10 - 19 - 2009 Rob Harron BU NT Seminar 4:15 pm.

Greenberg's $\mathcal{L}$-invariant of the symmetric 6-th power of an ordinary modular form.

(Symm. square - last time)
refresh.

Intro. Iwasawa thy $\rightsquigarrow$ p-adic interpolation.

$\mathcal{L}$-invariant $\leftarrow$ measure failure.

ex $E/\mathbb{Q}$. ell. curve with good ord or semistable red. at p.

Let α_p & β_p its Frob_p eigenvalue

assume $|\alpha_p|_p = 1$.

Mazur - Swinnerton-Dyer
M - T - T.

Construct $L_p(s, E, \alpha_p)$ p-adic analytic ftn.

s.t. $L_p(1, E) = (1 - \alpha_p^{-1}) \cdot (1 - \frac{\beta_p}{p}) \cdot \frac{L(1,E)}{\Omega_E}$

Selmer grp.

main conj

BSD.

M-T-T : 80's

if E has split multiplicative reduction at p.

then $\alpha_p = 1$, $\rightsquigarrow$ trivial (exceptional) zero. //

so $L_p(1,E) = 0$

(even when $L(1,E) \neq 0$.)

Conj. (M-T-T)

E, split multi red. & $L(1,E) \neq 0$.

Then

$$\boxed{L_p'(1,E) = \mathcal{L}_p(E) \cdot \frac{L(1,E)}{\Omega_E}}. \quad //$$

where $\mathcal{L}_p(E) \neq 0$.

$\downarrow$

$\mathcal{L}$-invariant.

$$= \frac{\log_p q}{\mathrm{ord}_p q} \qquad q : \text{Tate parameter}.$$

Greenberg - Stevens. proved this.

$p \geq 3$

octd

$p = 2, 3$

$\downarrow$

Hasse invariants //

use Hida theory & Look at 2-variable p-adic L-functions.

$$L_p(s, k, E)$$

$\uparrow$

wt variable.

~~(p-specialization)~~

$$E \rightsquigarrow f.\ \cancel{\rightsquigarrow f_p} \in S_2(pN)$$

$\overset{\text{Hida}}{\rightsquigarrow} \mathbf{f}$: p-adic family of m.f. varying wt.

$$E \rightsquigarrow \underset{\textstyle \sum a_n \cdot q^n}{\overset{}{f}} \rightsquigarrow \underset{\textstyle \sum a_n(k) \cdot q^n.}{\mathbb{f}}$$

$$\alpha_p \longrightarrow a_p \longrightarrow a_p(k)$$

$\mathbb{f}_2 = f.$

for $k \in \mathbb{Z}_{\geq 2}$, $\mathbb{f}_k \in (\text{wt } k \text{ m.f.})$

relate $\mathcal{L}_p(E)$ to some Galois cohomology.

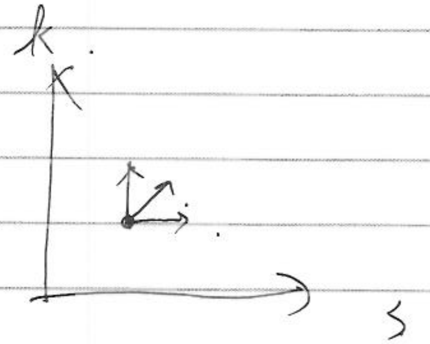
to obtain a relation

$$\boxed{\mathcal{L}_p(E) = - 2 \cdot \frac{a_p'(2)}{a_p}}$$

$a_p' \overset{\text{let}}{=} a_p'(2)$

Want to generalize this

relate to $\left.\dfrac{\partial L_p(s, k, E)}{\partial k}\right|_{\substack{s=1 \\ k=2}}.$



p-adic 2-variable ftn'l eqtn.

Coates - Perrin-Riou : conjectural shape for p-adic interpolation of $L(s, M)$

↑ ordinary motive.

Greenberg : ~~can~~ gives an "arithmetic" candidate for $\mathcal{L}_p(M)$

Ex. (Symm. square)

$f \in S_k(\Gamma_0(N))$. $p \nmid N$.

