Report on motivic zeta- and L-functions

Yves André, CNRS, ENS Ulm Paris

(Boston, June 9th, 2008)

Abstract. It oftens occurs that Taylor coefficients of (dimensionally regularized) Feynman amplitudes with rational parameters, expanded at an integral dimension $D = D_0$, are not only "periods" (Belkale, Brosnan) but actually "multiple zeta values" (Broadhurst, Kreimer).

In order to determine, at least heuristically, whether it is so in concrete instances, the philosophy of motives suggests an arithmetic method (Kontsevich): counting points of related algebraic varieties modulo sufficiently many primes p and checking that the number of points varies polynomially in p.

On the other hand, Kapranov has introduced an object, the "motivic zeta function", the role of which is precisely to "interpolate" between zeta functions of reductions modulo different primes p.

We will survey the developments of this idea.

$\Gamma \qquad V = \{\text{vertices}\}$

$\qquad\quad E = \{\text{edges}\}$

Feynman amplitude

$$\int_{\mathbb{R}^{D.\#E}} \prod_{e\in E} (1+|p_e|^2)^{-1} \prod_{v\in V} \delta\Big(\sum_{e\to v} p_e - \sum_{v\to e} p_e\Big) \prod_{e\in E} d^D p_e$$

$$\text{"="} \quad \frac{\pi^{b_1(\Gamma).D/2} \cdot \Gamma(\#E - b_1(\Gamma)D/2)}{\Gamma(\#E)} \cdot \underbrace{\int_{\Delta_{\#E}} P_\Gamma^{-D/2} \prod_{e\in E} dx_e}_{\text{Igusa integral}}$$

$0 \leq x_i \leq 1$

$\Sigma x_i \leq 1$

$$P_\Gamma = \sum_{T} \prod_{e\notin T} x_e$$

(T : spanning tree)

Belkale · Brosnan: Taylor coef. of the Igusa integral, at $D = D_0$ (e.g. 4), are periods (of varieties closely related to $X_\Gamma : P_\Gamma = 0$).

I. Introduction: Periods, multiple zeta values, Hasse-Weil zeta functions.

Let us consider a graph hypersurface (Kirchhoff hypersurface) X_Γ, given by equation

$$\sum_T \prod_{e \notin T} x_e = 0$$

where T runs through the spanning trees of a given graph Γ ((x_e) is a set of indeterminates indexed by the edges of Γ).

(e. g. for a cyclic graph of length n, $X_\Gamma = \mathbb{A}^{n-1}$).

Kontsevich speculated that the periods of X_Γ are (linear combinations of) multiple zeta values. According to Grothendieck's period conjecture, this would imply that the motive of X_Γ is a mixed Tate motive over $\mathbb{Z}$.

If this is the case, $\sharp X_{\Gamma}(\mathbb{F}_q)$ should be polynomial in q; equivalently, the poles of the Hasse-Weil zeta function of $X_{\Gamma} \otimes \mathbb{F}_q$ should be integral powers of q.

This was disproved in general by Belkale and Brosnan. But this leaves the interesting general question of controlling $\sharp X(\mathbb{F}_q)$ uniformly in q, of equivalently, of the variation of $Z(X \otimes \mathbb{F}_q, t)$ with q.

→ Kapranov zeta function.

II. Kapranov zeta functions.

$SmP(k) \subset Var(k)$

"Poor man's motives" (Drinfeld) : $K_0(Var(k)$:

- generators: $[X]$ $(X \in Var(k))$,

- relations: $[X - Y] = [X] - [Y]$ $(Y \hookrightarrow X)$.

- ring under $\times$.

- $\mathbb{L} := [\mathbb{A}^1]$.

Example: $[GL_n] = (\mathbb{L}^n - 1)\dots(\mathbb{L}^n - \mathbb{L}^{n-1})$.

