Preliminary Exam – 1999

Morning Part

Instructions: No calculators or crib sheets are allowed. Do as many problems as you can. Justify your answers as much as you can but very briefly.

1. For positive real x, let $B_n(x) = 1^x + 2^x + ... + n^x$. Prove or disprove the convergence of
$$\sum_{n=2}^{\infty} \frac{B_n(\log_n 2)}{(n \log_2 n)^2}.$$

2. Let $y(t)$ be the solution of
$$\frac{dy}{dt} = p(y), \quad y(0) = 0$$
where p is an odd degree polynomial

   True of false.

   (a) There is a choice of $p(y)$ such that $y(t) \to -\infty$ as $t \to -\infty$.

   (b) For every $p(y)$, $y(t) \to -\infty$ as $t \to -\infty$.

   (c) There is a choice of $p(y)$ such that $y(t) \to \infty$ as $t \to \infty$.

   (d) For every $p(y)$, $y(t) \to \infty$ as $t \to \infty$.

   (e) There is a choice of $p(y)$ such that $y(t) \to -\infty$ as $t \to -\infty$ and $y(t) \to \infty$ as $t \to \infty$.

   (f) For every $p(y)$, $y(t) \to -\infty$ as $t \to -\infty$ and $y(t) \to \infty$ as $t \to \infty$.

3. For
$$\frac{dy}{dt} = a(t)y, \quad y(0) = 1$$
verify that
$$y(t) = e^{\int_0^t a(s)\,ds}$$
is a solution. (Be sure to state all necessary theorems from Calculus.)

4. Consider the initial value problems
$$\begin{aligned} \frac{dy}{dt} &= f(y)\sin^2 y, \quad y(0) = y_0 \\ \frac{dy}{dt} &= f(y), \quad y(0) = y_0 \end{aligned}$$
where $f(y)$ is a smooth, nowhere negative function. If $y_1(t)$ is a solution of the first problem and $y_2(t)$ is a solution of the second problem, verify that $y_1(t) \leq y_2(t)$ for all t.

5. Consider the linear system

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{bmatrix} a & 2 \\ b & 3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

   (a) For what values of a and b do solutions oscillate (i.e., do both $x(t)$ and $y(t)$ pass through zero infinitely many times). Sketch the region in the ab plane.

   (b) For what values of a and b do all solutions tend to the origin. Sketch the region in the ab plane.

6. Consider the equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + 3y = 0.$$

   For what value of p in the range $0 < p \leq 10$,

   (a) Some of the solutions tend to zero at the quickest possible rate (and how fast is this)?

   (b) Most of the solutions tend to zero at the quickest possible rate (and how fast is this)?

   (c) All of the solutions tend to zero at the quickest possible rate (and how fast is this)?

7. Suppose A is an $n \times n, (n \geq 4)$ matrix with eigenvalue λ. Let A_t be the matrix A plus the matrix which is equal to t in the $2, 3$ location and $-t$ in the $1, 4$ location. Will λ be an eigenvalue of A_t? (Hint: give a condition on A which guarentees that λ will not be an eigenvalue of A_t for $t \neq 0$.)

8. Consider the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 & -3 & 0 \\ 1 & -1 & 2 & -3 & -4 & 1 \\ 1 & 0 & -1 & 0 & 1 & -1 \end{bmatrix}$$

   Note that $(2, 1, -1, -2, 1, 2)$ is an eigenvector for this matrix. What is the corresponding eigenvalue and is it the eigenvalue with the largest real part?

9. (a) Let

$$\gamma : \mathbf{R} \to \mathbf{R^2}.$$

   Now consider the map

$$,_1(t, s) = \gamma(t) + s\gamma'(t)$$
$$,_2(t, s) = \gamma(t) + s\gamma''(t).$$

   For what values of t and s are these maps locally one-to-one? Give a sketch (i.e., pictorial representation) of $,_1$ and $,_2$ including points where it is not onto. In particular, apply the above to $\gamma(t) = (t, t - t^3)$.

