phi(e)$ (up to sign) if $\phi(e) \notin \Delta$. The edge adjacent to x is not contracted by ϕ.*
4. *For each $v \in C_0(T)$, $v \neq x$ and $val(v) = 1$, we have $\phi(v) \in \Delta$. Moreover,*
   - (a) *If $\phi|_{e_v}$ is an embedding, then $\phi(e_v)$ in the monodromy invariant direction.*
   - (b) *If $\phi(e_v) \in \Delta$ is contracted, then the integral primitive tangent vector v_e associate to e is in the monodromy invariant direction in T_yB, $y = Exp_{\phi(e_v)}(\varepsilon v_e)$ for some small $\varepsilon > 0$.*
5. *For each $v \in C_0(T)$, $v \neq x$ and $val(v) = 2$, we have $\phi(v) \in \Delta$. Moreover,*

(a) *the edges e_v^+, e_v^- adjacent to v are not contracted by ϕ.*

(b) *$\phi(e_v^{\pm})$ is in the monodromy invariant direction.*

(c) *$w(e_v^+) = w(e_v^-)$.*

6. *For each $v \in C_0(T)$, $val(v) \geq 2$, we have the following assumption:*

   (balancing condition) Each outgoing tangent at u along the image of each edge adjacent to v is rational with respect to the above integral structure on $T_{\phi(v)}B$. Denote the outgoing primitive tangent vectors by v_i and the corresponding weight by w_i, then

$$\sum_i w_i v_i = 0. \tag{2}$$

Remark 3.5. *When $B = \mathbb{R}^2$ and a tropical curve with no stop and no contracted edges reduces to the tropical curves defined in* [23].

We will view two tropical discs (ϕ, T, w) and (ϕ', T', w') as the same if there exists a homeomorphism $f : T \to T'$ such that $\phi' \circ f = \phi$ and $w' \circ f = w$. The main difference of the definition of the tropical discs from the existing literature is that one need to allow the contracting edges. All the complication actually comes from the singularities of the affine structure.

The balancing condition in the Definition 3.4 allows one to define the multiplicity for each trivalent vertex [23]: Let v be a trivalent vertex and and three outgoing vectors to be v_1, v_2, v_3 with the corresponding weight w_1, w_2, w_3. Define the weight associate to v to be

$$Mult_v(\phi) = w_1 w_2 |v_1 \wedge v_2|. \tag{3}$$

Here we choose an isomorphism $\wedge T_{\phi(v)}B \cong \mathbb{Z}^2$, so $Mult_v(\phi) \in \mathbb{N}$. The balancing condition guarantees that $Mult_v(\phi)$ is well-defined for a trivalent vertex v.

Before we define the tropical disc counting invariant, we need to associate each tropical disc (ϕ, T, w) on B_ϑ with stop at $u \in B_0$ a relative class in $H_2(X, L_u)$. Let (ϕ, T, w) be a tropical disc with root x and $\phi(x) = u$. If $|C_0(T)| = 2$, then the image of another vertex v is in Δ and $\phi(e)$ is an embedding in the monodromy invariant direction for the unique edge e. We define $[\phi] \in H_2(X, L_u)$ to be the relative class of the Lefschetz thimble associate to $L_{\phi(v)}$ with the sign such that $\int_{[\phi]} \omega_\vartheta > 0$. We will assign a relative class $[\phi] \in H_2(X, L_u)$ for each tropical disc (ϕ, T, w) with stop at u by induction on $|C_0(T)|$. Assume $|C_0(T)| = k+1 > 2$ and let v' be the unique vertex share the edge with x. By deleting the edge e connecting v' and x, we get tropical

[2] If $\phi(v) \in \Delta$, one should replace $\phi(v)$ by $Exp_{\phi(e_v)}(\varepsilon v_e)$.

discs $(\phi_1, T_1, w_1), \cdots, (\phi_l, T_l, w_l)$ with stop at $\phi(v')$ and $|C_0(T_i)| < |C_0(T)|$. Thus, there is a relative class $[\phi_i] \in H_2(X, L_{\phi(v')})$ defined by induction hypothesis. Then we define $[\phi] \in H_2(X, L_u)$ to be the parallel transport of $\sum_{i=1}^{l}[\phi_i] \in H_2(X, L_{\phi(v')})$ along the affine line segment $\phi(e)$.

The following definition is auxiliary to define which tropical discs are "admissible".

