Differential Geometry on the Renormalization Bundle

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Renormalization: A classical example

Consider an object suspended in a fluid. Applying a force F and measuring its acceleration gives its inertial mass using

$$F = m_i a$$

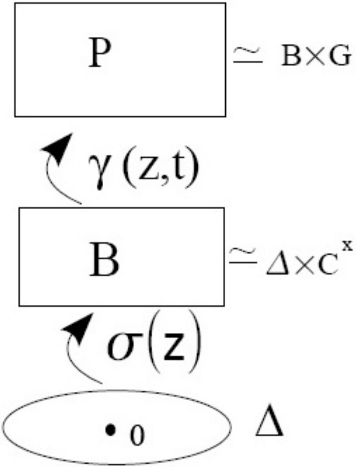
The object interacts with the surrounding fluid, so $m_i > m$, mass measured outside any fluid, m, the bare mass. Its inertial mass is

$$m_i = m + \alpha M$$

(Archimedes' principle).

In this scenario, the inertial mass is the renormalized mass. The bare mass is m, the unrenormalized mass, and the M is the interaction mass, or the counterterm. If the interaction cannot be turned off then the bare mass cannot be measured.

The Renormalization Bundle



Δ = complex dimension

C^{x} = renormalization mass

G = Spec H

Outline

- Define
  - Feynman Diagrams
  - The Hopf algebra of Feynman diagrams
  - Renormalization process
- Build the renormalization bundle
- Generalizing the renormalization bundle

Feynman Diagrams

Background: Lagrangian

- In general:

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_V$$

  - $\mathcal{L}_F$ quadratic form involving an exterior derivative (Δ)
  - $\mathcal{L}_V$ is a polynomial (minimal degree $= 3$)

Background: Lagrangian

- In general:

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_V$$

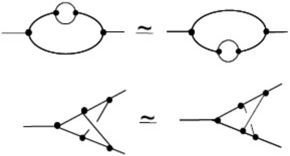
  - $\mathcal{L}_F$ quadratic form involving an exterior derivative (Δ)
  - $\mathcal{L}_V$ is a polynomial (minimal degree $= 3$)

- For this talk:

$$\mathcal{L} = \frac{1}{2}(|d\phi|^2 - m^2\phi^2) + g\phi^3$$

  - $\mathcal{L}_F = \frac{1}{2}(|d\phi|^2 - m^2\phi^2)$
  - $\mathcal{L}_V = g\phi^3$
  - ϕ is a hermitian scalar field
  - The space-time dimension is 6

Background: Feynman Diagrams



Definition

A Feynman graph is an abstract representation of a field interaction. It is drawn as a connected, not necessarily planar, graph with possibly differently labeled edges. The orientation of the embedding of the graph in the plane does not matter.

1. All vertices have valence 3
2. Vertices of valence one are replaced by *half edges* with no vertex at the end

Background: Feynman Integrals

Any particular diagram of interaction is associated to an integral

$$f(p_1, \ldots, p_n) \int_{\mathbb{R}^6} \frac{d^6 k}{\prod \Delta_i}$$

where

- Δ is the Laplacian
- Δ^{-1} is the associated Green's kernel
- Conservation of momentum

$$\sum p_i = 0$$

- **This integral is often divergent.** Thus we need to renormalize it.

Background: Graphs of interest

When are these integrals (superficially) divergent?

When they correspond to graphs with 2 or 3 external edges.

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Definition

A one particle irreducible, 1PI, graph is a connected Feynman graph such that the removal of any internal edge still results in a connected graph.

Background: Subgraph

Definition

An admissible subgraph of a divergent Feynman graph, γ, is a subgraph that can also be expressed as a divergent 1PI Feynman graph.

A subgraph of Γ is

1. A subset of vertices of Γ
2. A subset of interior edges meeting these vertices

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An admissible subgraph, can be expressed as a divergent 1PI Feynman diagram:

1. If an edge of Γ meets *1* vertex of γ, it is represented by *1* external edge of γ. If it meets γ at *2* vertices, and is not an edge of γ, then it is represented by *2* external legs of γ.
2. Each connected component of γ is 1PI
3. γ *has 2 or 3 external legs*

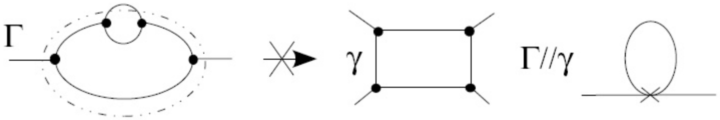
Background: Contracted graph

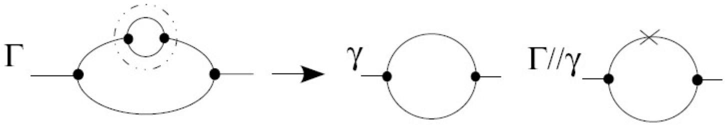
Definition

Let $\gamma = \gamma_1 \coprod \ldots \coprod \gamma_n$ A contracted graph is the Feynman graph derived by replacing each connected admissible subgraph, with a vertex v_{γ_i}. The resulting contracted graph is written $\Gamma//\gamma$.

