9-21-2009 Jay P. BU NT Semenar 4:15 PM.

A Non-ordinary Control Theorem.

Mazur ~'70

$E/\mathbb{Q}$ good ordinary at p

$K_n := K(\mu_{p^n})$, $n \le \infty$.

$\Gamma_n := Gal(\mathbb{Q}_n/\mathbb{Q}) \overset{?}{=} Gal(\mathbb{Q}_{p,n}/\mathbb{Q}_p)$

Let $n < \infty$.

$$Sel_p(E/\mathbb{Q}_n) := \ker\left(H^1(G_{\mathbb{Q}_n}, E[p^\infty]) \to \bigoplus_v \frac{H^1(G_{\mathbb{Q}_{n,v}}, E[p^\infty])}{\text{img. of Kummer}}\right)$$

$$0 \to E(\mathbb{Q}_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to Sel_p \to Ш(E/\mathbb{Q}_n)[p^\infty] \to 0$$

finite ?? Assume!

and $Sel_p(E/\mathbb{Q}_\infty) = \varinjlim_{n<\infty} Sel_p(E/\mathbb{Q}_n)$.

Then $Sel_p(E/\mathbb{Q}_n) \longrightarrow Sel_p(E/\mathbb{Q}_\infty)^{Gal(\mathbb{Q}_\infty/\mathbb{Q}_n)}$.

have ker / coker finite and bdd w.r.t. $n < \infty$.

Short, modern sketch of proof.

$T = T_p E$.

E, ordinary :

$$0 \longrightarrow \overset{1}{T^+} \longrightarrow \overset{\text{rank } 2}{T|_{G_{\mathbb{Q}_p}}} \longrightarrow \underset{\text{unram.}}{\overset{1}{T^-}} \longrightarrow 0 \qquad (*)$$

Let $(n \leq \infty)$

$$T_n^* = T^* \otimes_{\mathbb{Z}_p} \Lambda_n \qquad \Lambda_n = \mathbb{Z}_p[[\Gamma_n]]$$

note $I_n := \ker[\Lambda_\infty \to \Lambda_n]$ is principal.
$= (f_n)$, so

$$[\overset{-1}{\Lambda_\infty} \longrightarrow \overset{0}{\Lambda_\infty}] \xrightarrow[\text{quasi-isom.}]{\sim} [\overset{0.}{\Lambda_n}].$$

↓
next page.

mimicing

(analogue with derived ctg)

Consider $\underline{R}\Gamma_f(\mathbb{Q}_n, T) = \mathrm{Cone}[\underline{R}\Gamma(\mathbb{Q}, T_n) \to \underline{R}\Gamma(\mathbb{Q}_p, T_n^-)][-1]$

(1) by Shapiro's lemma,

$$\underline{R}\Gamma(\mathbb{Q}, T_n) \cong \underline{R}\Gamma(\mathbb{Q}_n, T) \qquad n<\infty.$$

"a little p-adic Hodge theory"

$$H_f^2(\mathbb{Q}_n, T)^\vee = \mathrm{Sel}_p^{str}(E/\mathbb{Q}_n) \qquad n \le \infty.$$

under global duality ($\because$ p=p more than local duality).

and Sel_p, Sel_p^{str} differ by a finite bdd amount.

(2) satisfies "control" ← derived $\otimes$

$$\underline{R}\Gamma_f(\mathbb{Q}, T_\infty) \overset{\mathbb{L}}{\otimes}_{\Lambda_\infty} \Lambda_n \xrightarrow[\text{quasi}]{\sim} \underline{R}\Gamma_f(\mathbb{Q}, T_n)$$

(this just follows from

$$[\Lambda_\infty \xrightarrow{f} \Lambda_\infty] \xrightarrow[\text{quasi}]{\sim} [\Lambda_n].)$$

$$H^{\ge 3} = 0 \Rightarrow H_f^2(\mathbb{Q}, T_\infty) \otimes_{\Lambda_\infty} \Lambda_n \to H_f^2(\mathbb{Q}_n, T)$$

get Mazur's result after $\vee$ dual

Generalize this process!!

To go ordinary $\rightsquigarrow$ non-ordinary
bdd $\rightsquigarrow$ unbdd.

