Let \( a, b \in G \) where \( ab = ba \) and \( |a| = m \) and \( |b| = n \).
If \( \langle a \rangle \cap \langle b \rangle = \{e\} \) show \( \exists x \in G \) st. \( |x| = \text{lcm}(m,n) \).

Claim: \( x = ab \) is such an element.

Say \( |x| = k \). As such \( x^k = (ab)^k = e \)
But since \( ab = ba \) then \( (ab)^k = a^k b^k \) and so
\[
a^k b^k = e
\]

Observe now that since \( |a| = m \) and \( |b| = n \) then
\[
(ab)^{\text{lcm}(m,n)} = a^{\text{lcm}(m,n)} b^{\text{lcm}(m,n)} = e \cdot e = e
\]
Therefore \( k \mid \text{lcm}(m,n) \) \( \text{(X)} \)

Now, if \( (ab)^k = e \) then \( a^k b^k = e \rightarrow a^k = b^{-k} \)
But if so, then \( b^k \in \langle a \rangle \) and \( a^k \in \langle b \rangle \) (i.e. \( a^k \) is a power of \( b \) so is \( b^{-k} \)
However, we assume that \( \langle a \rangle \cap \langle b \rangle = \{e\} \)
So we must have \( a^k = e \) and \( b^{-k} = e \) (which implies \( b^k = e \))
Therefore, since \( |a| = m \) and \( |b| = n \) then \( m | k \) and \( n | k \) and so \( k \)
is a common multiple of \( m \) and \( n \) so \( \text{lcm}(m,n) \mid k \) \( \text{(X\#)} \)

But \( \text{(X)} \) and \( \text{(X\#)} \) together imply that \( k = \text{lcm}(m,n) \)

Note: You can apply this to question 49.
You also need the following fact:

If \( \langle a \rangle \) and \( \langle b \rangle \) are subgroups of \( G \) then if \( \gcd(|a|,|b|) = 1 \) then \( \langle a \rangle \cap \langle b \rangle = \{e\} \).