Example

Inadmissible subgraph





Admissible subgraph

Background: Summary

1. Defined the Feynman diagrams
2. Defined subgraphs and contracted graphs

The Hopf Algebra

Hopf Algebra: Construction

Definition

The Hopf algebra $\mathcal{H}$ is generated by the vector space of indecomposable elements, $\mathbb{C} < x_\Gamma | \Gamma \in \{1\text{PI graphs of } \mathcal{L}\} >$.

$$\begin{aligned} m: \quad \mathcal{H} \otimes \mathcal{H} &\rightarrow \mathcal{H} \\ x_\Gamma \otimes x_{\Gamma'} &\rightarrow x_\Gamma x'_\Gamma \end{aligned}$$

Disjoint union of the 1PI graphs.

$$\begin{aligned} \eta: \quad \mathbb{C} &\rightarrow \mathcal{H} \\ 1_\mathbb{C} &\rightarrow 1_\mathcal{H} \end{aligned}$$

$$1_\mathcal{H} = x_\emptyset$$

$\mathcal{H}$ is a commutative algebra.

Hopf Algebra: Construction

$$\begin{array}{rccl} \Delta: & \mathcal{H} & \rightarrow & \mathcal{H} \otimes \mathcal{H} \\ & x_\Gamma & \rightarrow & 1 \otimes x_\Gamma + x_\Gamma \otimes 1 + \sum_{(\Gamma)} x_\gamma \otimes x_{\Gamma//\gamma} \end{array}$$

Δ is defined such that $\Delta(x_1 x_2) = \Delta(x_1)\Delta(x_2)$.

$$\begin{array}{rccl} \varepsilon: & \mathcal{H} & \rightarrow & \mathbb{C} \\ & x_\Gamma & \rightarrow & \begin{cases} x_\Gamma & \Gamma = \emptyset \\ 0, & \text{else} \end{cases} \end{array}$$

$\mathcal{H}$ is non-co-commutative coalgebra.

Hopf Algebra: Construction

$$\begin{aligned} S : \mathcal{H} &\rightarrow \mathcal{H} \\ x_\Gamma &\rightarrow -x_\Gamma - \sum_{(\Gamma)} m(S(x_\gamma) \otimes x_{\Gamma//\gamma}) \end{aligned}$$

$$S(x_\Gamma x_{\Gamma'}) = S(x_{\Gamma'})S(x_\Gamma)$$

Definition

$x_\Gamma \in \mathcal{H}$ is primitive if $\Delta(x_\Gamma) = x_\Gamma \otimes 1 + 1 \otimes x_\Gamma$.

Hopf Algebra: Grading

There is a grading on $\mathcal{H}$ given by the loop number: For x a monomial in $\mathcal{H}$, $x \in \mathcal{H}^n \Leftrightarrow \dim H_1(x) = n$. $\mathcal{H}^0 = \mathbb{C}$.

For x a monomial in $\mathcal{H}$, $x \in \mathcal{H}^n$

$$Y(x) = nx$$

Hopf Algebra: Affine Group Scheme

- $G = \mathrm{Spec}\ \mathcal{H}$
- Hopf algebra relations $\leftrightarrow$ group axioms.

$$\begin{aligned}(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta &\quad\leftrightarrow\quad \text{multiplication}\\ (id \otimes \varepsilon)\Delta = id &\quad\leftrightarrow\quad \text{identity}\\ m(S \otimes id)\Delta = \varepsilon\eta &\quad\leftrightarrow\quad \text{inverse}\end{aligned}$$

- G is an affine group scheme.
- G is a covariant functor.

$$\begin{aligned}\mathcal{C}(\mathbb{C} - alg) &\quad\rightarrow\quad \mathrm{Hom}_{alg}(\mathcal{H}, *)\\ A &\quad\rightarrow\quad \mathrm{Hom}_{alg}(\mathcal{H}, A)\end{aligned}$$

