Algebras and varieties related to finite subgroups of Sp(2n)

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Abstract

This talk is an introduction to my recent work with Victor Ginzburg.

The Cherednik algebra H_n is the algebra over $\mathbf{C}$ generated by the symmetric group S_n and two sets of generators $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$, where the defining relations are the obvious commutation relations between S_n and x_i, y_i, and the relations:

$$[x_i, x_j] = [y_i, y_j] = 0,$$

$$[x_i, y_j] = -s_{ij}, i \neq j$$

$$[x_i, y_i] = s_{i1} + ... + s_{in}.$$

(where s_{ij} is the permutation of i and j). This algebra has a filtration given by $deg(S_n) = 0$, $deg(x_i) = deg(y_i) = 1$, and it is known from Cherednik's work that $gr(H_n)$ is the smash product $\mathbf{C}[S_n] \bullet \mathbf{C}[\mathbf{x}, \mathbf{y}]$ (the Poincare-Birkhoff-Witt theorem).

Ginzburg and I proved the following:

Theorem. 1. Let Z_n be the center of H_n. Then $gr(Z_n) = \mathbf{C}[\mathbf{x}, \mathbf{y}]^{S_n}$

2. Let $M_n = Spec(Z_n)$. Then M_n is a smooth, affine, symplectic algebraic variety of dimension $2n$.

3. There exists an algebraic vector bundle V on M_n of dimension $n!$ such that $H_n = \mathrm{End}(V)$. The group $S_n \subset H_n$ acts on fibers of this bundle as in the regular representation. In particular, all irreduicble representations of H_n are of dimension $n!$ and are parametrized by points of M_n.

4. The variety M_n is isomorphic to the Kazhdan-Kostant-Sternberg-Wilson "Calogero-Moser space", which is the set of pairs of n by n matrices X, Y such that $[X, Y] + 1$ has rank 1, modulo conjugation.

Some, but not all, of these results can be generalized to the case when S_n is replaced by any Coxeter group and even any group generated by symplectic reflections: some as theorems, some as conjectures.

I will try to describe some of these results and also their quantum analogs.