Overview of the Conference on Motives, Quantum Field Theory and Pseudodifferential Operators

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1. Mathematical Topics

While quantum field theory, as used in physics, lacks a rigorous mathematical foundation at present, it is a broad field with remarkable predictive power both in mathematics and in particle physics. In particular, perturbative QFT has unmatched precision in the laboratory. In the past decade, deep algebraic and analytic structures within perturbative QFT have given unexpected and still unexplained overlaps with analytic number theory. As explained below, these overlaps indicate a deeper connection between the theory of motives and perturbative QFT, and the exploration of this connection is the main algebraic theme of the conference. A corresponding analytic theme will focus on the connections between perturbative QFT and pseudodifferential operators (ΨDOs). Finally, while the developing interactions between noncommutative geometry and the areas of QFT, motives and ΨDO theory are still preliminary, we expect these interactions to be a main influence on these three areas by the time of the conference.

We now sketch the relations among these fields. In this overview, QFT means perturbative quantum field theory, whose main subject is the computation of Feynman integrals. These are finite dimensional but typically divergent integrals whose suitably renormalized answers can be matched with physical experiment, and do so with remarkable precision for e.g. quantum electrodynamics. Since their introduction as *ad hoc* rules 60 years ago, the Feynman rules for computing these integrals have become increasingly unmanagable, as one wishes both to apply Feynman rules to more sophisticated QFTs and to compute more accurate ("higher loop order") integrals. In particular, the exponential growth in the calculations as a function of loop order implies that getting bigger, faster computers is not a long term approach.

In the late 1990s, Kreimer and Connes-Kreimer uncovered the fundamental Hopf algebra structure in Feynman rules, and in fact realized the universal nature of the Hopf algebra of Feynman diagrams. As a result, many previously intractable Feynman integrals could be more efficiently organized and computed by implementing the Hopf algebra structure. More importantly from the theoretical viewpoint, algebraic machinery became available in QFT, so that e.g. classical identities in QFT (Slavnov-Taylor identities) were interpreted as a Hochschild one-cocyle on this Hopf algebra. It is fair to say that this approach was a driving force behind the rejuvenation of QFT during this period, particularly in Europe. At present, algebraists with expertise in Hopf algebras play a key role in the development of QFT, and ideas from QFT, suitably abstracted, are filtering into the Hopf algebra literature.

Around the same time, it was realized that the actual values computed by Feynman integrals had unexpected number theoretic significance. In particular, these integrals often (but not always!) produced rational multiples of multiple zeta values (MZVs) for no apparent *a priori* reason. A conjectural link between Feynman integrals and MZVs is given by the shuffle-type relations common to both the Kreimer Hopf algebra and MZVs. While this has been worked out to some extent, the shuffle relations seem to be more of a common occurance than an underlying explanation. It is more likely that a deeper explanation for the appearance of MZVs in QFT is the theory of motives, where a major success is the explanation of relations in the algebra of MZVs via calculating dimensions of certain motivic cohomology groups. As a first step towards uncovering a QFT-motivic connection, Bloch-Esnault-Kreimer developed a motivic interpretation of some simple Feynman diagrams in terms of a blow-up procedure familiar as a standard example of motivic methods.

These connections will certainly be extended by the time of the conference, yielding a new relationship between two major fields of active research. A major obstacle towards progress is the historical divide between these two fields. Both QFT and Grothendieck-style algebraic geometry require a vast background, but in very different areas, with QFT usually based on analytic and/or C^*-algebra techniques, and motives based on the GAGA approach.

A major theme of this conference is to bring experts in these two fields together both for a series of introductary lectures and for more technical talks. The result should be both expanded interest in this area and the realization that experts and students in each field can indeed work in the other.

A second theme of the conference is the relation between QFT and pseudodifferential operator (ΨDO) theory. This relationship is much more direct: Feynman integrals are iterated integrals of Green's functions, which are kernels of classical ΨDOs. Certain locality properties essential to QFT should have explanations in terms of known asymptotic expansions for ΨDOs, in particular in terms of the Wodzicki residue. In addition, in work of Paycha and others, the symbol calculus for zeta functions of elliptic operators is being extended to produce multizeta functions for elliptic operators. In this program, renormalization techniques common in QFT are used, and the shuffle relations mentioned above are still valid. So once again, the Hopf algebraic structure seems to interact with analytic calculations. Although a direct motivic-ΨDO link is premature speculation at this point, it may not be by June 2008.

Finally, noncommutative geometry is linked to all three topics of QFT, motives and ΨDOs, although the precise nature of the linkage is unclear. Connes and Marcolli have produced papers relating noncommutative geometry both to motives and to QFT, and ΨDO techniques lie at the heart of the Connes-Moscovicii local index theorem. We have invited several prominent noncommutative geometers to this conference, mainly because of the evident overlap of subject interest. Moreover, there is a cultural aspect to including noncommutative geometry: the subject seems somewhat underrepresented in the US outside of specific universities, and this conference would expose aspects of noncommutative geometry to junior and senior US mathematicians.

2. **Organization**

In Week I, we will have two sets of five introductory, 60 minute lectures, one set on QFT and one set on motives. The rest of the day will be devoted to two 60 minute lectures by senior mathematicians on more technical topics. In Week II, we will have one set of five 60 minute lectures on (Hopf) algebraic and analytic structures in perturbative QFT, with the rest of the day devoted to three 60 minute talks by a mixture of senior and junior faculty. This schedule provides both introductions to new PhDs and grad students to two traditionally distant areas in mathematics, and cutting edge results in the interactions among QFT, motives, ΨDOs and noncommutative geometry, while leaving enough time for discussions among the participants.