- $G(A) = A$ valued points of G

Hopf Algebra: Restricted Dual

Indecomposable elements of $\mathcal{H}^\vee$ are the generators of

$$\mathbb{C} < \delta_{x_\Gamma} | \Gamma \in \{1\text{PI graphs of } \mathcal{L}\} >$$

$\delta_{x_\Gamma}(x_{\Gamma'})$ is the Kronecker delta function.
Multiplication is the convolution product:

$$\delta_{x_\Gamma} \star \delta_{x_{\Gamma'}}(x) = (\delta_{x_\Gamma} \otimes \delta_{x_{\Gamma'}})(\Delta x)$$

Γ, Γ' 1PI, $x \in \mathcal{H}$.
Comultiplication shows the indecomposables are primitive:

$$\Delta \delta_{x_\Gamma}(x \otimes y) = \delta_\Gamma(xy) = \delta_{x_\Gamma}\varepsilon(y) + \varepsilon(x)\delta_{x_\Gamma}(y)$$

Hopf Algebra: Milnor-Moore

Theorem

Milnor-Moore *Given a connected, graded, cocommutative, locally finite Hopf algebra, H, there is a Hopf algebra isomorphism, $H \simeq \mathcal{U}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra generated by the indecomposable elements of H.*

The Milnor-Moore theorem holds on the restricted dual of $\mathcal{H}$.

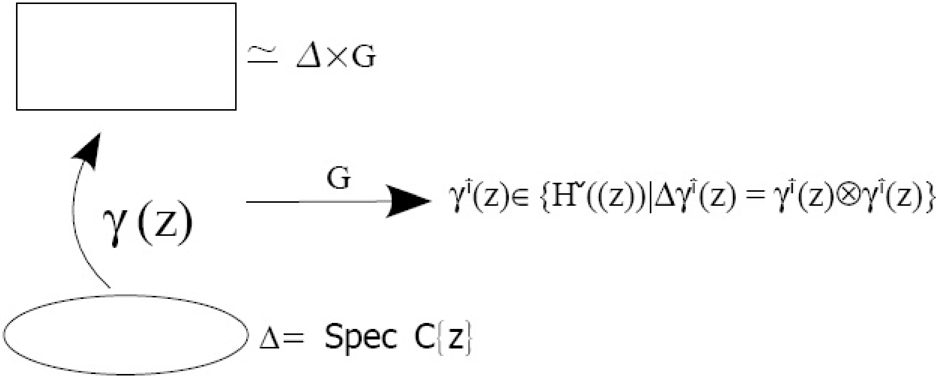
$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{H}^\vee = \bigoplus_n (\mathcal{H}^n)^\vee = \bigoplus_n \mathcal{H}_n$$

This gives a grading on $\mathcal{H}^\vee$.

$$\mathfrak{g} = \mathrm{Lie}\ G(\mathbb{C})$$

$\{\text{Generators of } \mathfrak{g}\} \sim \{\text{Primitives of } \mathcal{H}^\vee\} \sim \{\text{Indecomposables of } \mathcal{H}\} \sim \{\text{1PI graphs}\}$

Hopf Algebra: Contravariant Relation



Hopf Algebra: Summary

- Defined $\mathcal{H}$ the Hopf algebra of Feynman graphs
- Defined G of $\mathcal{H} = \mathbb{C}[G]$
- $G(A) = \mathrm{Hom}_{alg}(\mathcal{H}, A)$
- Defined $\mathcal{H}^\vee \simeq \mathcal{U}(\mathrm{Lie}\ G(\mathbb{C}))$
- G takes $\{\text{sections}\} \to \{\gamma^\dagger(z) \in \mathcal{H}^\vee((z)) | \Delta\gamma^\dagger(z) = \gamma^\dagger(z) \otimes \gamma^\dagger(z)\}$

Renormalization

Renormalization: For Quantum Field Theory

The fields in quantum field theory interact with themselves, but this interaction cannot be turned off.

- $U_\Gamma \leftarrow$ divergent (unrenormalized) Feynman integrals associate to 1PI graphs.
- $R_\Gamma \leftarrow$ renormalized integral.
- $C_\Gamma \leftarrow$ counterterm.
- Renormalization separates $U(\Gamma)$ into an iterated product of R_Γ, and a counterterm, C_Γ.

Renormalization: The Process

The process of extracting finite values from these divergent integrals is twofold.