Prop. (Loojenga-Bittner) (car. $k = 0$):

$$K_0(Var(k) = \mathbb{Z}\langle SmP(k)\rangle/\text{blow-up relations}$$
$$[\tilde{X}] - [E] = [X] - [Y].$$

Ex: locally trivial fibration (Zariski)

$X \to S$, fiber Y $\rightsquigarrow [X]=[S][Y]$.

e.g. GL_n- fibrations are locally trivial.

It is often convenient to localize

$K_0(Var(k))$ by $\mathbb{L}^n - \mathbb{L}$ $n>1$.

eg. (McWilliams - Belkale - Brosnan):

$$[Sym^n_r] = \begin{cases} \prod_1^s \frac{\mathbb{L}^{2i}}{\mathbb{L}^{2i}-1} \prod_0^{2s-1} (\mathbb{L}^{n-i}-1) & \text{if } 0\leq r=2s\leq n \\ \prod_1^s \frac{\mathbb{L}^{2i}}{\mathbb{L}^{2i}-1} \prod_0^{2s} (\mathbb{L}^{n-i}-1) & \text{if } 0\leq r=2s+1\leq n \end{cases}$$

(n: dim, r: rank)

Fix a ring homomorphism

$$\mu : K_0(Var(k) \to R.$$

Kapranov: $\boxed{Z_\mu(X,t) := \sum_0^\infty \mu[S^n X]\, t^n} \in R[[t]].$

(X quasiprojective, say).

$Z_\mu(X+X',t) = Z_\mu(X,t).Z_\mu(X',t).$

Examples: - $k = \mathbb{F}_q$, $\mu(X) = \sharp X(k)$, $Z_\mu(X,t)$ is the Hasse-Weil zeta function $Z(X,t) \in \mathbb{Z}[[t]] \cap \mathbb{Q}(t)$.

- $k = \mathbb{C}$, $\mu = \chi_c$, $Z_\mu(X,t) = (1-t)^{-\chi_c(X)}$ (MacDonald)

Prop. (Kapranov) X smooth projective curve of genus g:

$Z_\mu(X,t) = \frac{P_\mu(X,t)}{(1-t)(1-\mathbb{L}t)}$, deg $P_\mu(X,t) = 2g$, and
$Z_\mu(X,t) = \mathbb{L}^{g-1}t^{2g-2}Z_\mu(X,\mathbb{L}^{-1}t^{-1})$.

(Larsen, Lunts) X product of two curves of genus > 1: $Z_\mu(X,t)$ is *not* rational for $\mu = id$.

In the sequel, we try to remedy this by working with a μ which is "sufficiently universal", but for which one can hope that $Z_\mu(X,t)$ is always rational. Concretely, we will work with genuine motives instead of "poor man's motives".

rationality of $Z_p(X)$: X curve

$$X^n \longrightarrow J(X)$$

$$(x_1,\dots,x_n) \longmapsto [x_1]+\dots+[x_n]-n[x_0]$$

factors through $S^n X \longrightarrow J(X)$ proj. bundle if $n \geq 2g-1$.

$$S^n(X) \hookrightarrow S^{n+1}(X)$$

$$x_1+\dots+x_n \longmapsto x_0+x_1+\dots+x_n$$

complement: vector bundle of rk $n+1-g$ over $J(X)$

$$\Rightarrow \quad [S^{n+1}X]-[S^n X] = [J(X)]\,\mathbb{L}^{n+1-g}$$

$$\Rightarrow \quad Z_p(X,T)(1-T)(1-\mathbb{L}T) \text{ polynomial.}$$

by telescoping

Chow motives

linearization of $SmP(k)$

morphisms : algebraic correspondences

(of dim.= the dim of the source)

modulo rational equivalence

---

take pseudo-abelian envelop

(= add kernels of idempotents)

and invert $\tilde{h}(\mathbb{P}^1)$

Chow motives $\longrightarrow$ Numerical motives (modulo numerical eq.)