Definition 3.6. *1. Let $v_1, \cdots, v_n \in M \cong \mathbb{Z}^2$ be primitive vectors (not necessarily distinct) and $\mathfrak{d}_{ij} \in M \otimes \mathbb{R}$ be the lines in the direction v_i with weight w_{ij}, $j = 1, \cdots, l_i$. Assume that $w_{ij} \leq w_{ij'}$ if $j \leq j'$. We order $\mathfrak{d}_{ij}$ such that $\mathfrak{d}_{i_1 j_1} < \mathfrak{d}_{i_2 j_2}$ if*

(a) $i_1 < i_2$ or

(b) $i_1 = i_2$ and $j_1 < j_2$.

2. *We say that the lines $\{\mathfrak{d}_{ij}\}$ are in the standard position if the intersection of $\mathfrak{d}_{i_1 j_1}$ and $\mathfrak{d}_{i_2 j_2}$ is on the far right side of the line $\mathfrak{d}_{ij}$ if $\mathfrak{d}_{ij} > \mathfrak{d}_{i_1 j_1}$ and $\mathfrak{d}_{ij} > \mathfrak{d}_{i_2 j_2}$.*

3. *Let (ϕ, T, w) be a tropical curve in $\mathbb{R}^2$ with no contracted edges. We say (ϕ, T, w) is in the standard position with respect to $\{(v_i, w_{ij})\}$ if T has $|\sum_i l_i| + 1$ unbounded edges such that all except one unbounded edges are mapped into some $\mathfrak{d}_{ij}$ with weight w_{ij} by ϕ. The exceptional unbounded edge has direction v and weight w such that $wv = \sum_{i,j} v_i w_{ij}$.*

The following definition explains which tropical discs will contribute to the tropical discs counting invariants in Definition 7.1.

Definition 3.7. *A tropical disc (ϕ, T, w) with stop $u \in B_0$ is called an admissible tropical disc if the following holds:*

1. *For every vertex $v \in C_0(T)$, its valency $val(v) \leq 3$.*

2. *Assume $e \in C_1(T)$ is contracted to a point $\phi(e) \in B$. The preimage of $\phi(e)$ is a disjoint union of subtrees of T. Let T_e be the connected subtree containing e. Let $e_0, e_1, \cdots, e_m \in C_1(T)$ be the edges adjacent to T_e and e_0 is the one closest to the root. Denote the weight of e_i by w_i. Let T' be the tree obtained by adding edges $e_0, \cdots, e_m$ with weight $w_1, \cdots w_m$ and $\tilde{T}$ be the tree by replacing each $e_i \in C_1(T')$ by an unbounded edge with weight w_i. $T \backslash T'$ is a disjoint union of subtrees of T. Let T_i be the connected subtree containing e_i, $i = 1, \cdots m$. Then $\phi_i = (\phi|_{T_i}, T_i, w|_{T_i})$ defines a tropical disc with stop at $\phi(e)$. For each e_i there exists a relative class $[\phi_i] \in H_2(X, L_{\phi(e)})$. Let $v_i \in T_{\phi(e)}B_0$ be the primitive vector such that $v_i|Z_{[\phi_i]}| > 0$ and $v_i Arg Z_{[\phi_i]} = 0$. Let $(v_i, Z_{[\phi_i]}), i = 1, \cdots, n$ be such distinct pairs and with the order such that*

(a) *$ArgZ_{[\phi_i]}(\phi(e)-\varepsilon v_i) \leq ArgZ_{[\phi_j]}(\phi(e)-\varepsilon v_j)$ for $\varepsilon \ll 0$, if $i<j$.*

(b) *The above equality holds and $i<j$ imply $|Z_{[\phi_i]}(\phi(e))| < |Z_{\phi_j]}(\phi(e))|$.*

Assume that $w_{ij}, j=1,\cdots,l_i$ are the weights of the edges attached to the pairs $(v_i, Z_{[\phi_i]})$ and ordered in the way that $w_{ij} \leq w_{ij'}$ if $j<j'$. Then there exists $\tilde{\phi}: \tilde{T} \to \mathbb{R}^2$ with no contracted edges and weight $\tilde{w}: C_1(\tilde{T}) \to \mathbb{N}$,

$$\tilde{w}(e') = \begin{cases} w(e'), & e \in T' \\ w_i, & e = e_i, \end{cases}$$

such that the balancing condition (2) is satisfied. Moreover, the tropical curve $(\tilde{\phi}, \tilde{T}, \tilde{w})$ is in the standard position with respect to $\{v_i, w_{ij}\}$.

