Integrable ellipsoidal billiards with separable potentials, billiards on quadrics, and the Poncelet theorem.

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Abstract

One of the best known discrete integrable systems is the billiard inside an n-dimensional ellipsoid. Veselov showed that generic complex invariant manifolds of the algebraic billiard map are open subsets of (coverings of) hyperelliptic Jacobians and that the restriction of the map to such manifolds is described by the shift by a constant vector (an algebraic addition law). Later, Dragovic and Radnovic explicitly formulated generalized Cayleys conditions of periodicity of the map in terms of the moduli of underlying spectral curves.

On the other hand, there exist integrable generalizations of the billiard in the presence of an infinite number of separable polynomial potentials. We show that in the case of quartic and higher degree potentials the above algebraic geometrical description does not hold anymore. In particular, generic complex invariant manifolds become non-Abelian subvarieties (strata) of Jacobians, the billiard map does not have an algebraic form, and it is infinitely valued. Nevertheless, it is possible to describe the billiard by introducing an analog of addition law on the strata.

The above phenomenon is not quite exceptional: one can give several

examples of integrable maps which possess algebraic first integrals, but which are not algebraic themselves.

A relevant example is given by potentialless billiard on a quadratic surface Q with elastic impacts along its intersection with a confocal quadric. For this problem, Chang established an analog of the celebrated Poncelet theorem. We derive some explicit sufficient conditions of periodicity of the billiard trajectory on Q, which are related to the reducibility of hyperelliptic curves and their Jacobians.