Mirror Symmetry, SYZ & Enumerative Geometry

\[ X = \{ f(x_0, \ldots, x_5) = 0 \} \subseteq \mathbb{P}^4_{(x_0, \ldots, x_5)}, \quad \deg f = 5 \]

Calabi-Yau 3-folds \( K_X \cong \mathcal{O}_X \)

\( n_d = \# \) of rational curves of degree \( d \) in \( X \)

= \# of homog. poly. \( x_i(s,t) \) of degree \( d \)

\[ s \quad t \quad f(x_i(s,t), \ldots, x_5(s,t)) = 0. \]

This is a classical problem in enumerative geometry.

\[ n_1 = 2875, \quad n_2 = 609 \times 5, \quad n_3 = ?? \]

(Candelas-de la Ossa-Green-Plesser ‘90)

generating function of \( n_d \) \( \leftarrow \) period integral \( (\text{residue calculation}) + \text{ODE} \)

on another CY 3-fold \( \tilde{X} \)

mirror of \( X \)

Q: How to find the minor \( \tilde{X} \)?

(Strominger-Yau-Zaslow ‘96)

\[ \text{Lagrangian} \quad \omega_L = 0 \quad \Rightarrow \quad \pi \quad \Rightarrow \quad \text{Special} \quad \text{Im} \omega_L = 0 \]

\( \text{near large complex structure limit (LCSL)} \)

complex structure of \( \tilde{X} \) received "quantum correction" from Calabi-Yau boundary of \( \pi \).
There might exist singular fibres

\[
\begin{array}{c}
\text{compactification} \\
X \cong (T^*\mathcal{Y}, \omega) \downarrow \downarrow \text{quantum correction} \\
(B, \mathcal{O}) \quad \dashrightarrow \quad (\check{X}, \mathcal{O})
\end{array}
\]

fibres
\[
L = \frac{H^1(L, \mathbb{R})}{H^1(L, \mathbb{Z})} \leftarrow \frac{\mathcal{L}^*}{\mathcal{L}} \leftarrow \frac{H^1(L, \mathbb{R})}{H^1(L, \mathbb{Z})} \leftarrow
\]

\[L^* = \text{Hom}(\pi_1(L), U(1))\]

\[= \text{flat } U(1) \text{- connection on } L\]

before "quantum correction" \[\Xi_A = \exp \left( -\int_A \omega \right) \mathcal{H}_U(\mathcal{A})\]

where \[A \in \mathcal{H}_U(X, L)\]

Notice the implicit dependence on the symplectic form

Q: How do we get the "quantum corrected" mirror \(\check{X}\) ?

ex. (Gross, Auroux) \[X = \mathbb{C}^2 \setminus \{xy = \epsilon\}\]

\[\omega = \frac{i}{2} (dx \wedge dx + dy \wedge dy)\]

\[\Omega = \frac{dx \wedge dy}{xy - \epsilon}\]

\[\lambda : \text{symplectic area}\]

\[\int \omega\]
Q: What are the Rho discs w/ boundary on L_{r,a}?

$$(\mathbb{D}, \partial \mathbb{D}) \xrightarrow{f} (X, L_{r,a})$$

- $p \circ f = u \circ o$, $p'(u) = \frac{\zeta}{2}$
- $f = \text{const}$ by maximal principle

- $p \circ f = o$, $p'(o) = \frac{\zeta}{2}$

Composition of $\circ \circ \circ = \text{id}$

$\Rightarrow$ well-defined gluing

$X = \{ uv = 1 + z_1 \}$
another CY

Siu-Cheng generalized the mirror construction to toric Calabi-Yau manifolds.
LCSL & Enumerate Geometry

LCSL "=" collapsing of SYZ fibration

- Projection of fibro. curves converge to 1-skeletons at LCSL tropical curves certain adiabatic limit in geometric analysis

- Suitably defined fibro. curve counting is deformation invariant. Gromov-Witten invariants

\[ \implies \text{Enumeration of 1-skeleton recovers} \]
\[ \text{the enumeration of fibro. curves.} \]

\[
X = (\mathbb{C}^*)^n \xrightarrow{\text{moment map}} \mathbb{R}^n \xrightarrow{\text{Legendre transform}} \Delta = \Delta^0 = \mathbb{R}^n \xrightarrow{\text{Log}} (\log |x_1|, \ldots, \log |x_n|)
\]

Mikhalkin
\[ \sum H_t(x, y) = \left( \left| \frac{x_1}{x_2} \right|, \left| \frac{x_3}{x_4} \right| \right) \]
as \( t \to \infty \), \( \omega_\gamma \to 0 \)

What happens to fibro. curves as \( t \to \infty \)?

\[ \{ x + y + 1 = 0 \} \subseteq (\mathbb{C}^*)^2 \]

\[
\text{corner locus of } \max\{x, y, 1\} = \sqrt{v(x \circ y \circ 1)}
\]
tropical curve

\[ 1 \cdot (1, 1) + 1 \cdot (0, -1) + 1 \cdot (-1, 0) = 0 \]
balancing condition
Moral: LCSL $\rightarrow$ tropical geometry

- toric degeneration
- Bezout theorem $\leftrightarrow$ Pick’s theorem of lattice points.
- Two points determines a line.
- Five points determines a conic.

(Mikhalkin, Nishinou-Siebert) Weighted count of tropical curves

= enumeration of Polv. curves

log Gromov-Witten invariants

(Collins-Jacob-L. ’19) $X = \mathbb{P}^2 \setminus E$, $E$: smooth elliptic curve

Conjecture of Auroux ’07 $\implies$ SYZ fibration

$B \cong \mathbb{R}^2_{\text{top}}$

(Lau-Lee-L. ’20) As an affine manifold,

$B = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ $|\det \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}| = 3$

There are $3 \times 3$ rational curves intersecting $E$ at 1 point.

They are simply the tangent lines of the 3-torsion points of $E$.

(L. ’20) Enumeration of $A^1$-curves
To sum up,

- Mirror Symmetry
- Enumerative geometry
- SYZ

Geometric analysis: behavior of Ricci-flat metrics

Thank You