(b) Let

$$\zeta : \mathbf{R} \to \mathbf{R^3}.$$

Consider the map

$$Z_1(t,s) = \zeta(t) + s\zeta'(t)$$
$$Z_2(t,s) = \zeta(t) + s\zeta''(t)$$
$$Z_3(t,s) = \zeta(t) + s(\zeta'(t) \times \zeta''(t)).$$

The images of these maps are surfaces in $\mathbf{R^3}$. For each of these maps, for what values of t and s is the map locally one-to-one. Give sketches of typical surfaces of this type which include points which are not locally one-to-one. In particular, apply the above to

$$\zeta(t) = (\cos t, \sin t, t).$$

10. Find the Fourier series of the function $f : [-\pi, \pi] \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & x \in [-\pi, 0] \\ 1 & x \in (0, \pi] \end{cases}$$

11. Recall that the inner product used to define the Legendre polynomials is

$$< f_1, f_2 >= \int_{-1}^{1} f_1(x) f_2(x)\, dx$$

and that the first four Legendre Polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

How far (with length given by the inner product above) is the function

$$f(x) = x^3$$

from the subspace of the function space spanned by $P_0(x), P_2(x)$?

12. Let $C^\infty[0,1]$ denote the space of functions $f : \mathbf{R} \to \mathbf{R}$ which are infinitely differentiable. Let $C^\infty[0,1]$ have the metric $d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Show that the map $D : C^\infty[0,1] \to C^\infty[0,1]$ given by $Df(x) = f'(x)$ is not a continuous map with respect to this metric.

13. Verify the formula

$$\int\int_S 1\, dx\, dy = \int\int_{F(S)} \det(DF^{-1}|_{(u,v)})\, du\, dv$$

where $(u,v) = F(x,y)$ for $F : \mathbf{R^2} \to \mathbf{R^2}$ continuously differentiable, S is a subset of $\mathbf{R^2}$ and F is one to one on S.

Hint: This requires little computation–First verify it for F a linear map, then argue that it applies to nonlinear maps by looking at very small regions where F is well approximated by its linear part.

14. Compute the derivative of the determinant function at a non-singular matrix C in the direction D. Hint: The derivative is

$$\lim_{t \to 0} \frac{\det(C + tD) - \det(C)}{t}.$$

We can factor $\det(C)$ out of this limit–then we look at the resulting equation and recall that any term in the numerator with t^2 or higher power of t will go to zero in the limit. The result is

$$D\det|_C(D).$$

1999 Preliminary Exam

Afternoon Exam: 3 hours

Do as many problems as possible, choose enough
problems from different areas of algebra
(several problems test the same concept!)
Justify your answers as much as you can, briefly

1. The Fibonacci sequence $\{F_k\}_{k\geq 0}$: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ obeys the rule $F_{k+2} = F_{k+1} + F_k$, so that by letting

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

we have $u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

(i) Find a formula for the entries (you don't have to *compute* the entries) of A^{100} (hint: diagonalize A)

(ii) Show $F_k = \dfrac{1}{\sqrt{5}} \left[\left(\dfrac{1+\sqrt{5}}{2} \right)^k - \left(\dfrac{1-\sqrt{5}}{2} \right)^k \right] =$ nearest integer to $\dfrac{1}{\sqrt{5}} \left(\dfrac{1+\sqrt{5}}{2} \right)^k$

(iii) Find $\displaystyle\lim_{k\to\infty} \frac{F_k}{F_{k+1}}$

2. Explain why a 5 by 5 matrix (a_{ij}) $1 \leq i, j \leq 5$ with a 3 by 3 zero submatrix must be singular. (A submatrix is the set of all a_{ij} for

$$i \in \{i_1, i_2, i_3\} = \{1, 2, 3, 4, 5\} \quad \text{and} \quad \{j_1, j_2, j_3\} \subset \{1, 2, 3, 4, 5\})$$

3. Show that the trace of a (square) nilpotent matrix is 0 (the trace of a square matrix (a_{ij}), $1 \leq i, j \leq n$ is the sum of the diagonal elements $\sum_{i=1}^{n} a_{ii}$. A matrix A is said to be nilpotent if $A^m = 0$ for some positive integer m, where 0 denotes the zero matrix.)