1. **regularize** the integral: rewrite them in terms of a set of parameters that yields a sensible value away from predetermined limit.
2. **renormalize** away any divergences that still occur after regularization.

Dimensional Regularization analytically continues the dimension of the theory, to a complex ϵ ball around 6. $z = D - 6$.

BPHZ Renormalization is a recursive formula for extracting finite values from dimensionally regularized divergent integrals.

Renormalization: Dimensional Regularization

Rewrite the divergent Feynman integrals:

$$f(p_1, \ldots, p_n) \int_{\mathbb{R}^6} \frac{d^6 k}{\prod \Delta_i} = \frac{\imath A_D}{(2\pi)^D} \int_0^\infty dr\, r^{D-1} f(-r^2)$$

$A_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} =$ area of a unit sphere in D dimensions.

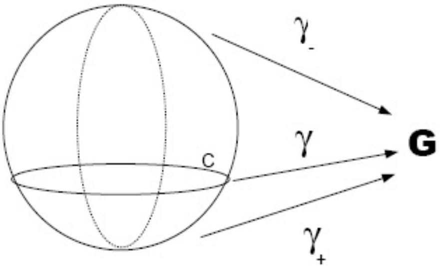
$$= \frac{\imath}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \int_0^\infty dr\, r^{D-1} f(-r^2)$$

All poles are now captured in the dimension parameter.

Renormalization: Birkhoff Decomposition

Theorem

Birkhoff Decomposition Theorem *Let C be a smooth simple curve in $\mathbb{CP}^1$ separating it into two connected components: $\infty \in C_-$, $0 \in C_+$. For G a simply connected complex Lie group and $\gamma : C \to G$, there are holomorphic maps $\gamma_\pm : C_\pm \to G$ such that $\boxed{\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z)}$ on C. This decomposition is unique up to the normalization $\gamma_-(\infty) = 1$.*



Renormalization: Birkhoff Decomposition variant

- $\Delta =$ the infinitesimal disk of complex dimension around $z = 0$
- $\gamma : \Delta \to G \quad \to \quad \gamma(z) = \gamma_-^{-1}(z)\gamma_+(z)$
- By the functor G,

$$\gamma^\dagger(z) = \gamma_-^{\dagger\star-1}(z) \star \gamma_+^\dagger(z)$$

  - $\gamma_-^{\dagger\star-1}(z) \in G(\mathbb{C}[z])$
  - $\gamma_+^\dagger(z) \in G(\mathbb{C}\{z\})$
  - $\gamma^\dagger(z) \in G(\mathbb{C}((z)))$

Renormalization: BPHZ

Theorem

Connes Kreimer *For $x_\Gamma \in \mathcal{H}$, Birkhoff decomposition gives*

$$\gamma_-^\dagger(z)(x_\Gamma) = -\pi(\gamma^\dagger(z)(x_\Gamma) + \sum_{(x_\Gamma)} \gamma_-^\dagger(z)(x'_\Gamma) \star \gamma^\dagger(z)(x''_\Gamma))$$

$$\gamma_+^\dagger(z)(x_\Gamma) = \gamma^\dagger(z)(x_\Gamma) + \gamma_-^\dagger(z)(x_\Gamma) + \sum_{(x_\Gamma)} \gamma_-^\dagger(z)(x'_\Gamma) \star \gamma^\dagger(z)(x''_\Gamma)$$

$\gamma^\dagger(z)(x_\Gamma) = U_\Gamma(z)$ $\gamma_+^\dagger(z)(x_\Gamma) = R_\Gamma(z)$ $\gamma_-^\dagger(z)(x_\Gamma) = C_\Gamma(z)$

The BPHZ renormalization process gives:

$$C_\Gamma = -\pi(U_\Gamma + \sum_{(\Gamma)} C'_\Gamma U''_\Gamma)$$

$$R_\Gamma = U_\Gamma + C_\Gamma + \sum_{(\Gamma)} C'_\Gamma U''_\Gamma$$

Renormalization: A Rota-Baxter Perspective

Definition

A Rota-Baxter Algebra is pair (A, R), where A is an algebra over a commutative ring k and R is a linear operator on A such that for x, $y \in A$,

$$R(x)R(y) + \theta R(xy) = R(R(x)y) + R(xR(y))$$

where $\theta \in k$ is the weight.