Given a tree T, we say a vertex $v \in C_0^{ext}(T)$ if $val(v)=1$ and we denote $C_0^{int}(T) = C_0(T) \backslash C_0^{ext}(T)$. With above notations, the tropical discs counting can be defined as follows, motivated by the work of Gross-Pandharipande-Siebert [7]:

Definition 3.8. *1. Let $\phi: T \to B_\vartheta$ be an admissible tropical disc with the stop $u \in B_0$. Then we define its weight of ϕ to be*

$$Mult(\phi) := \prod_{v \in C_0^{int}(T)} Mult_v(\phi) \prod_{v \in C_0^{ext}(T)\backslash\{u\}} \frac{(-1)^{w_v-1}}{w_v^2} \prod_{T_e:\phi(e) \text{is a point}} |Aut(\mathbf{w}_{T(e)})|,$$

where the notation is explained below:

(a) *Here we use the notation in Definition 3.7. Then we set $\mathbf{w}_{T_e} = (\mathbf{w}_1, \cdots, \mathbf{w}_n)$. The last product doesn't repeat the factor if $T_e = T_{e'}$.*

(b) *For a set of weight vectors $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_n)$ and $\mathbf{w}_i = (w_{i1}, \cdots, w_{il_i})$, for $i = 1, \cdots, n$. We set*

$$a_n^i = |\{w_{ij} | w_{ij} = n\}|$$

and

$$|Aut(\mathbf{w})| = \prod_i \prod_{\substack{n \in \mathbb{N} \\ a_n^i \neq 0}} (a_n^i)!,$$

which is the subgroup of the permutation group $\prod_i \Sigma_{l_i}$ stabilizing $\mathbf{w}$.

2. *Let $u \in B_0$ and $\gamma \in H_2(X, L_u)$. We define the tropical discs counting invariant $\tilde{\Omega}^{trop}(\gamma; u)$ to be*

$$\tilde{\Omega}^{trop}(\gamma; u) := \sum_\phi Mult(\phi),$$

where the sum is over all admissible tropical discs on B_{ArgZ_γ} with stop at u such that $[\phi] = \gamma$.

4 Open Gromov-Witten Invariants on K3 Surfaces and Wall-Crossing

Recall that $X_\vartheta \to B$ is a K3 surface with special Lagrangian fibration. Our goal is to define a counting for the holomorphic discs with boundary on a fibre L_u, $u \in B_0$. Given a relative class $\gamma \in H_2(X, L_u)$, we denote the $\mathscr{M}(X_\vartheta, L_u)$ to be the moduli space of stable holomorphic discs (up to isomorphisms) in relative class γ. The first issue of defining such counting is that the virtual dimension of the moduli space is -1, which reflects the fact that there is no holomorphic disc with respect to a generic almost complex structure. Inspired by the work of Bryan-Leung [1], it is natural to consider the following moduli space instead

$$\mathscr{M}_\gamma(\mathfrak{X}, L_u) := \bigcup_{u \in S^1} \mathscr{M}_\gamma(X_\vartheta, L_u).$$

The virtual dimension of the new moduli space $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$ is zero and it makes sense to count. The second issue to define the counting is that generally the moduli spaces of holomorphic discs admit real codimension one boundaries and thus there are no well-defined virtual fundamental classes.

To understand how hyperKähler geometry helps to resolve the issue coming from real codimension one boundaries of the moduli space, we need the help of the central charge function (see Definition 3.1). Assume that $\partial\gamma \neq 0$ and $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$ admits non-empty real codimension one boundary, then there exists γ_1, γ_2 such that $\gamma = \gamma_1 + \gamma_2$ and both γ_1, γ_2 are represented as holomorphic discs with respect to the same complex structure in the S^1-family. From Proposition 3.2, we have

$$\mathrm{Arg} Z_{\gamma_1}(u) = \mathrm{Arg} Z_{\gamma_2}(u). \tag{4}$$

Since Z_{γ_i} are holomorphic functions, the locus cut out by the equation (4) are real codimension one and locally divides the base into chambers. Set

$$W'_\gamma = \bigcup_{\substack{\gamma = \gamma_1 + \gamma_2 \\ \mathrm{Arg} Z_{\gamma_1} \neq_c \mathrm{Arg} Z_{\gamma_2}}} \{u \in B | \mathscr{M}_{\gamma_1}(X_\vartheta, L_u), \mathscr{M}_{\gamma_2}(X_\vartheta, L_u) \text{ are non-empty for some } \vartheta \in S^1.\}$$

Each locus in the union is locally a closed subset of locus in the form defined in (4). Moreover, Gromov compactness theorem guarantees that the union is a finite union. In particular, the complement of W'_γ is a real analytic Zariski open subset on B.

Theorem 4.1. [20] *Assume that*

1. *the relative class γ has its boundary $\partial\gamma$ not null-homologous,*

2. $u \notin W'_\gamma$ *and*

3. γ *cannot be expressed as* $d\gamma' + \gamma_0$, *where* $d \in \mathbb{N}$, $\gamma' \in H_2(X, L_u)$ *and* $\gamma_0 \in H_2(X)$ *such that* $\int_{\gamma_0} \Omega = 0$.