$CHM(k)_F \longrightarrow NM(k)_F$

$\otimes$

semisimple category (Jannsen)

Kimura - O'Sullivan conjecture :

any $M \in CHM(k)_F$ decomposes

$M_+ \oplus M_-$ $\qquad S^n M_- = \bigwedge^n M_+ = 0$ for $n \gg 0$.

rmk: the analogous cj. for $NM(k)_F$ amounts to the sign conj. : the even Künneth projector $H^*(X) \twoheadrightarrow H^{even}(X) \hookrightarrow H^*(X)$ is algebraic.

III. Motivic zeta functions.

Fix a field F of "coefficients" of car. 0.

We now work with Chow motives:

$K_0(CHM(k)_F)$ $([M \oplus M'] = [M] + [M'])$

$\cong K_0(DM_{geom}(k)_F)$ (Bondarko) $(M' \to M \to M'' \to M'(1))$

Ring under $\otimes$

Example: (Manin)

$$f := t + \sum_{2}^{\infty} [\bar{M}_{0,n+1}] \frac{t^n}{n!}, \quad (1 + t - f)\frac{df}{dt} = 1 + f$$

in $K_0(CHM(k)_F)[[t]]$.

$$\boxed{Z_{mot}(M,t) := \sum_{0}^{\infty} [S^n M]\, t^n} \in (CHM(k)_F)[[t]].$$

$$Z_{mot}(M + M', t) = Z_{mot}(M,t).Z_{mot}(M',t).$$

Remarks: - (car$k = 0$) via prop. 1 (motives satisfy blow-up relations), there is a natural homom.

$$\mu : K_0(Var(k)) \to K_0(CHM(k)_F),$$

$Z_\mu = Z_{mot}$ ($\mathbb{L} \mapsto F(-1)$, Lefschetz motive).

- $k = \mathbb{F}_q$, there is a natural homom.

$$\nu : K_0(CHM(k)_F) \to \mathbb{Z}, \ \ [X] \mapsto \sharp X(k)$$

(trace of Frobenius) , $\nu(Z_{mot}) = Z$ (Hasse-Weil-type zeta function).

Prop. If M is finite-dimensional in the sense of Kimura ($M = M_+ \oplus M_-$, $S^n M_- = \Lambda^n M_+ = 0$ for $n >> 0$), then $Z_{mot}(M, t)$ is rational.

Examples: motives of abelian type (products of curves, Fermat hypersurfaces, etc...

Functional equation (in the finite-dimensional case):

Prop. (Kahn)

$$Z_{mot}(M^\vee, t^{-1}) = (-1)^{\chi_+(M)} . \det M . t^{\chi(M)} . Z_{mot}(M, t)$$

(where $\det M = \Lambda^{\chi_+} M_+ \otimes (S^{-\chi_-} M_-)^{-1}$.)

IV. Motivic zeta functions of curves.

X smooth projective curve/ $k = \mathbb{F}_q$

$Z(X,t) = \sum_{D \geq 0} t^{\deg D}$ (sum over effective divisors)

$= \sum_{\mathcal{L}} h^0(\mathcal{L})_q . t^{\deg \mathcal{L}}$ (sum over line bundles; $n_q = 1 + q + \ldots + q^{n-1}$)

Substitute $q \mapsto u$ (indeterminate)

Pellikan: $Z(X,t,u) := \sum_{\mathcal{L}} h^0(\mathcal{L})_u . t^{\deg \mathcal{L}} = \dfrac{P(X,t,u)}{(1-t)(1-ut)}$

Baldassarri-Deninger-Naumann:

$$Z_{mot}(X,t,u) := \sum_{n,d} [Pic_n^d]n_u . t^d = \frac{P_{mot}(X,t,u)}{(1-t)(1-ut)} \in K_0(CHM(k)_F)[[t,u]]$$

(where Pic_n^d classifies line bundles with $h^0(\mathcal{L}) \geq n$).

$$\begin{array}{ccccc} & \overset{u\mapsto q}{\nearrow} & Z_{mot}(X,t) & \overset{\nu}{\searrow} & \\ Z_{mot}(X,t,u) & & & & Z(X,t). \\ & \underset{\nu}{\searrow} & Z(X,t,u) & \underset{u\mapsto q}{\nearrow} & \end{array}$$

V. Motivic Artin L-functions. (Dhillon-Minac)

V: fin. dim. vector space/F

$M \in CHM(k)_F \mapsto V \otimes M$:

$Hom(V \otimes M, M') = Hom_F(V, Hom(M, M'))$.