$\rho_{k,p} : G_{\mathbb{Q}} \longrightarrow GL_2(K)$

$$\rho_{k,p}|_{G_p} \sim \begin{pmatrix} \chi^{k-1} \cdot \delta^{-1} & \varphi \\ 0 & \delta \end{pmatrix}$$

χ : p-adic cyclotomic char

δ : unram. char $\mathrm{Frob}_p \mapsto a_p \cdot$ unit.

$$(\rho_2, V_2) := (\mathrm{Sym}^2 \rho_{f,p})(1-k_0)$$

$$\parallel$$

$$\mathrm{Ad}^0 \rho_{f,p}.$$

$$\rho_2|_{G_p} \sim \begin{pmatrix} \chi^{k_0-1}\cdot\delta^{-2} & * \\ 0 & \chi^{1-k_0}\cdot\delta^2 \end{pmatrix}$$

F^+V_2, F^-V_2, 1 "means" there's a trivial zero of order 1.

(1, χ contributes 1.

$\begin{pmatrix} \chi & *^{0} \\ 0 & 1 \end{pmatrix}$ ~~const~~ contributes 1)

ex χ^2

χ

$\begin{pmatrix} \chi & * \\ & 1 \end{pmatrix}$ → order 3.

1

χ^{-1}

$$W_2 = F^-V_2 / F^+V_2.$$

$\wr$

trivial 1-dim'l $G_{\mathbb{Q}_p}$-module.

Greenberg's $\mathcal{L}$-invariant.

$$\mathcal{L}_p^{Gr}(\rho_2).$$

Assume '$\mathrm{Sel}_{\mathbb{Q}}$' $(\rho_2 \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is finite.

Greenberg's conj. and ($=H^1_f$ & H^1_g of Bloch-Kato). thesis

$L(1,\rho_2) \neq 0$.

Flach - Weston - Kisin

use same def'n.

construct $H^{\text{exceptional}}_{\text{Global}} \subseteq H^1(G_{\mathbb{Q}}, \rho_2)$

a 1-dim'l subsp. w/ specific local behavior.

so that its image

$H^{\text{exc}}_{\text{loc}} \subseteq H^1(G_{\mathbb{Q}_p}, \rho_2)$

$$\begin{array}{ccc} H^1(\mathbb{Q}, \rho_2) & \longrightarrow & H^1(\mathbb{Q}_p, \rho_2) \\ \cup\!| & & \cup\!| \\ H^{\text{excep}}_{\text{Global}} & & H(\mathbb{Q}_p, F^-\rho_2) \to H^1(\mathbb{Q}_p, \mathcal{W}_2) \\ & & \cup\!| \qquad\qquad \cup\!| \\ & & \longmapsto H^{\text{exc}}_{\text{loc.}} \end{array}$$

is nonzero & "ramified" no ∞-slope.

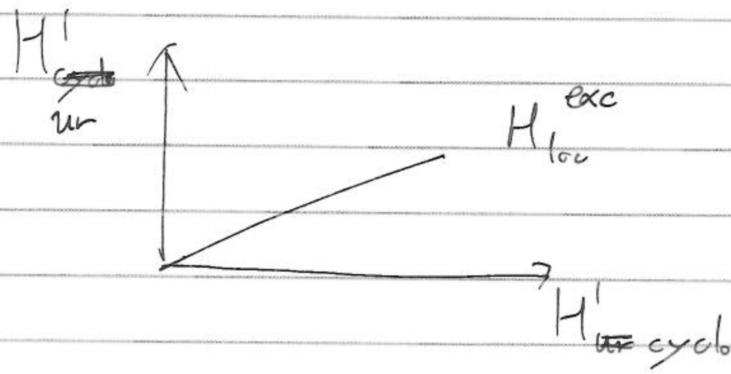
$$H^1(\mathbb{Q}_p, \mathcal{W}_2) \cong \operatorname{Hom}(G^{ab}_{\mathbb{Q}_p}, \mathcal{W}_2) \ni \varphi.$$

2-dim'l v.sp.

coordinates: take any principal unit u.

let $\mathrm{rec}: \mathbb{Q}_p^\times \longrightarrow G^{ab}_{\mathbb{Q}_p}$.

$$\left(\frac{-1}{\log_p u} \varphi(\mathrm{rec}(u)), \varphi(\mathrm{Frob}_p) \right)$$



$\mathcal{L}_p(\rho_2)$ = the slope of this line.