4. How many generators are there in the group $(\mathbb{Z}_{18}, +)$? Same question for $(\mathbb{Z}_n, +)$ (use the prime decomposition of n).

5. What are all possible orders of the elements of the permutation group S_5?

6. Show that the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.

7. Which of the following are vector subspaces:

(i) in $\mathbb{R}^3$, the set of vectors $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with $b_1 b_2 = 0$.

(ii) in $\mathbb{R}^\infty$,

$$\begin{aligned} S_1 &= \{\text{all sequences which include infinitely many zeros}\} \\ S_2 &= \{\text{all convergent sequences } (x_j)\text{: the } x_j \text{ have a limit as } j \to \infty\} \\ S_3 &= \{\text{all arithmetic progressions } (x_j) : \ x_{j+1} - x_j \text{ is the same for all } J\} \\ S_4 &= \{\text{all geometric progressions } (x_1, kx_1, k^2x_1, \ldots) \text{ allowing all } k \text{ and } x_i\}. \end{aligned}$$

8. Let $I = (x^3 - 2x + 1)$ denote the ideal of the polynomial ring $\mathbb{Z}[x]$ generated by $x^3 - 2x + 1$. Is the ring $\mathbb{Z}[x]/(x^3 - 2x + 1)$ an integral domain?

Find an expression for the coset $(2x^7 - 7x^5 + 4x^3 - 9x + 1) + I$ in terms of the cosets $1 + I$, $x + I$ and $x^2 + I$.

9. Prove Fermat's and Wilson's theorems for a prime number p:

- $a^p \equiv a \pmod{p}$ for all $a \in \mathbb{Z}$
- $(p-1)! \equiv -1 \pmod{p}$

10. Let F be a field of characteristic different from 2. Show that if A is a square alternating matrix with coefficients in F, of size $n \times n$ with n an odd positive integer, then det $A = 0$ (alternating means that the transpose of A equals $-A$).

11. Find all finite subgroups of the multiplicative group of the complex numbers: $(\mathbb{C}\backslash\{0\}, \cdot)$.

12. Consider the following group (the quaternion group) Q: Q is generated by 3 elements i, j, k with the conditions: $i^2 = j^2 = k^2 = -1$ and

$$ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j$$

(i) Show that Q is isomorphic to the subgroup of $GL_2(\mathbb{C})$ (invertible complex 2×2 matrices) generated by

$$\begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(ii) Find all possible quotient groups of Q.

13. Find all possible quotient groups of $(\mathbb{Z}_n, +)$ (use the prime decomposition of n).

14. Compute the last two digits of 9^{1500}.

15. What are all the possible degrees for irreducible polynomials over the field F, in the following cases?

(i) $F = \mathbb{C}$
(ii) $F = \mathbb{Z}_p$ $(= \mathbb{Z}/p\mathbb{Z})$, any prime p
(iii) $F = \mathbb{R}$
(iv) $F = \mathbb{Q}$

16. Find the center Z of the dihedral group of order $2n$, $D_{2n}, (n \geq 2)$ and identify the quotient group D_{2n}/Z.

17. Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi(x, y) = x + y$. Show that π is a surjective homeomorphism and describe the Kernel and the fibers (i.e., inverse image of an element of the image) of π geometrically.

18. Consider the ring

$$\mathbb{Z}[\alpha] = \{\sum_{i \geq 0} n_i \alpha^i \mid n_i \in \mathbb{Z}, \text{ almost all } n_i = 0\}.$$

What is the degree over $\mathbb{Q}$ of the field of quotients of $\mathbb{Z}[\alpha]$ for: (i) $\alpha = 2^{1/3}$

(ii) $\alpha = \cos(2\pi/7) + i \sin(2\pi/7)$