$(\mathbb{C}((z)), \pi)$ is a Rota-Baxter algebra of weight 1:

$$\begin{aligned} \pi : \mathbb{C}((z)) &\rightarrow z^{-1}\mathbb{C}[z^{-1}] \\ \sum_{-n}^{\infty} a_i z^i &\rightarrow \sum_{-n}^{-1} a_i z^i \end{aligned}$$

Renormalization: A Rota-Baxter Perspective

Theorem

Fard, Guo, Kreimer *Let (A, R) be a complete filtered Rota Baxter algebra of non-zero weight. For $\gamma^\dagger \in G(A)$, one can write $\gamma^\dagger = e^a$ with $a \in Hom(\mathcal{H}, A)^{(1)}$. Let $u = \gamma^\dagger - (\eta \circ \varepsilon) \in Hom(\mathcal{H}, A)^{(1)}$.*

1. *$P : Hom(\mathcal{H}, A) \to Hom(\mathcal{H}, A)$ is a Rota-Baxter operator given by $P = R \circ f$. $(Hom(\mathcal{H}, A), P)$ is a filtered non-commutative, associative, unital Rota-Baxter algebra.*
2. *The Birkhoff decompositions are*

$$\gamma^\dagger_- = -R(\gamma^\dagger + \sum_\gamma \gamma^\dagger_- \gamma^\dagger) = (\eta \circ \varepsilon) - P(\gamma^\dagger_- \star u)$$

and

$$\gamma^\dagger_+ = \tilde{R}(\gamma^\dagger + \sum_\gamma \gamma^\dagger_- \gamma^\dagger) = (\eta \circ \varepsilon) - \tilde{P}(\gamma^\dagger_+ \star (\gamma^{\dagger \star -1} - (\eta \circ \varepsilon)))$$

Renormalization: The Renormalization Group

The renormalization group describes how the dynamics of a system depends on the scale at which it is probed.

The process of dimensional regularization transforms the coupling constant

$$g \mapsto t^z g$$

where $t \in \mathbb{C}^\times$. I will also write this as $t = e^s$ for $s \in \mathbb{C}$.

For $\mathcal{H}^\vee$,

$$\theta_s(\gamma^\dagger(z)(x)) = \gamma^\dagger(z)(e^{sY}(x))$$

θ_s is the renormalization group.

Renormalization:Renormalization group flow generator

- Renormalization group gives sections of the $P \to B$ bundle corresponding to $\gamma^\dagger(z,t) = t^Y \gamma^\dagger(z)(x)$
- Renormalization group flow: $F_t(\gamma(z,t)) = \frac{d}{dt}\gamma^\dagger(z,t)$.
- Renormalization group flow generator: $\beta(z,t)(\gamma(z,t)) = \lim_{z\to 0} F_1(\gamma(z,t))$

β is key in describing how the Lagrangian changes with the renormalization mass.

Renormalization: Important Physical Condition

The counterterms of a physical Lagrangian do not depend on the renormalization mass scale. This is expressed by the restriction to $G(\mathbb{C}((z)))$:

$$G^{\Phi}(\mathbb{C}((z))) = \{\gamma^{\dagger} | \frac{d}{ds}(\theta_{sz}\gamma^{\dagger})_{-} = 0\}$$

which is satisfied by examples in the physical world.

Renormalization: Summary

- Δ comes from Dimensional Regularization
- Sections of the bundle decompose as BPHZ renormalization
- $\mathbb{C}^\times$ comes from the renormalization group
- Interested in sections of $\gamma^\dagger(z,t) = t^Y \gamma^\dagger(z)(x)$
- Defined the renormalization group flow generator

The Renormalization Bundle

The Renormalization Bundle: Construction

The renormalization bundle can be written in two stages as

$$P \to B \to \Delta$$

Δ is the infinitesimal disk of complex dimensions
B is the trivial $\mathbb{C}^\times$ principal bundle over Δ
P is the trivial G principal bundle over B

Δ^* is the punctured disk. B^* and P^* are the corresponding bundles missing the fiber over 0.