Then $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$ *is compact without boundaries.*

Under the assumption of Theorem 4.1, the moduli space $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$ particularly has no real codimension one boundary and would admit a virtual fundamental class $[\mathscr{M}_\gamma(\mathfrak{X}, L_u)]^{vir}$ [4] and we define the open Gromov-Witten invariants [20] by

$$\tilde{\Omega}(\gamma; u) := \int_{[\mathscr{M}_\gamma(\mathfrak{X}, L_u)]^{vir}} 1. \tag{5}$$

In general, if we only have the first two assumption in Theorem 4.1, the moduli space $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$ might still have real codimension one boundary. From the general theory of Fukaya-Oh-Ohta-Ono[4](see also [20] for the modification in this particular case), there exists a Kuranishi structure on the moduli space $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$. The open Gromov-Witten invariant $\tilde{\Omega}(\gamma; u)$ can be defined using the smooth correspondences:

$$\tilde{\Omega}(\gamma; u) : Corr_*(\mathscr{M}_\gamma(\mathfrak{X}, L_u), pt; tri, tri)(1).$$

We will refer the reader to [4][3] for the definition and properties of the smooth correspondence.

Although the definition of the moduli space $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$ depends on the choice of the Kähler from ω, we have the following properties of the open Gromov-Witten invariants:

Theorem 4.2. [20] *Assume that* $\partial\gamma \neq 0$ *and* $u \notin W'_\gamma$, *then the open Gromov-Witten invariants* $\tilde{\Omega}(\gamma; u)$ *satisfy*

1. $\tilde{\Omega}(\gamma; u)$ *does not depend on the choice of* $[\omega]$.

2. $\tilde{\Omega}(\gamma; u)$ *is locally constant in u.*

3. *(reality condition)* $\tilde{\Omega}(-\gamma; u) = \tilde{\Omega}(\gamma; u)$.

Locally the wall W'_γ divides the base into chambers and the open Gromov-Witten invariant $\tilde{\Omega}(\gamma; u)$ is a constant inside each of the chamber. The open Gromov-Witten invariants may jump when u varies from one chamber to another. Thus, we will not try to define the open Gromov-Witten invariants when u falls on the wall. To sum up, the appearance of real codimension one boundaries of the moduli space $\mathscr{M}_\gamma(\mathfrak{X}, L_u)$ is largely constrained cohomologically, which is due to the hyperKähler geometry. The ambiguity from the real codimension one boundaries is interpreted as the wall-crossing of the invariants when the boundary conditions of holomorphic discs vary. We will discuss how the open Gromov-Witten invariants $\tilde{\Omega}(\gamma;u)$ and the tropical discs invariants $\tilde{\Omega}^{trop}(\gamma;u)$ jump in Section 6.

5 Near an I_1-Type Singular Fibre

When the symplectic area of a holomorphic disc with boundary on a special Lagrangian torus fibre is small, the gradient estimate of holomorphic discs guarantees that the image holomorphic discs is contained in a tubular neighborhood of a singular fibre. Let $u_0 \in \Delta$ and L_{u_0} be a I_1-type singular fibre. For an small open ball U such that $U \cap \Delta = \{u_0\}$, we denote $X_U := \pi^{-1}(U)$ to be the pre-image of the projection. It is straight-forward to check that $H_2(X_U, L_u) \cong \mathbb{Z}$ for each $u \in U \backslash \{u_0\}$ and $H_2(X_U, L_u)$ is generated by the Lefschetz thimble.

For a fixed $\vartheta \in S^1$, there are two affine rays $l_\pm$ in U emanating from u_0 such that the tangents are monodromy invariant. From Proposition 3.3, only the torus fibres above $l_\pm$ can bound holomorphic discs with small symplectic area. For $u \in l_\pm$, the union of vanishing cycles over the affine segments between u_0 and u can be perturbed to a smooth holomorphic disc in X_ϑ, if X_ϑ is closed enough to the large complex limit point [20]. Moreover, the open Gromov-Witten invariants are calculated via cobordism argument and localization:

Theorem 5.1. [20] *Let $u_0 \in \Delta$. For each $d \in \mathbb{N}$, there exists an open neighborhood $\mathscr{U}_d$ of u_0 in B such that*

$$\tilde{\Omega}(d\gamma;u) = \begin{cases} \frac{(-1)^{d-1}}{d^2}, & \textit{if } \gamma \textit{ represents the Lefschetz thimble }, \\ 0, & \textit{otherwise.} \end{cases}$$

for any $u \in \mathscr{U}_d$.