<u>Picture</u> : $p > 2$, $\Gamma_\infty = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$

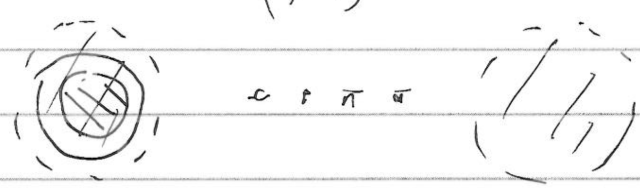
$$\simeq \mathbb{Z}_p^\times \simeq \underbrace{\mu_{p-1}}_{\Delta \simeq \mathbb{Z}/(p-1)} \times \underset{\gamma^{\mathbb{Z}_p}}{\underset{\wr\!\wr}{(1+p\mathbb{Z}_p)}}.$$

$$\Lambda_\infty = \mathbb{Z}_p[[T]]^{(p-1)} \quad (T = \underset{\text{made a choice.}}{\gamma} - 1)$$

$$\mathcal{W} = \mathrm{Spf}(\Lambda_\infty)^{an}.$$

"$|T| <$"
rigid open disc

$(p-1)$



Λ_∞ = all ftn's on $\mathcal{W}$ bdd by 1.

Λ_n = residue rings of special pts of $\zeta - 1$ ($\zeta \in \mu_{p^n}$)

$\mathcal{W} = \bigcup_m \mathcal{W}_m$, $\mathcal{W}_m$: "$|T| \le p^{-1/m}$".

$$\Lambda_{an} := \text{all fns} = \Gamma(\mathcal{W}, \underset{\text{str. sheaf}}{\mathcal{O}})$$

$$= \mathbb{Q}_p\{\{T\}\}^{(p-1)}$$

convergent for $|T|<1$ (possibly unbdd)

$$= \varprojlim_{m<\infty} \Gamma(\mathcal{W}_m, \mathcal{O})$$

$$= \mathbb{Q}_p\langle T, T^m/p\rangle^{(p-1)}$$

cvgt for $|T| \leq p^{-1/m}$. (actually, all bdd)

Each $f_n \in \Lambda_\infty \to \Lambda_{an}$. defines a divisor with fin. support.

admissible cover $\rightsquigarrow$ contains any finite set.

So $\Lambda_{an}/f_n[1/p] \xrightarrow{\sim} \Lambda_{an}/f_n$

$$\to \Lambda_{an,m}/f_n \text{ for } m \gg 0 \ (n)$$

To get ord $\rightsquigarrow$ nonordinary
bdd $\rightsquigarrow$ unbdd.
$\Lambda_\infty \rightsquigarrow \Lambda_{an}$
($\Lambda_{an,m}$; practically)

$$T|_{G_{\mathbb{Q}_p}} \rightsquigarrow \underline{D} = \underline{D}^\dagger_{rig}(T|_{G_{\mathbb{Q}_p}})$$

(φ, Γ)-modules

$$\underline{D}^\dagger_{rig} : Rep_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \xrightarrow{\sim} \underline{M}^{ét}(\varphi, \Gamma_\infty)/_{B^\dagger_{rig,\mathbb{Q}_p}}.$$

$$\subset \underline{M}(\varphi, \Gamma_\infty) \quad \text{: may have non-étale slope.}$$

<u>Advantage</u> (a choice of Frobenius eigenvalue) gives

$$0 \to D^+ \to D \to D^- \to 0.$$

HT wt 0.

regardless of whether ordinary or not !!

$$D^*_{\mathcal{W},m} := D^* \otimes_{B^\dagger_{rig,\mathbb{Q}_p}} D^\dagger_{rig}(\Lambda_{an,m}) \quad ; \quad Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \Gamma_\infty \subset \Lambda_\infty^\times$$

$\circlearrowleft$ Gal.

$$\Lambda_\infty^\times \to \Lambda_{an}^\times \to \Lambda_{an,m}^\times$$

family of (φ, Γ_∞)-modules over $\mathcal{W}_m$ attached ~~to~~ by Breuil-Colmez.

Galois acts via this char.

<u>Thm</u> $\exists$ good theory of Galois cohomology with coeff. over a rigid space. X.

(1) it patches to form coherent analytic sheaves on X.

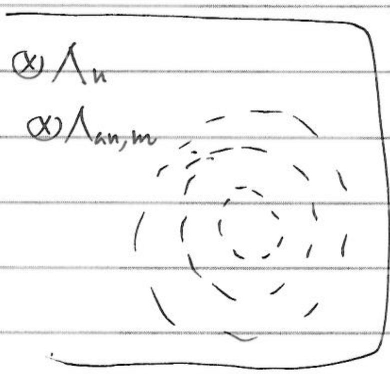
(2) $\underline{R}\Gamma(G_{\mathbb{Q}_p}, V_{/X}) \xrightarrow{\sim} \underline{R}\Gamma(\underline{D}^{\dagger}_{rig}(V_{/X}))$

"Herr complex".