(Hida 2004, same result w/ slight different assumptions)

<u>Now</u>. $\rho_6 := (\mathrm{Sym}^6 \rho_{f,p})(3(1-k_0))$

Why? odd powers $\rightarrow$ Tate curve

$$\mathcal{L}(\mathrm{Sym}^{2n+1} \rho_{f,p}(n)) = -2a_p'.$$

$\left.\begin{matrix} 4? \\ 2\cdot \text{even} \end{matrix}\right)$ critical. (Deligne's sense)

<u>Idea</u>. $\rho_6|_{G_p} \sim \begin{pmatrix} \chi^{3(k_0-1)}\cdot\delta^{-6} & & & & \\ & \chi^{2(k_0-1)}\cdot\delta^{-4} & & * & \\ & & \chi^{k_0-1}\cdot\delta^{-2} & & \\ & & & 1 & \\ & & & & \chi^{1-k_0}\cdot\delta^{2} \\ & & & & \vdots \end{pmatrix}$ F^+ F^-

$\sim$ $\cancel{\chi^{3(k_0-1)}}$

m odd

ρ_{2m} $\quad W = F^-V_0 / F^+V_0.$

construct $c \in H^1(G_{\mathbb{Q}}, \rho_6)$ generates H^{exc}_{global}.

Look at $\mathrm{Sym}^3 \rho_{f,p} = \rho_3$.

($\mathrm{Ad}^0 \rho_3$: ρ_6 occurs as a direct summand) ✓

want a deformation of ρ_3.

call it $\tilde{\rho}_3$ over R, & p, prime

such that $\tilde{\rho}_3 \bmod p =: \tilde{\rho}_{3,p} \cong \rho_3$.

$r := \tilde{\rho}_3 \otimes \rho_{f,p}$.

$r_p = \rho_3 \otimes \rho_{f,p} \cong \mathrm{Sym}^4 \rho_{f,p} \oplus \mathrm{Sym}^2 \rho_{f,p} \otimes \det$.

use Ribet's approach to construct

ext'n $\mathcal{E} \in \mathrm{Ext}^1(\mathrm{Sym}^2 \rho_{f,p} \otimes \det, \mathrm{Sym}^4 \rho_{f,p})$

$$\begin{aligned}&\qquad\qquad \parallel \\ &\bigoplus_{i=0}^{2} H^1(\mathbb{Q}, \underbrace{\mathrm{Sym}^{2(3-i)} \rho_{f,p} \otimes \det^{i-3}}_{\rho_6})\end{aligned}$$

get c_6.

What is $\tilde{\rho}_3$?

If take $\tilde{\rho}_3 = \mathrm{Sym}^3 \rho_{\mathcal{F}}$, $\mathcal{F}$ Hida family containing f.

get $c_6 = 0$.

<u>idea</u>. get a richer deformation.

<u>Thm</u>. (Ramakrishna – Shahidi, 2007)

k_0 even, f <u>not</u> CM.

$\exists \pi$ on $GSp(4)$ s.t. $L(s,\pi) = L(s, \mathrm{Sym}^3 f)$

<u>Thm</u>. (Tilouine – Urban, 1999) Hida thy on $GSp(4)$

($k_0 \geq 4$)

Two variable deformation of ρ_3.

$$\tilde{\rho}_3 \sim \begin{pmatrix} \theta_1\theta_2\mu_1 & & & \\ & \theta_2\mu_2 & & * \\ & & \theta_1\mu_1^{-1} & \\ & 0 & & \mu_2^{-1} \end{pmatrix}$$

Let $a_p^{(1,1)} := \left.\frac{\partial \mu_1}{\partial s_1}\right|_p (\mathrm{Frob}_p)$

Thm (Harron, 2009)

Assume $Sel_{\mathbb{Q}}(\rho_6 \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ finite.

Then

$$\mathcal{L}_p^{Gr}(\rho_6) = -10 \cdot a_p^3 \cdot a_p^{(1,1)} - \frac{12 \cdot a_p'}{a_p}.$$

$$\mathcal{L}_p^{Gr}(\rho_6) = -10\, a_p^3 \cdot a_p^{(1,1)} + 6\, \mathcal{L}_p^{Gr}(\rho_2)$$

Wild guess:

$$\left(\text{if } a_p^{(1,1)} = \frac{-a_p'}{a_p^4}, \text{ then they're equal.} \right)$$

Idea: for higher power, ...