It worth noticing that the generating function of the open Gromov-Witten invariants

$$\sum_{d \in \mathbb{N}} d\tilde{\Omega}(d\gamma;u)x^d = \log(1+x)$$

is exactly the slab function of the initial ray in the Gross-Siebert program [9]. Similar observation is known in the case of toric Calabi-Yau manifolds [22].

On the other hand, the affine line segment between u_0 and u uniquely determines a tropical disc (ϕ, T, d): with T be a rooted tree with only two vertices v and root x, ϕ be an embedding from T to the affine line segment between u_0 and

u and $d \in \mathbb{N}$. On the other hand, this is the only tropical disc with its image contained in U. Therefore, the Theorem 5.1 also holds for the tropical discs invariants $\tilde{\Omega}^{trop}(\gamma;u)$ by Definition 7.1. In particular, we have

$$\tilde{\Omega}(\gamma;u) = \tilde{\Omega}^{trop}(\gamma;u)$$

for u close enough to u_0.

6 Wall-Crossing Formula for the Invariants

To understand the how do the open Gromov-Witten invariants $\tilde{\Omega}(\gamma;u)$ (and $\tilde{\Omega}^{trop}(\gamma;u)$) change as the u varies, we will first introduce the Kontsevich-Soibelman algebra [18]: Fix a sector $\mathscr{S}$ with angle less than π. Let $\Lambda^{\mathscr{S}}$ to be the ring in terms of formal variable T,

$$\Lambda^{\mathscr{S}} := \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} | \mathrm{Arg}\lambda_i \in \mathscr{S}, \lim |\lambda_i| = \infty\}.$$

When $\mathscr{S}$ reduced to a ray, then $\Lambda^{\mathscr{S}}$ is isomorphic to the standard Novikov ring. There is a natural filtration $F^{\lambda}\Lambda^{\mathscr{S}}$ of $\Lambda^{\mathscr{S}}$ given by

$$F^{\lambda}\Lambda^{\mathscr{S}} := \{\sum_{i=0}^{\infty} a_i T^{\lambda} \in \Lambda^{\mathscr{S}} | a_i = 0 \text{ if } |\lambda_i| < \lambda\}.$$

For each $u \in B_0$, we consider the Lie algebra structure on the associate, commutative algebra $\mathscr{L}_u := \Lambda^{\mathscr{S}}[H_1(L_u)]$ with the bracket structure given by

$$[z^{\partial\gamma_1}, z^{\partial\gamma_2}] = \langle\gamma_1,\gamma_2\rangle z^{\partial\gamma_1+\partial\gamma_2}, \tag{6}$$

where $\gamma_1, \gamma_2 \in H_2(X, L_u)$. For a primitive $\gamma \in H_2(X, L_u)$, we associate an automorphism of $\mathbb{C}[H_1(L_u)]$ given by

$$\begin{aligned} \mathscr{K}_{\gamma}(u) : & \mathscr{L}_u \longrightarrow \mathscr{L}_u \\ & z^{\partial\gamma'} \mapsto z^{\partial\gamma'} f_{\gamma}(u)^{\langle\gamma,\gamma'\rangle}, \end{aligned} \tag{7}$$

where $f_{\gamma}(u)$ is the generating function of the open Gromov-Witten invariants

$$\log f_{\gamma}(u) := \sum_{d\in\mathbb{N}} d\tilde{\Omega}(d\gamma;u)(T^{Z_{\gamma}(u)} z^{\partial\gamma})^d. \tag{8}$$

Straight-forward computation shows that $\mathscr{K}_{\gamma}(u)$ commutes with $\mathscr{K}_{\gamma'}(u)$ if $\langle\gamma_1,\gamma_2\rangle = 0$. One can similarly define $\mathscr{K}_{\gamma}^{trop}(u)$ and $f_{\gamma}^{trop}(u)$.

The jumping of the invariants $\tilde{\Omega}(\gamma;u)$ and $\tilde{\Omega}^{trop}(\gamma;u)$ can be described by the following theorem:

Theorem 6.1. [21] *Given a path $\phi(t)$ between $u,u' \in B_0$. Let $Par_{u,u'}$ denotes the isomorphism $\mathscr{L}_u \to \mathscr{L}_{u'}$ induced by the parallel transport along $\phi(t)$. Assume that there exists no $\gamma \in H_2(X, L_{\phi(t)})$ such that $\mathscr{M}_\gamma(X, L_{\phi(t)})$ is nonempty and $ArgZ_\gamma(u(t)) \in \partial\mathscr{S}$. Then*

$$Par_{u,u'}\left(\prod_{ArgZ_\gamma(u)\in\mathscr{S}} \mathscr{K}_\gamma(u)\right) = \prod_{ArgZ_\gamma(u')\in\mathscr{S}} \mathscr{K}_\gamma(u').$$