G: finite group, $\rho : G \to GL(V)$,

$$\boxed{L_{mot}(M, \rho, , t) := Z_{mot}((V \otimes M)^G, t)}$$

Def. extends to characters χ of G ($\mathbb{Z}$-linear combinations of ρ's)

One has the same formalism (as for usual Artin L-functions)

$L_{mot}(M, \chi + \chi', t) = L_{mot}(M, \chi, t).L_{mot}(M, \chi', t)$

$L_{mot}(M, \chi', t) = L_{mot}(M, Ind_{G'}^{G}\chi', t) \quad (G' < G)$

$L_{mot}(M, \chi'', t) = L_{mot}(M, \chi, t) \quad (G' \triangleleft G$, χ coming from char. χ'' of G/G')

$Z_{mot}(M, t) = \prod_{\chi \text{ irr.}} L_{mot}(M, \chi, t)^{\chi(1)}$.

Example: G acting on a curve $X \rightarrow$ action of G on the motive of X via $(g^*)^{-1} \rightarrow L_{mot}(X, \chi, t)$

$k = \mathbb{F}_q$, $\nu(L_{mot}(X, \chi, t)) =$ Artin non-abelian L-function $L(X, \chi, t)$:

$\log L(X, \chi, t) = \sum \nu_n(X) \frac{t^n}{n} \quad \nu_n(X) = \frac{1}{\sharp G} \sum \chi(g^{-1}) \sharp Fix(gF^n)$

This leads to motivic Artin symbols, motivic Cebotarev theorem... (D. M.)

Interlude

G connected split semi-simple algebraic group $/k$.

$T \subset G$ max. torus, $\dim = r$ $\qquad S(X(T)) \circlearrowleft W$

$S(X(T))_k^W$ generated in deg. $d_1, \dots d_r$

(eg. $G = SL_n$, $\quad d_1 = 2, \dots, d_r = n$)

(*) $\quad (T-1)^r \sum_{w \in W} T^{\ell(w)} = \prod_1^r (T^{d_i} - 1)$

(**) $\quad \sum d_i = \frac{1}{2}(\dim G + r)$

Prop (Behrend / Dhillon)

in $K_0(Var(k))[\frac{1}{\mathbb{L}}]$ or $K_0(CHM(k)_F)$,

$$[G] = \mathbb{L}^{\dim G} \prod_1^r (1 - \mathbb{L}^{-d_i}).$$

B Borel, U unip' rad.

$[G] = [G/B][T][U]$

$[U] = \mathbb{L}^u$, $\quad u = \frac{1}{2}(\dim G - r)$

$[T] = (\mathbb{L} - 1)^r$,

$[G/B] = \sum_{w \in W} \mathbb{L}^{\ell(w)}$ $\qquad$ (Bruhat decomposition)

(*) + (**) $\rightsquigarrow$ formula.

$\mathbb{L}^{-\dim G} \prod^r (1 - \mathbb{L}^{-d_i})^{-1}$

algebraic group $/k$.

$T \subset G$ max. torus, $\dim = r$ $\qquad S(X(T)) \circlearrowleft W$

$S(X(T))_{\mathbb{Q}}^{W}$ generated in deg. $d_1, \dots d_r$

(eg. $G = SL_n$, $\qquad d_1 = 2, \dots, d_r = n$)

$$(*) \quad (T-1)^r \sum_{w \in W} T^{\ell(w)} = \prod_1^r (T^{d_i} - 1)$$

$$(**) \quad \sum d_i = \tfrac{1}{2}(\dim G + r)$$

<u>Prop</u> (Behrend / Dhillon)

in $K_0(Var(k))[\frac{1}{\mathbb{L}}]$ or $K_0(CHM(k)_F)$,

$$[G] = \mathbb{L}^{\dim G} \prod_1^r (1 - \mathbb{L}^{-d_i}).$$

B Borel, U unip. rad.