The Renormalization Bundle: The Connes-Marcolli Connection

- P^* has a trivial connection on it.
- A section of P^* pulls back its connection form to

$$\omega = \gamma^{\dagger -1}(z,t) d\gamma^{\dagger}(z,t) \in (\mathfrak{g}(\mathcal{A}) \rtimes \mathbb{C}) \otimes \Omega^1$$

- Interested in sections corresponding to $t^Y \gamma^{\dagger}(z) \in G(\mathbb{C}((z)))[t, t^{-1}]$.
  - Look like the renormalization group has acted on these.
  - Pull back ω to flat, $\mathbb{C}^{\times}$ invariant connection forms.
  - This last condition means that

$$\omega(z,t) = t^Y \omega(z,1)$$

The Renormalization Bundle: Defining the Connection Form

There is an element $\tilde{R}(\gamma^\dagger) = \gamma^{\dagger\star -1} \star (\gamma^\dagger(z) \circ Y) \in \mathfrak{g}(\mathcal{A})$ that determines the connection one form ω by

$$\gamma^\dagger(z) = Te^{-\int_0^\infty \theta_{-t}\tilde{R}(\gamma^\dagger)dt}$$

The Renormalization Bundle: $G(\mathcal{A})$ Gauge Equivalence

Let $\omega = \gamma^{\dagger -1} d\gamma^\dagger$ and $\omega = \gamma'^{\dagger -1} d\gamma'^\dagger$

The two connection 1-forms $\omega \sim \omega'$ are equivalent if and only if $\gamma'^\dagger(z, 1) = \gamma^\dagger(z, 1)\phi(z, 1)$ for some holomorphic function $\phi \in G(\mathcal{A})$. In other words,

$$\omega \sim \omega' \Leftrightarrow \gamma_-^\dagger(z, 1) = \gamma_-'^\dagger(z, 1)$$

The Renormalization Bundle: Equisingularity

Equisingularity is a geometric generalization of the "physica" condition $\frac{d}{ds}(\theta_{sz}\gamma^{\dagger})_{-} = 0$.

Definition

If two section of the bundle $P^* \to \Delta^*$, $\gamma^{\dagger}(z, \sigma_1(z))$, and $\gamma^{\dagger}(z, \sigma_2(z))$ have the property

1. $\gamma_{-}^{\dagger\star-1}(z, \sigma_1(z)) \sim \gamma_{-}^{\dagger\star-1}(z, \sigma_2(z)) \quad \forall \sigma_i(z) \text{ s.t. } \sigma_1(0) = \sigma_2(0)$
2. the pull back of the connection form ω is $\mathbb{C}^{\times}$ invariant

the pullback of the connection form ω is equisingular.

The Renormalization Bundle: Connes Marcolli Main Theorem

Theorem

If γ defines an equisingular connection one form, then $\gamma^\dagger \in G^\Phi(\mathcal{A})$. That is, any equisingular connection form

$$\gamma^\dagger \sim Te^{-z^{-1}\int_0^\infty \theta_{-t}\beta(\gamma_-^{\dagger\star-1})dt}$$

$\beta(\gamma_-^{\dagger\star-1}) \in \mathfrak{g}$ is the renormalization group flow generator.

Renormalization Bundle: Summary

Connes and Marcolli have taken the renormalization process for a scalar field theory, and interpreted it geometrically.

1. Defined a Hopf algebra for the theory
2. Created a bundle over the regularization parameter space
3. Identified renormalization with the sections of this bundle
4. "Physical" sections correspond to connection forms uniquely determined by the renormalization group generator

Future hopes and dreams

Generalizations: The Renormalization Bundle for Other Theories

The Rota-Baxter perspective to the Birkhoff Decomposition allows for the decomposition of a more general class of regularization scheme. For instance, one can make small changes to this bundle by changing Δ to Δ^n to account for renormalization schemes with multiple parameters (ζ function renormalization). Or, one can change the number of renormalization mass parameters (for theories like QCD.)

Generalizations: Zeta Function Renormalization

The propagators in the Feynman diagrams are defined as the Green's functions for the Laplacian in the Minkowski metric. One can build an analog of this in on a compact manifold. Let Δ_M be the Laplacian on a compact manifold M in d dimensions. Then the Feynman integral becomes

$$\int_{\mathbb{R}^d} f(k) \frac{d^d k}{\prod \Delta_{Mi}}$$

Presumably, this can be renormalized via zeta function renormalization onto a Rota-Baxter algebra V. Following the methods of this talk, one hopes to then define a renormalization group generator for Feynman integrals over a generalized background manifold.

Generalizations: QFT Over Curved Space Time

Without explicitly stating so, I have been ignoring the external leg structure of the Feynman graphs in this talk. To include these, one needs to reattach the external legs in the bundle

$$P \times \mathbb{R}^n \to B \times \mathbb{R}^n \to \Delta \times \mathbb{R}^n$$

where $\mathbb{R}^n$ (or rather its Fourier transform) contains the information about the external momenta of the interacting fields. To complete the picture in generalized space time, rewrite the bundle

$$P \times M \to B \times M \to \Delta \times M$$

and reattach the legs.