Here both of the products[3] *are taken in the order that $ArgZ_\gamma$ is increasing. The analogue statement holds for $\mathscr{K}_\gamma^{trop}$.*

Theorem 6.1 generalized the wall-crossing formula in [20] when γ_1, γ_2 are both primitive. When u varies across W'_γ and $\gamma = \gamma_1 + \gamma_2$ is the only possible degeneration of holomorphic discs, then the jump of $\tilde{\Omega}(\gamma; u)$ is given by

$$\langle \gamma_1, \gamma_2 \rangle \tilde{\Omega}(\gamma_1; u) \tilde{\Omega}(\gamma_2; u).$$

By induction on the symplectic area of holomorphic discs, one can get more identities. For instance, the jump of $\tilde{\Omega}(\gamma_1 + 2\gamma_2; u)$ is given by

$$2\langle \gamma_1, \gamma_2 \rangle \tilde{\Omega}(\gamma_1, ; u) \tilde{\Omega}(2\gamma_2; u) + \frac{1}{2} \langle \gamma_1, \gamma_2 \rangle \tilde{\Omega}(\gamma_1; u) \tilde{\Omega}(\gamma_2; u)^2. \tag{9}$$

Recall that from the definition of open Gromov-Witten invariants, $\tilde{\Omega}(\gamma; u) \neq 0$ implies that there exists a holomorphic disc in the relative class $\gamma \in H_2(X, L_u)$. In particular, Theorem 6.1 implies the existence of holomorphic discs in a lot of situation when the standard gluing theorem does not due to the highly non-transversality.

The Theorem 5.1 and the Theorem 6.1 together gives the correspondence theorem between the open Gromov-Witten invariants and the counting of tropical discs, which generalized the work of Mikhalkin [23] and Nishinou-Siebert [25].

Theorem 6.2. *Given $u \in B_0$ and $\gamma \in H_2(X, L_u)$ such that $u \in W'_\gamma$, then we have*

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}^{trop}(\gamma; u).$$

It also worth mentioning that the open Gromov-Witten invariants $\tilde{\Omega}(\gamma; u)$ are defined via De Rham models and are $\mathbb{R}$-valued a priori. On the other hand, the tropical discs counting invariants $\tilde{\Omega}^{trop}(\gamma; u)$ are defined in $\mathbb{Q}$. The correspondence theorem indicates that $\tilde{\Omega}(\gamma; u) \in \mathbb{Q}$, which is not obvious from the definition.

Motivate by the Gopakumar-Vafa conjecture [13] and the work of Gaiotto-Moore-Neitzke [8], we are aiming for integer-valued invariants.

[3] In general, this is an infinite product and converges with respect to the non-Archimedean topology induced by the filtration $F^\lambda \Lambda^{\mathscr{S}}$.

Definition 6.3. *A quadratic refinement is a continuous homomorphism*

$$c : \Gamma \longrightarrow \mathbb{Z}_2$$

such that for each $\gamma_1, \gamma_2 \in \Gamma$*,*

$$c(\gamma_1 + \gamma_2; u) = (-1)^{\langle \gamma_1, \gamma_2 \rangle} c(\gamma_1; u) c(\gamma_2; u).$$

Remark 6.4. *It is well-known that the quadratic differential on a Riemann surface is canonically one-to-one to the spin structures on the Riemann surface* [16]*. We expect that the choices the quadratic differentials corresponds to certain choices of orientations of the relevant moduli spaces of holomorphic discs.*

Consider the following basic transformation

$$\theta_\gamma(u) := \exp ad\big(Li_2(c(\gamma; u) T^{Z_\gamma(u)} z^{\partial \gamma})\big),$$

where Li_2 is the dilogarithm function $Li_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2}$. Straight-forward computation shows that the explicit action of θ_γ is given by

$$\theta_\gamma(u) : z^{\partial \gamma'} \mapsto z^{\partial \gamma'} (1 - c(\gamma; u) T^{Z_\gamma(u)} z^{\partial \gamma})^{\langle \gamma, \gamma' \rangle}.$$