$[G] = [G/B][T][U]$

$[U] = \mathbb{L}^u$, $\quad u = \frac{1}{2}(\dim G - r)$

$[T] = (\mathbb{L}-1)^r$,

$[G/B] = \sum_{w \in W} \mathbb{L}^{\ell(w)}$ $\qquad$ (Bruhat decomposition)

$(*) + (**) \rightsquigarrow$ formula.

$\rightsquigarrow$ $\boxed{[BG] = [G]^{-1} = \mathbb{L}^{-\dim G} \prod_1^r (1 - \mathbb{L}^{-d_i})^{-1}}$ in suitable localization

VI. Towards motivic Tamagawa numbers.
(Behrend-Dhillon)

X: projective smooth curve$/k$ of genus g,
$K = k(X)$.

G: simply connected split semisimple algebraic group$/k$, e.g. SL_n.

$Bun_{G,X}$: moduli stack of G-torsors on X (smooth of dim. $(g-1).\dim G$)

admits a infinite stratification by pieces $[Xi/GL_{n_i}]$ of dim. $\to -\infty$

$[Bun_{G,X}] := \sum[X_i][GL_{n_i}]^{-1} \in K_0[CHM(k)_F]\hat{}$ (suitable completion w.r.t. $\mathbb{L}^{-1}$)

$([GL_n]^{-1} = [BGL_n] = \mathbb{L}^{-n^2}(1+\ldots) \in \mathbb{Z}[[\mathbb{L}^{-1}]])$

Conj. (Behrend-Dhillon)

$$[Bun_{G,X}] = \mathbb{L}^{(g-1).\dim G} \prod_i Z_{mot}(X, \mathbb{L}^{-d_i})$$

(d_i: exponents of $G+1$, e.g. $2, 3, \ldots, n$ for SL_n)

Prop. (B. D.) this is true for $X = \mathbb{P}^1$ and any G, and for $G = SL_n$ and any X.

Case SL_n. Two specializations of the motivic formula:

- $k = \mathbb{C}$, $\mu = \chi_c$ via gauge theory à la Yang-Mills (Atiyah-Bott 1982, Teleman)

computation of $H^*(Bun_{G,X}) = H^*(G)^{\otimes 2g} \otimes H^*(BG) \otimes H^*(\Omega G)$

P_w: weighted Poincaré pol. $\sum_{i,j} \dim(W_i H^j) t^i$

$$P_w(Bun_{G,X}, t) = \prod_2^n \frac{(1+t^{2i-1})^{2g}}{(1-t^{2i})(1-t^{(2i-2)})}.$$

• $k = \mathbb{F}_q,\ \mu = \nu,\ K = k(X)$.

$Bun_{G,X} \sim$ transformation groupoid of $G(K)$ on $G(\mathbb{A}_K)/\mathcal{K}$

$$(\mathcal{K} := \prod_x G(\hat{\mathcal{O}}_{X,x}))$$

$$\sharp Bun_{G,X}(k) = \frac{vol(G(K)\backslash G(\mathbb{A}_K))}{vol(\mathcal{K})}$$

with Tamagawa measure

$$vol(\mathcal{K}) = q^{(g-1).(n^2-1)} \prod_2^n \zeta_K(i)$$

$vol(G(K)\backslash G(\mathbb{A}_K)) = \tau(SL_n) = 1$ (Harder).

* * *

* using Tamagawa numbers, one can show that $Bun_{G,X}$ is geometr. connected.
* If G is no longer simply connected ?

<u>conj</u> :

$$[Bun_{G,X}] = \tau(G)\, \mathbb{L}^{(g-1).\dim G} \prod_i Z_{mot}(X, \mathbb{L}^{-d_i})$$

$\tau(G) = |\pi_1(G)|$ is the number of connected components of $Bun_{G,X}$,

and has a motivic-measure-theoretic interpretation.

—.