$V = T[1/p]$, $T = T_p E$. , $E/\mathbb{Q}$ good reduction at p.

Consider : $\underline{R}\Gamma_f(\mathbb{Q}, V \otimes \Lambda_{an,m})$

$$= \mathrm{Cone}\left[\underline{R}\Gamma(\mathbb{Q}, V\otimes\Lambda_{an,m}) \to \underline{R}\Gamma(D^-_{W,m})\right][-1].$$



(1) Shapiro's lemma.

$$\underline{R}\Gamma_f(\mathbb{Q}, V\otimes \Lambda_{an,m}/\underset{\substack{\| \\ (f_n)}}{I_n}) = \underline{R}\Gamma_f(\mathbb{Q}_n, V). \qquad m >> 0.$$

p-adic Hodge theory

$$H^2_f(\mathbb{Q}_n, V)^\vee = H^1_f(\mathbb{Q}_n, V)$$
$$= \mathbb{I}_p(\mathrm{Sel}_p^{str}).$$

strict Selmer gp.

Bloch-Kato. ✓.

Greenberg.

$$\mathbb{I}_p X = \varprojlim_p X$$

e.g. $\mathbb{I}_p(\mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Q}_p$.

(2) Control:

derived $\otimes$-product.

$$\underline{R\Gamma}_f(\mathbb{Q}, V\otimes\Lambda_{an,m}) \otimes^{\mathbb{L}}_{\Lambda_{an,m}} \Lambda_n \xrightarrow{\sim} \underline{R\Gamma}_f(\mathbb{Q}_n, V\otimes\Lambda_n)$$

$m \gg 0$ (n).

$$H^2_f(V\otimes\Lambda_{an,m}) \otimes_{\Lambda_{an,m}} \Lambda_n \xrightarrow{\simeq} H^2_f(\mathbb{Q}_n, V)$$

Summing Up.

Let $X = \left(\varprojlim_m H^2_f(V\otimes\Lambda_{an,m})\right)^\vee$

$= \varinjlim_m H^2_f(V\otimes\Lambda_{an,m})^\vee$ big cpt. type space?

Then $\Phi_p(\mathrm{Sel}(E/\mathbb{Q}_p)) \xrightarrow{\sim} X^{\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}_n)}$ $\forall n < \infty$.

Remarks

(1) differences? between natural integral structures on two sides?
they differ increasingly, at a rate corresponding to slopes of φ?

(2) I chose a Frobenius eigenvalue α of E
to make X.
Let β be the other
Conj. X locally cotorsion (Each $H^2_f(V\otimes\Lambda_{an,m})$ is torsion.)
get char. ideal,
can compare to $L_{p,\alpha}(E)$
$L_{p,\beta}(E)$?

Get a main conj. whether or not $a_p = 0$.

Control for Galois cohomology of D^- ?

$$\underline{R\Gamma}(D^-) \otimes^{\mathbb{L}}_{\Lambda} B \xrightarrow[\text{quasi}]{\simeq} \underline{R\Gamma}(D^- \hat{\otimes}_A B)$$

correct (φ, Γ)-module from the restriction.

$$\underline{R\Gamma}(D^-) \otimes^{\mathbb{L}}_{\Lambda} B = [C_{Herr}(D^-)] \otimes^{\mathbb{L}}_A B = [C_{Herr}(D^-) \otimes_A B]$$

$\leftarrow$ derived $\otimes$.

four copies of D^- $\underset{\text{flat}}{\simeq} (B^{+}_{rig} \otimes A)$

$$\underline{R\Gamma}(D^- \hat{\otimes}_A B) = [C_{Herr}(D^- \hat{\otimes}_A B)] = [C_{Herr}(D^-) \hat{\otimes}_A B]$$

$$[C_{Herr}(D^-) \otimes_A B] \xrightarrow{\sim} [C_{Herr}(D^-) \hat{\otimes}_A B].$$

2 scenerios when get $\cong$ for sure.

(1) B is a finite A-alg:

then $\otimes = \hat{\otimes}$.

$$\Lambda_{an, m} \twoheadrightarrow \Lambda_n[1/p]. \quad \checkmark$$

(2) When $H^i(D^-)$ are finitely generated $/A$.