Then the transformation $\mathscr{K}_\gamma(u)$ can be uniquely decomposed into

$$\mathscr{K}_\gamma(u) = \prod_{d \in \mathbb{N}} \theta_{d\gamma}(u)^{\Omega(d\gamma; u)}, \tag{10}$$

for some $\Omega(d\gamma; u) \in \mathbb{Q}$. From equation (7)(8)(10), we have

$$\tilde{\Omega}^{trop}(d\gamma) = -\sum_{k|d} c(\gamma)^d \frac{\Omega^{trop}(\frac{d}{k}\gamma)}{k^2}, \tag{11}$$

for any $d \in \mathbb{Z}$. The information of $\tilde{\Omega}(d\gamma; u)$ can be converted to $\Omega(\gamma; u)$ by Möbius inversion formula,

$$\Omega^{trop}(d\gamma; u) = -\sum_{k|d} c(\gamma; u)^{\frac{d}{k}} \mu(k) \frac{\tilde{\Omega}^{trop}(\frac{d}{k}\gamma; u)}{k^2}.$$

One of the motivations of introducing such decomposition is the following open analogue of the Gopakumar-Vafa conjecture:

Conjecture 6.5. [20] *Assume* $u \notin W'_\gamma$*, then*

1. $\Omega(\gamma;u) \in \mathbb{Z}$.

2. $\Omega(d\gamma;u) = 0$ *for sufficiently large* $d \in \mathbb{N}$.

For instance, under the notation of Theorem 5.1, we have

$$\Omega(\gamma;u) = \begin{cases} 1, & \text{if } \pm\gamma \text{ represents the Lefschetz thimble },\\ 0, & \text{otherwise.} \end{cases}$$

7 Refinement of the Invariants and Refined Wall-Crossing Formula

There is a q-deformation of the wall-crossing formula [2][15][17] as follows: we first enlarge $\mathscr{L}_u$ by

$$\mathscr{L}_{u,q} := \Lambda^{\mathscr{S}}[H_1(L_u)] \otimes \mathbb{C}[q^{\pm\frac{1}{2}}, ((q^n-1)^{-1})_{n\geq 1}].$$

Then we replace the commutative product on $\mathscr{L}_{u,q}$ by

$$z^{\partial_1} z^{\partial\gamma_2} = q^{\frac{1}{2}\langle\gamma_1,\gamma_2\rangle} z^{\partial\gamma_2} z^{\partial\gamma_1}$$

and the bracket becomes

$$[z^{\partial\gamma_1}, z^{\partial\gamma_2}] = (q^{\frac{1}{2}\langle\gamma_1,\gamma_2\rangle} - q^{-\frac{1}{2}\langle\gamma_1,\gamma_2\rangle}) z^{\partial\gamma_1+\partial\gamma_2}.$$

The study of special function theory of q-commuting variables provides the q-deformed dilogarithm function

$$Li_2(z;q) := \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)},$$

with semi-classical limit back to the dilogarithm function

$$\lim_{q^{\frac{1}{2}}\to 1} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) Li_2(q^{\frac{1}{2}} z;q) = Li_2(z).$$

Thus, it is natural to define the analogue of θ_γ by

$$\theta_{\gamma,q}(u) := exp\big(ad(Li_2(c(\gamma;u)q^{\frac{1}{2}} T^{Z_\gamma(u)} z^{\partial\gamma}, q))\big)$$

as an automorphism of $\mathscr{L}_{u,q}$. Generally, we denote

$$\theta_{\gamma,q,n}(u) := exp\big(ad(Li_2(c(\gamma;u)q^{\frac{n+1}{2}} T^{Z_\gamma(u)} z^{\partial\gamma}, q))\big).$$

Straight-forward computation shows that

$$\theta_{\gamma,q}(u) = Ad \exp\left(\frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{d\in\mathbb{N}} \frac{(-1)^{d-1}}{d[d]_q} (T^{Z_\gamma(u)} z^{\partial\gamma})^d\right),$$

when γ is the relative class of a Lefschetz thimble. This motivates that we should have

$$\tilde{\Omega}_q^{trop}(d\gamma;u) = \frac{(-1)^{d-1}}{d[d]_q}, \tag{12}$$

for the γ being the relative class of Lefschetz thimble, where

$$[d]_q = \frac{q^{\frac{d}{2}} - q^{-\frac{d}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, d \in \mathbb{N}.$$

The following is the generalization of tropical disc counting invariant with q-deformation motivate from the work of [10][5].

Definition 7.1. *1. Let $\phi : T \to B_\vartheta$ be an admissible tropical disc with the stop $u \in B_0$. Then we define its q-deformed weight of ϕ to be*

$$Mult_q(\phi) := \prod_{v\in C_0^{int}(T)} [Mult_v(\phi)]_q \prod_{v\in C_0^{ext}(T)\setminus\{u\}} \frac{(-1)^{w_v-1}}{w_v[w_v]_q} \prod_{T_e:\phi(e) \text{ is a point}} |Aut(\mathbf{w}_{T_e})|,$$

where the notation is the same as in Definition 7.1.

2. Let $u \in B_0$ and $\gamma \in H_2(X, L_u)$. We define the tropical discs counting invariant $\tilde{\Omega}_q^{trop}(\gamma;u)$ to be

$$\tilde{\Omega}_q^{trop}(\gamma;u) := \sum_\phi Mult_q(\phi) \in \mathbb{Q}[q^{\pm\frac{1}{2}}, (q^n-1)^{-1}]_{n\geq 1}.$$

where the sum is over all admissible tropical discs on B_{ArgZ_γ} with stop at u such that $[\phi] = \gamma$.

It is easy to see that under the semi-classical limit $q^{\frac{1}{2}} \to 1$, the q-deformed tropical discs counting invariants $\tilde{\Omega}_q^{trop}(\gamma;u)$ reduce $\tilde{\Omega}^{trop}(\gamma;u)$. For each primitive $\gamma \in H_2(X, L_u)$, we associate an automorphism of $\mathscr{L}_{u,q}$ given by

$$\mathscr{K}_{\gamma,q}^{trop}(u) := Ad \exp\left(\frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{d\in\mathbb{N}} d\tilde{\Omega}_q^{trop}(d\gamma;u)(T^{Z_\gamma(u)} z^{\partial\gamma})^d\right). \tag{13}$$

In the examples the author is aware of, the transformation $K^{trop}_{\gamma,q}(u)$ can be expressed in terms of $\theta_{\gamma,q}(u)$ similar to (10),

$$\mathscr{K}^{trop}_{\gamma,q}(u) = \prod_{d\in\mathbb{N}} \theta_{d\gamma,q}(u)^{\sum_n (-1)^n \Omega_n(d\gamma;u)},$$

for some rational number $\Omega^{trop}_n(\gamma;u)$. Then the q-deformed multiple cover formula becomes

$$\tilde{\Omega}^{trop}_q(d\gamma;u) = \sum_{n\in\mathbb{Z}} (-1)^{n+1} \left(\sum_{k|d} \frac{c(\gamma;u)^d}{k[k]_q} \Omega_n(\frac{d}{k}\gamma;u) \right) q^{\frac{kn}{2}}.$$

Conjecture 7.2. *$\Omega_n(\gamma;u)$ are always positive integers.*

Similar to the proof of tropical counterpart of Theorem 6.1 and together with the Corollary 4.9 [5], we have the wall-crossing formula for q-deformed tropical discs counting invariants:

Theorem 7.3. [21] *Under the same assumption and notation of Theorem 6.1, then*

$$Par_{u,u'}\left(\prod_{Arg Z_\gamma(u)\in\mathscr{S}} \mathscr{K}^{trop}_{\gamma,q}(u) \right) = \prod_{Arg Z_\gamma(u')\in\mathscr{S}} \mathscr{K}^{trop}_{\gamma,q}(u').$$

Example 7.4. *Under the same condition before the equation (9) and $\tilde{\Omega}^{trop}_q(\gamma_1,u) = \tilde{\Omega}^{trop}_q(\gamma_2,u) = 1$, then we have*

$$\Delta\tilde{\Omega}^{trop}_q(\gamma_1+\gamma_2;u) = q^{\frac{1}{2}} + q^{-\frac{1}{2}}.$$

and

$$\Delta\Omega^{trop}_n(\gamma;u) = \begin{cases} 1, & \text{if } \gamma=\gamma_1+\gamma_2 \text{ and } n=\pm 1 \\ 1, & \text{if } \gamma = k\gamma_1+(k+1)\gamma_2 \text{ for some } k\in\mathbb{Z} \text{ and } n=0 \\ 0, & \text{otherwise.} \end{cases}$$

Remark 7.5. *From the Correspondence theorem (Theorem 6.2) and Definition 7.1, it suggests that the open Gromov-Witten invariants $\tilde{\Omega}(\gamma;u)$ also admits an q-deformation*

$$\tilde{\Omega}_q(\gamma;u) \in \mathbb{Q}[q^{\pm\frac{1}{2}}, (q^n-1)^{-1}]_{n\geq 1},$$

such that

$$\lim_{q^{\frac{1}{2}}\to 1} \tilde{\Omega}_q(\gamma;u) = \tilde{\Omega}(\gamma;u).$$

We don't know the geometric meaning of $\tilde{\Omega}_q(\gamma;u)$. Especially, the terms $q^{\pm\frac{1}{2}}$ in Example 7.4 corresponds to vectormultiplets in [8]. *We are interested in the corresponding elements in symplectic geometry will leave it for the future work.*

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